# The vector potential of a steady azimuthal current density. Once again

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We give an integral expression for the vector potential of a time-independent, steady azimuthal current density. Our derivation is substantially simpler and somewhat more general than others given in the literature. As an illustration, we recover the results for the vector potential of a circular current loop as an orthogonal expansion in spherical and cylindrical coordinates. Additionally, we obtain closed analytical expressions for the vector potential and the magnetic induction of a circular current loop in terms of Legendre functions of the second kind, that are simpler than the results in terms of complete elliptic integrals given in textbooks.

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## 1. Introduction

We consider a magnetostatic [1,2] system defined by a known time-independent current density $\boldsymbol{J}(\boldsymbol{r})$ which is in a steady state,

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}(\boldsymbol{r}) = 0. \tag{1}$$

We further assume, for simplicity, that $\boldsymbol{J}(\boldsymbol{r})$ is spatially localized, that is, there exists $R > 0$ such that $\boldsymbol{J}(\boldsymbol{r}) = 0$ if $|\boldsymbol{r}| > R$. Under these conditions, the vector potential $\boldsymbol{A}(\boldsymbol{r})$ in Coulomb gauge,

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}(\boldsymbol{r}) = 0, \tag{2}$$

satisfies the vector Poisson equation with boundary conditions given by,

$$\boldsymbol{\nabla}^2\boldsymbol{A}(\boldsymbol{r}) = -\mu_0\boldsymbol{J}(\boldsymbol{r}), \qquad \lim_{|\boldsymbol{r}|\to\infty}\boldsymbol{A}(\boldsymbol{r}) = 0\,. \tag{3}$$

From these equations it is clear that if $\boldsymbol{J}$ points in a fixed direction, $\boldsymbol{J} = J\hat{\boldsymbol{J}}$, then $\boldsymbol{A}$ must be parallel to $\boldsymbol{J}$, $\boldsymbol{A} = A_J\hat{\boldsymbol{J}}$. Indeed, for any unit vector $\hat{\boldsymbol{k}}$ satisfying $\hat{\boldsymbol{k}}\cdot\boldsymbol{J} = 0$, we obtain from (3) that $\hat{\boldsymbol{k}}\cdot\boldsymbol{A}$ satisfies the Laplace equation with a homogenous boundary condition at infinity, and must therefore vanish: $\hat{\boldsymbol{k}}\cdot\boldsymbol{A} = 0$.

If the current density points in the azimuthal direction,

$$\boldsymbol{J}(\boldsymbol{r}) = J_\varphi(\boldsymbol{r})\hat{\boldsymbol{\varphi}}, \qquad \hat{\boldsymbol{\varphi}} = -\sin(\varphi)\hat{\boldsymbol{x}} + \cos(\varphi)\hat{\boldsymbol{y}}, \tag{4}$$

with $\varphi$ the azimuthal coordinate in spherical or cylindrical coordinates, the steady-state condition (1) yields

$$\frac{\partial J_\varphi}{\partial\varphi}(\boldsymbol{r}) = 0. \tag{5}$$

In this case, because the unit vector $\hat{\boldsymbol{\varphi}}$ is not constant, it is not obvious that $\boldsymbol{A}$ is parallel to $\boldsymbol{J}$. Here, we show that for a stationary current density of the form (4), satisfying (5), we have

$$\boldsymbol{A}(\boldsymbol{r}) = A_\varphi(\boldsymbol{r})\hat{\boldsymbol{\varphi}},$$
$$A_\varphi(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\cos(\varphi'-\varphi)}{|\boldsymbol{r}-\boldsymbol{r}'|}J_\varphi(\boldsymbol{r}'). \tag{6}$$

Several comments on this equation are in order. First, on the $z$ axis the denominator becomes independent of $\varphi'$ and, therefore, the integral vanishes: $A_\varphi(z\hat{\boldsymbol{z}}) = 0$. Second, the denominator depends on $\varphi'$ only through $\varphi'-\varphi$, like the numerator, which means that we could set $\varphi$ to any fixed value in the integrand without changing the integral. Thus, $\boldsymbol{A}(\boldsymbol{r})$ does not depend on $\varphi$. The equality (6) with $\varphi = 0$ is proven in spherical coordinates in [3], and in cylindrical coordinates in Sec. 10.5.3 of [2]. The proof given below is substantially simpler than that in Ref. [3], and more general than the proofs in both [2, 3], as it is valid in spherical and cylindrical coordinates and, in fact, in any orthogonal system of curvilinear coordinates containing the azimutal angle $\varphi$. Below, we apply (6) to recover the known results for the vector potential of a circular current loop as an expansion in spherical harmonics, and as an integral transform of Bessel functions in cylindrical coordinates. We use the latter form, in turn, to find a new, simpler exact solution in closed analytical form for the vector potential and the magnetic induction of the circular current loop in terms of Legendre functions of the second kind, that are simpler than the results in terms of complete elliptic integrals given in textbooks.

This note is intended as reading material for graduate or advanced undergraduate students, to complement the treatment of magnetostatics in textbooks such as [1,2]. The paper is organized as follows. In the next Sec. , we provide an elementary proof of (6) that is valid for any coordinate system containing the azimuthal angle $\varphi$. In Sec. 3 we discuss the well-known example of the vector potential of a circular current loop as an orthogonal expansion in spherical (Sec. 3.1) and cylindrical (Sec. 3.2) coordinates. We also discuss closed analytical forms for the vector potential and magnetic induction in Sec. 3.3, where we derive a simpler expression than those found in textbooks. Finally, in Sec. 4, we provide a brief summary

## 2. Vector potential

The solution to Eqs. (3) with the condition (2) is given by [1, 2],

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{7}$$

From (4), (5) we then have,

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int dV_2' J_\varphi(\boldsymbol{r}') \int_{-\pi}^{\pi} d\varphi' \frac{\hat{\boldsymbol{\varphi}}'}{|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{8}$$

with

$$\int dV_2' = \begin{cases} \displaystyle\int_0^\infty dr' r'^2 \int_0^\pi d\theta' \sin(\theta') & \text{in spherical coordinates} \\ \displaystyle\int_0^\infty d\rho' \rho' \int_{-\infty}^{\infty} dz' & \text{in cylindrical coordinates} \end{cases}. \tag{9}$$

It will be useful below to define the notation

$$R(\alpha) = \left(r^2 + r'^2 - 2rr'\cos(\theta)\cos(\theta') - 2rr'\sin(\theta)\sin(\theta')\cos(\alpha)\right)^{\frac{1}{2}}, \tag{10a}$$

in spherical coordinates, and

$$R(\alpha) = \left(\rho^2 + z^2 + \rho'^2 + z'^2 - 2zz' - 2\rho\rho'\cos(\alpha)\right)^{\frac{1}{2}}, \tag{10b}$$

in cylindrical ones. The function $R(\alpha)$ in (10) has three properties that will be needed below: (i) it is a $2\pi$-periodic function of $\alpha$, (ii) it is an even function of $-\pi < \alpha < \pi$, and (iii) in both spherical and cylindrical coordinates it is true that

$$|\boldsymbol{r}-\boldsymbol{r}'| = R(\varphi' - \varphi). \tag{11}$$

We can then rewrite (8) as,

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int dV_2' J_\varphi(\boldsymbol{r}') \int_{-\pi}^{\pi} d\varphi' \frac{-\sin(\varphi')\hat{\boldsymbol{x}} + \cos(\varphi')\hat{\boldsymbol{y}}}{R(\varphi'-\varphi)}. \tag{12}$$

Let $\varphi'' = \varphi' - \varphi$, then

$$= \frac{\mu_0}{4\pi}\int dV_2' J_\varphi(\boldsymbol{r}') \int_{-\pi+\varphi}^{\pi+\varphi} d\varphi'' \frac{-\sin(\varphi''+\varphi)\hat{\boldsymbol{x}} + \cos(\varphi''+\varphi)\hat{\boldsymbol{y}}}{R(\varphi'')}. \tag{13}$$

We now use the trigonometric relation

$$-\sin(\varphi''+\varphi)\hat{\boldsymbol{x}} + \cos(\varphi''+\varphi)\hat{\boldsymbol{y}} = -\sin(\varphi'')\hat{\boldsymbol{n}} + \cos(\varphi'')\hat{\boldsymbol{\varphi}}, \tag{14}$$

with $\hat{\boldsymbol{\varphi}}$ as in (4) and

$$\hat{\boldsymbol{n}} = \begin{cases} \sin(\theta)\hat{\boldsymbol{r}} + \cos(\theta)\hat{\boldsymbol{\theta}} & \text{in spherical coordinates} \\ \hat{\boldsymbol{\rho}} & \text{in cylindrical coordinates} \end{cases}, \tag{15}$$

to rewrite (13) as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int dV_2' J_\varphi(\boldsymbol{r}') \int_{-\pi+\varphi}^{\pi+\varphi} d\varphi'' \frac{-\sin(\varphi'')\hat{\boldsymbol{n}} + \cos(\varphi'')\hat{\boldsymbol{\varphi}}}{R(\varphi'')}, \tag{16}$$

Because the integrand is $2\pi$-periodic in $\varphi''$, we have,

$$= \frac{\mu_0}{4\pi} \int dV_2' J_\varphi(\boldsymbol{r}') \int_{-\pi}^{\pi} d\varphi'' \frac{-\sin(\varphi'')\hat{\boldsymbol{n}} + \cos(\varphi'')\hat{\boldsymbol{\varphi}}}{R(\varphi'')}, \tag{17}$$

and because the term proportional to $\hat{\boldsymbol{n}}$ in the integrand is an odd function of $\varphi''$, its integral vanishes and we have,

$$= \frac{\mu_0}{4\pi} \hat{\boldsymbol{\varphi}} \int dV_2' J_\varphi(\boldsymbol{r}') \int_{-\pi}^{\pi} d\varphi'' \frac{\cos(\varphi'')}{R(\varphi'')}. \tag{18}$$

We can now make the change of variable $\varphi'' = \varphi' - \varphi$ to obtain,

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \hat{\boldsymbol{\varphi}} \int dV_2' J_\varphi(\boldsymbol{r}') \int_{-\pi+\varphi}^{\pi+\varphi} d\varphi' \frac{\cos(\varphi' - \varphi)}{R(\varphi' - \varphi)}. \tag{19}$$

By invoking the $2\pi$-periodicity of the integrand as a function of $\varphi'$, we get

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \hat{\boldsymbol{\varphi}} \int dV_2' J_\varphi(\boldsymbol{r}') \int_{-\pi}^{\pi} d\varphi' \frac{\cos(\varphi' - \varphi)}{R(\varphi' - \varphi)}, \tag{20}$$

and using (9), (11) we arrive at (6).

## 3. The circular current loop

As a simple application of (6), we consider the vector potential of a planar circular current loop of radius $a$ and current $I$ in spherical coordinates, as discussed in Sec. 5.5 of [1], as well as in cylindrical ones, as discussed in Example 10.6 of [2].

### 3.1. Spherical coordinates

In spherical coordinates, the current density of a circular loop lying at the origin on the $xy$ plane is given by (4) with

$$J_\varphi = \frac{I}{a} \sin\theta' \delta(\cos\theta') \delta(r' - a). \tag{21}$$

Substituting this value in (6) we get,

$$A_\varphi = \frac{\mu_0}{4\pi} a I \int_{-\pi}^{\pi} d\varphi' \left[ \frac{\cos(\varphi' - \varphi)}{|\boldsymbol{r} - \boldsymbol{r'}|} \right]_{\substack{r'=a \\ \theta'=\pi/2}} . \tag{22}$$

This integral is computed in [1] both in closed form in terms of elliptic functions in Eq. (5.37), and as an expansion in Legendre associated functions in Eq. (5.46). We consider the former in Sec. 3.3. below, and in this Sec. we briefly comment on the latter.

We employ the expansion of $1/|\boldsymbol{r} - \boldsymbol{r'}|$ in spherical harmonics, as given in Eq. (3.70) of [1] or (4.83) of [2], to obtain

$$\begin{aligned} A_\varphi &= \mu_0 a I \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} \int_{-\pi}^{\pi} d\varphi' \cos(\varphi' - \varphi) Y_{\ell,m}(\theta, \varphi) Y_{\ell,m}^*(\pi/2, \varphi') \\ &= \mu_0 a I \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell,m}(\theta, 0) Y_{\ell,m}(\pi/2, 0) \int_{-\pi}^{\pi} d\varphi' \cos(\varphi' - \varphi) e^{-im(\varphi' - \varphi)}, \end{aligned} \tag{23}$$

where $r_< = \min\{a, r\}$ and $r_> = \max\{a, r\}$ and in the second line we used $Y_{\ell,m}(\theta, \varphi) = Y_{\ell,m}(\theta, 0) e^{im\varphi}$. The integration over the azimuthal angle in (23) is immediate

$$\int_{-\pi}^{\pi} d\varphi' \cos(\varphi' - \varphi) e^{-im(\varphi' - \varphi)} = \pi(\delta_{-1m} + \delta_{1m}). \tag{24}$$

Substituting this in (23) and using $Y_{\ell,-m}(\theta, 0) = (-1)^m Y_{\ell,m}(\theta, 0)$ yields

$$A_\varphi = 2\pi \mu_0 a I \sum_{\ell=1}^{\infty} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell,1}(\theta, 0) Y_{\ell,1}(\pi/2, 0). \tag{25}$$

TABLE I. Values of $2\pi/\mu_0 I A_\varphi$, for the indicated values of $\rho/a$, $z/a$, obtained from (30), (31), (33) with [4].

| $\rho/a$ | $z/a$ | (30) | (31) | (33) |
|---|---|---|---|---|
| 0.35 | 0.0 | 0.577159 | 0.577159 | 0.577159 |
| 3.50 | 0.0 | 0.132367 | 0.132367 | 0.132367 |
| 0.35 | 1.74 | 0.065912 | 0.065912 | 0.065912 |
| 3.50 | 1.74 | 0.091995 | 0.091995 | 0.091995 |
| $10^{-6}$ | 1.74 | $1.94335\times10^{-7}$ | $1.94335\times10^{-7}$ | $1.94735\times10^{-7}$ |

We can expand the spherical harmonics into an associated Legendre function of $\cos\theta$ and imaginary exponential of $\varphi$ as in Eq. (3.53) of [1] or (C.18) of [2], to obtain the equivalent form

$$A_\varphi = \frac{\mu_0 a I}{2}\sum_{\ell=1}^{\infty}\frac{1}{(\ell+1)\ell}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell^1(0)P_\ell^1(\cos\theta). \tag{26}$$

This is our result for the vector potential of the circular current loop. Since $P_\ell^1(0)$ vanishes for even $\ell$, the sum in (25) runs only over odd values of $\ell$. Furthermore, by using the expression for $P_\ell^1(0)$ given in Eq. (5.45) of [1], we can show that (25) is mathematically equivalent to the expression for the vector potential given in Eq. (5.46) of [1]. To obtain the magnetic field we have to take the curl of (6), (25); this has been explicitly done in [1] for the circular loop, and in [3] for the potential (6). There is no need to repeat those calculations here.

### 3.2. Cylindrical coordinates

In cylindrical coordinates, the current density of a circular loop lying at the origin on the $xy$ plane is given by (4) with

$$J_\varphi = I\delta(z')\delta(\rho'-a). \tag{27}$$

Substituting this value in (6) we get,

$$A_\varphi = \frac{\mu_0}{4\pi}aI\int_{-\pi}^{\pi}d\varphi'\left[\frac{\cos(\varphi'-\varphi)}{|\boldsymbol{r}-\boldsymbol{r'}|}\right]_{\substack{\rho'=a\\ z'=0}}. \tag{28}$$

We use now the expansion of $1/|\boldsymbol{r}-\boldsymbol{r'}|$ as a trigonometric series of Laplace transforms given in Problem 3.16 of [1], which we repeat here for convenience,

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r'}|} = \sum_{m=-\infty}^{\infty}e^{-im(\varphi'-\varphi)}\int_0^\infty dk J_m(k\rho')J_m(k\rho)e^{-k|z-z'|}. \tag{29}$$

Substituting (29) in (28) and using the angular integral (24) and the relation $J_{-1}(x) = -J_1(x)$ yields,

$$A_\varphi(\boldsymbol{r}) = \frac{\mu_0 I a}{2}\int_0^\infty dk J_1(ka)J_1(k\rho)e^{-k|z|}, \tag{30}$$

which expresses the vector potential (6) as an integral transform in cylindrical coordinates. Equation (30) recovers the result in Example 10.6 in Sec. 10.5.4 of [2].

### 3.3. Analytically closed expressions: vector potential

The integral in (30) is a Laplace transform that can be readily evaluated with the software system Mathematica [4] in analytically closed form in terms of complete elliptic integrals,

$$A_\varphi(\boldsymbol{r}) = \frac{\mu_0 I}{2\pi}\frac{\sqrt{(a+\rho)^2+z^2}}{\rho}\left(\frac{a^2+\rho^2+z^2}{(a+\rho)^2+z^2}K(\kappa)-E(\kappa)\right),\quad \kappa=\sqrt{\frac{4a\rho}{(a+\rho)^2+z^2}}. \tag{31}$$

Here, $K$ and $E$ are the complete elliptic integrals of the first and second kind, respectively, as defined in Sec. 13.8 of [5] (see also [6] for quick reference). From a practical point of view, we point out that in Mathematica one has,

$$K(\kappa) = \texttt{EllipticK}[\kappa^2],\quad E(\kappa) = \texttt{EllipticE}[\kappa^2], \tag{32}$$

notice the exponents in the arguments. The result (31) can be converted to spherical coordinates by substituting $\rho = r\sin\theta, z = r\cos\theta$. Doing so leads to Jackson's textbook result as given in its Eq. (5.37) [1].

On the other hand, the Laplace transform in (30) is evaluated in analytically closed form in Eq. (13) of Sec. 4.14 of the table of integral transforms [7], which yields the simpler expression,

$$A_\varphi(\boldsymbol{r}) = \frac{\mu_0 I}{2\pi}\sqrt{\frac{a}{\rho}}Q_{\frac{1}{2}}(\xi), \quad \xi = \frac{\rho^2+z^2+a^2}{2a\rho}. \tag{33}$$

In this equation $Q_{\frac{1}{2}}$ is a Legendre function of the second kind, as defined in chapter III of [8]. $Q_{\frac{1}{2}}(x)$ is analytic in the complex $x$ plane cut along the real axis from $-\infty$ to 1, and has a logarithmic branch point at $x = 1$. Notice that in (33) the argument is $\xi > 1$ for $\rho \neq a$ or $z \neq 0$. It is only at $\rho = a$, $z = 0$, where the current loop is located, that $\xi = 1$ and $Q_{\frac{1}{2}}(\xi)$ diverges. The implementation of this function in Mathematica is given by

$$Q_\nu(x) = Q_\nu^0(x) = \texttt{LegendreQ}[\nu, \texttt{0}, \texttt{3}, \texttt{x}], \tag{34}$$

where the argument "3" indicates the position of the cut. As an illustration of the mathematical equivalence of Eqs. (30), (31), (33), in Table I we give several numerical values for $(2\pi/\mu_0 I)A_\varphi$ from those equations, with the integral in (30) and the special functions in (31), (33) evaluated numerically with Mathematica. We find perfect equality of the three values to six decimals at all values of $\rho/a$, $z/a$, except for a minor numerical difference of order $10^{-10}$ near the $z$ axis, where $A_\varphi$ vanishes.

The equality between the two expressions (31) and (33) for $A_\varphi(\boldsymbol{r})$ leads to a neat relation between the Legendre function $Q_{\frac{1}{2}}$ and the complete elliptic integrals $K$, $E$ that can be written as,

$$Q_{\frac{1}{2}}(z) = \sqrt{\frac{2}{z+1}}\left[zK\left(\sqrt{\frac{2}{z+1}}\right) - (z+1)E\left(\sqrt{\frac{2}{z+1}}\right)\right], \tag{35}$$

and can be easily checked numerically.

### 3.4. Analytically closed expressions: magnetic induction field

The magnetic induction field is obtained as the curl of the vector potential (33) in cylindrical coordinates. The partial derivatives can be expressed in terms of the derivative [8]

$$\frac{d^m Q_\nu}{dz^m}(z) = (z^2-1)^{-\frac{m}{2}}Q_\nu^m(z), \quad m = 1, 2, 3, \ldots, \tag{36}$$

with $\nu = 1/2$ and $m = 1$. This way we are led to the magnetic induction field,

$$\begin{aligned} B_\rho &= -\frac{\partial A_\varphi}{\partial z} = -\frac{\mu_0 I}{2\pi}\frac{2\sqrt{a/\rho}\,z}{\sqrt{(\rho-a)^2+z^2}\sqrt{(\rho+a)^2+z^2}}Q_{\frac{1}{2}}^1(\xi), \\ B_z &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\varphi) = \frac{\mu_0 I}{2\pi}\frac{1}{\rho}\sqrt{\frac{a}{\rho}}\left(\frac{1}{2}Q_{\frac{1}{2}}(\xi) + \frac{\rho^2-a^2-z^2}{\sqrt{(\rho-a)^2+z^2}\sqrt{(\rho+a)^2+z^2}}Q_{\frac{1}{2}}^1(\xi)\right), \end{aligned} \tag{37}$$

with $\xi$ as in (33). As a verification, by using the power expansions as $\rho \to 0$ [8],

$$\begin{aligned} Q_{\frac{1}{2}}(\xi) &= \frac{\pi}{2}\frac{a^{\frac{3}{2}}}{(a^2+z^2)^{\frac{3}{2}}}\rho^{\frac{3}{2}} + O(\rho^{\frac{7}{2}}), \\ Q_{\frac{1}{2}}^1(\xi) &= -\frac{3\pi}{4}\frac{a^{\frac{3}{2}}}{(a^2+z^2)^{\frac{3}{2}}}\rho^{\frac{3}{2}} + O(\rho^{\frac{7}{2}}), \end{aligned} \tag{38}$$

which can also be obtained with Mathematica, we find the magnetic induction field on the $z$ axis,

$$B_\rho(z\hat{\boldsymbol{z}}) = 0, \qquad B_z(z\hat{\boldsymbol{z}}) = \frac{\mu_0}{2\pi}\frac{I\pi a^2}{(a^2+z^2)^{\frac{3}{2}}}. \tag{39}$$

This way we recover the result on the $z$ axis that is more usually obtained from the standard Amperian integral

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int d^3r'\boldsymbol{J}(\boldsymbol{r}') \wedge \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^{\frac{3}{2}}},$$

with $\boldsymbol{r} = z\hat{\boldsymbol{z}}$ and $\boldsymbol{J}$ from (27).

## 4. Final remarks

In this brief note, we present Eq. (6) for the vector potential of a steady-state azimuthal current density and provide an elementary proof in Sec. 2. Our proof is simpler than the spherical-coordinate derivation in Ref. [3] and more general than the cylindrical-coordinate approach in Ref. [2]. In Sec. 3, we examine the vector potential of a circular current loop and recover its expansions in orthogonal eigenfunctions in spherical coordinates [1] and in cylindrical ones [2]. Furthermore, we derive a new, simpler closed-form expression for this case, given by (33) in terms of the Legendre function of the second kind, $Q_{\frac{1}{2}}$. This expression, together with (31), leads to the relation (35) between this Legendre function and the complete elliptic integrals $E$ and $K$. By direct evaluation of the curl of (33) we obtain the magnetic induction field $\boldsymbol{B}(\boldsymbol{r})$ of the circular current loop in analytically closed form in (37) in cylindrical coordinates.


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