Comment on "The one-dimensional harmonic oscillator damped with Caldirola–Kanai hamiltonian"

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Several conceptual errors in a recently published paper (*Rev. Mex. Fís. E* **64** (2018) 47) dealing with the damped one-dimensional harmonic oscillator are pointed out.

Keywords: Analytical mechanics; Hamilton–Jacobi equation; damped oscillator.

Se señalan varios errores conceptuales en un artículo recientemente publicado (*Rev. Mex. Fís. E* **64** (2018) 47) relacionado con el oscilador armónico unidimensional amortiguado.

Descriptores: Mecánica analítica; ecuación de Hamilton–Jacobi; oscilador amortiguado.

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1. Introduction

In Ref. [1] the solution of the equation of motion of a damped one-dimensional harmonic oscillator is obtained by means of the Hamilton–Jacobi equation applied to the so-called Caldirola–Kanai Hamiltonian. As discussed below, throughout that paper there are several conceptual errors related to the Hamilton–Jacobi equation as well as the basic Lagrangian and Hamiltonian formalisms.

In the third paragraph of the Introduction of Ref. [1] we find the statement that "Dissipative systems are non-Hamiltonian." Unfortunately, a precise definition of a Hamiltonian (or non-Hamiltonian) system is not given there. If the claim is that the equations of motion of such systems cannot be expressed in the form of the Hamilton equations, then the claim is wrong: any system of ordinary differential equations can be written in the form of the Hamilton equations (see, *e.g.*, Refs. [2, 3]); furthermore, in the case of a single second-order ordinary differential equation (such as the equation of motion considered in Ref. [1]) there exist an *infinite* number of Lagrangians (a result already known to Darboux in 1894 [4], well before the works of Caldirola and Kanai cited in Ref. [1], see also Ref. [5]). Actually, in this last case, it is possible to give a Hamiltonian formulation without obtaining first a Lagrangian, as shown in Ref. [6], where the procedure is applied precisely to the damped one-dimensional harmonic oscillator. By contrast, for systems with more than one degree of freedom, a Lagrangian may not exist (see, *e.g.*, Ref. [7]).

The paragraph after Eqs. (5) of Ref. [1] contains a minor mistake, expressing that "the transformation is canonical if Hamilton equations (5) are satisfied." However, Eqs. (5), which are just the Hamilton equations expressed in terms of a single set of coordinates q_i, p_i, do not make reference to a transformation; these equations contain a set of coordinates only.

After introducing the Hamilton–Jacobi equation for the Hamilton principal function S, in the last paragraph of Sec. 2 of Ref. [1] it is claimed that "The general solution for S depends on $n + 1$ constants, one of them being additive." This is wrong because the *general* solution of a first-order partial differential equation in $n + 1$ variables contains an arbitrary *function* of n variables. However, what is required in the Hamilton–Jacobi method is a *complete* solution of the Hamilton–Jacobi equation, which is a solution $S(q_i, \alpha_i, t)$ of the Hamilton–Jacobi equation containing n parameters, α_i, such that

$$\det\left(\frac{\partial^2 S}{\partial q_i \partial \alpha_j}\right) \neq 0.$$

A common mistake is to assume that any additional parameter in S *must* be an additive constant (a counterexample is given in Ref. [8]).

Some signs in Eqs. (7), (18) and (19) are wrong. Also, in some equations [Eqs. (6), (8), and (17)] the lower case p should be an upper case P.

In Sec. 4 of Ref. [1], through a complicated argument, a canonical transformation is proposed which leads to a new Hamiltonian, which is time-independent and therefore the corresponding Hamilton–Jacobi equation can be solved by separation of variables. (The initial Hamiltonian, by contrast, is time-dependent and the Hamilton–Jacobi equation would not admit separable solutions, but this fact is not mentioned in the paper.) The Hamilton–Jacobi equation is then solved and the solution is employed to find the original coordinate q as a function of time. (In passing, it may be pointed out that the Hamilton–Jacobi equation for the initial Hamiltonian can be solved directly, without the explicit use of a canonical transformation; by inspection, one can convince oneself that the change of variable $q' = q\, e^{\gamma t/2}$ simplifies the equation, in fact, one obtains Eq. (20) of Ref. [1].)

The greatest mistake in Ref. [1] is related with the lack of understanding of the meaning of the so-called "generalized momenta" (by the way, the plural of 'momentum' is 'momenta'). This mistake leads to the conclusion that the solution obtained by means of the Hamilton–Jacobi method (which is the same that one obtains by means of the Hamilton equations) is in some sense erroneous. In Sec. 5 of Ref. [1] it is remarked that the canonical momentum, $p(t)$, has a behavior "different from that expected one in a damped harmonic oscillator, where $p(t)$ increases as time increases." However, going to the basic definition of the generalized, or canonical, momentum, applied to the Lagrangian $L = \frac{1}{2}e^{\gamma t}(m\dot{q}^2 - m\omega^2 q^2)$ (here we use the Greek letter ω to denote the natural frequency of the oscillator, following the common usage), we have

$$p = \frac{\partial L}{\partial \dot{q}} = m e^{\gamma t} \dot{q}. \tag{1}$$

On the other hand, $q(t)$ goes to zero as t goes to infinity through a factor $e^{-\gamma t/2}$ (which can be obtained directly from the equation of motion, without Lagrangians or Hamiltonians) and, therefore, $p(t)$ grows exponentially as t goes to infinity, which is not contradictory (because the canonical momentum is not equal to the elementary linear momentum, as we can see in Eq. (1) above).

There is nothing wrong with Eq. (1) and all the other implications of the formalism; in particular, it is not necessary to invoke a time-dependent mass or anything outside the problem at hand; the formalism is consistent as it stands.

The conclusion at the end of Ref. [1], claiming that the system considered there "does not represent a dissipative system as the momentum $p(t)$ increases with time ..." is plainly wrong.

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