

# On a general correlation for the discharge rate of grains through slots in thin sidewalls of silos due to gravity

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In this work, we discuss the general validity of a phenomenological correlation for the mass flow rate of dry non-cohesive granular material that outflows, due to the gravity, from slots in very thin vertical sidewalls of bins. The correctness of such a formula is analyzed by comparing it to other correlations published elsewhere which reports cases of very large, similar and very small channel width  $W$  relative to its length  $D$ .

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## 1. Introduction

The Hagen formula is the basis to design hourglasses, *i.e.*, devices where the discharge rate of grains, driven by gravity, through the circular orifice in between the upper and the lower chambers is near-constant [1–3]. Such a formula dictates that the mass flow rate,  $\dot{m}$ , follows the relationship  $\dot{m} \sim D^{5/2}$  where  $D$  is the orifice diameter [4]. The same relation is maintained for circular orifices in flat bottomed bins [5].

In recent years we have reported phenomenological correlations that fit very well the experimental mass flow rate of cohesionless granular solids emerging from circular orifices in *the vertical sidewalls of vessels*. If such orifice is in very thin sidewalls, the mass flow rate also obeys approximately the Hagen formula [6].

The discharge of grains through slot-like orifices (rectangles, for instance) at the bottom or at the sidewalls of containers is also a commonly studied problem due to not only its scientific interest but instead, it is envisaged as a problem useful for many applications in industry and technology. Vertical slots have been used in devices for dispersing powders [7], horizontal, vertical and tilted slots are very frequent in oil industry (in slotted liners made from tubulars by saw-cutting slot configurations) where, during oil recovery, clogging by sand grains occurs close to the liners and even it is desirable, in some proportion, as a sand control mechanism to avoid high filtration of fines and controlling the permeability in the liner's vicinity [8, 9]. Once again, if the wall thickness is not thin the correlations will include its effect [6].

To complete the knowledge of the discharge rate of grains through slots, studies on the correctness of formulas of the flow rate across vertical sidewalls are highly necessary. In this work, we will show that in the limit of thin vertical sidewalls there are at least three different regimes where width  $W$ , length  $D$ , or both of them, control substantially the discharge rate correlation through the slots. We will show that

a single general correlation proposed by Brown and Richards [10] appears to embrace all of these cases.

The plan of this work is as follows. First, in Sec. 2 we establish the phenomenological correlations for orifices at the bottom and afterward, in Sec. 3, we analyze through the general formula the cases of elongated rectangular slots in sidewalls of bins. In Sec. 4, we compare the behavior of the general formula and other one also used by us [6] to describe the discharge rate from non-elongated rectangular orifices. Finally, in Sec. 5 we give the main conclusions of the study here tackled.

## 2. Discharge rates through circular and no circular orifices at the bottom

In its explicit form, the Hagen formula that expresses the flow rate of grains for circular orifices at the bottom of reservoirs, like silos and bins, is [4]

$$\dot{m} = c\rho g^{1/2} D^{5/2}, \quad (1)$$

where  $c$  is a dimensionless constant that depends on the particle properties,  $\rho$  is the bulk density of the grains, and  $g$  is the acceleration due to gravity. In some cases, the grain size,  $d$ , could affect the discharge rate and the equivalent to the Eq. (1), that considers this effect is the Hagen-Beverloo formula [4, 11] given by

$$\dot{m} = c\rho g^{1/2} (D - kd)^{5/2}, \quad (2)$$

where  $k$  is other dimensionless constant of order unity. The corrected term  $(D - kd)$  was explained by suggesting that the fall of a particle through an aperture would be hindered if it happened to be near the edge of the hole, and might not occur at all if it were closer than its radius. The overall effect would be to reduce the effective diameter of an orifice by at least one-grain diameter. In the following discussion, we will not take in to account this last quantity by assuming that

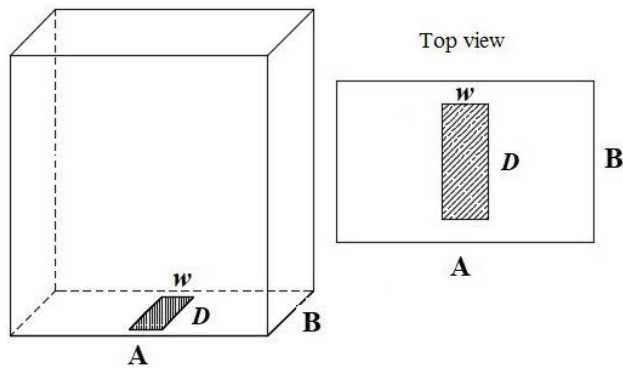


FIGURE 1. Depictions of rectangular orifices in vertical sidewalls: (a) vertically elongated slot, (b) non-elongated slot, and (c) horizontally elongated slot.

$D \gg d$ , orifices large enough respect to the grain size. Equations (1) and (2) also can be used as approximately valid for circular orifices on vertical sidewalls of silos, whenever the wall thickness is very thin [6].

By the way, for central slots at flat-bottomed silos, several authors [5, 12] have reported that the flow through an elongated slot of width  $W$  and length  $D$ , provided  $W \ll D$  (see Fig. 1), the mass flow rate holds as

$$\dot{m} = c\rho g^{1/2}DW^{3/2}, \quad (3)$$

here  $c$  is the corresponding non-dimensional constant depending on the properties of the granular material. Notice that the effect of the grain size is neglected.

Now, it is pertinent to ask if whether or not Eq. (3) will be approximately valid for slots in vertical, very thin, sidewalls. The rest of this work is devoted to elucidate such a question.

### 3. Discharge rates across vertical slots in sidewalls

Here we study discharge rates correlations for slot orifices of rectangular shape of width  $W$  and length (height)  $D$  made in sidewalls of vessels, as it is indicated in Fig. 2, there are three different cases of slots: Fig. 2(a) a vertically elongated slot where  $W \ll D$ , Fig. 2(b) a slot where  $D \sim W$  and, finally, a horizontally elongated slot that may span the overall width of the silo and  $D \ll W$ , Fig. 2(c). It is apparent that the high contrast between cases must strongly affect the overall flow of grains in vertical cases.

In order to embrace cases of grains discharged from slots depicted in Fig. 2, Davis and Foye [7] applied a correlation proposed by Brown and Richards [10]. Such a correlation can be written as

$$\dot{m} = c\rho g^{1/2}AD_H^{1/2}, \quad (4)$$

where  $A$  is the flow area (the area of the aperture), and  $D_H$  is the hydraulic diameter of the flow area (as in hydrodynamics  $D_H = 4A/P$ , where  $P$  is the perimeter [13]). Such a proposal was did because Brown and Richards [10] have noticed that the values of the exponent on  $D$  varied rather than maintain a fixed value of  $5/2$ , as in the classical Hagen formula.

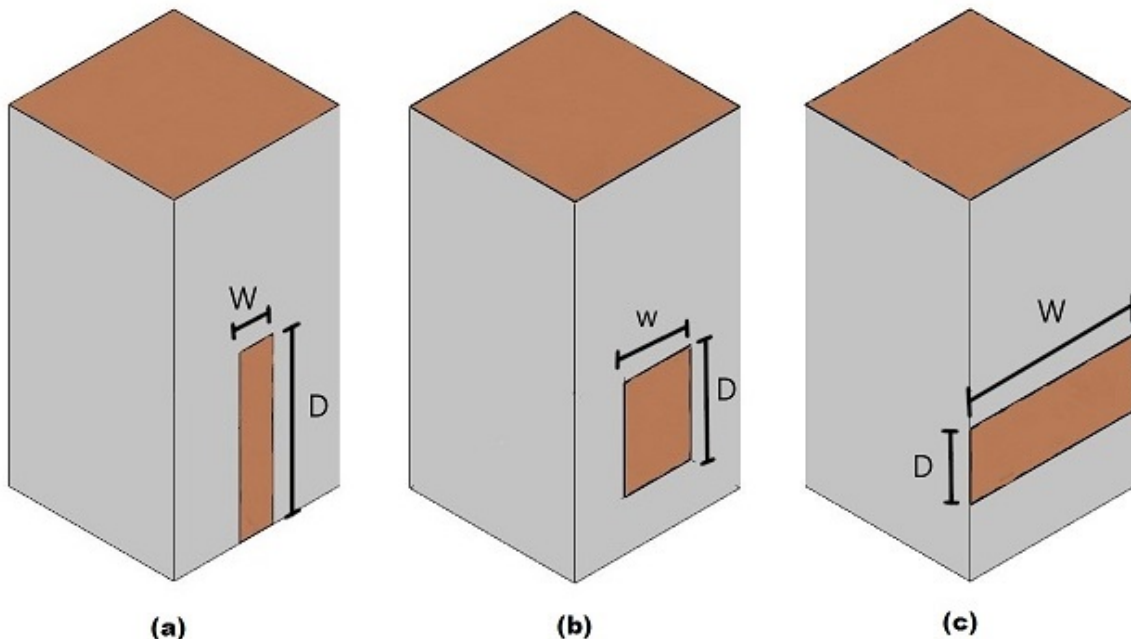


FIGURE 2. Depictions of rectangular orifices in vertical sidewalls: (a) vertically elongated slot, (b) non-elongated slot and (c) horizontally elongated slot.

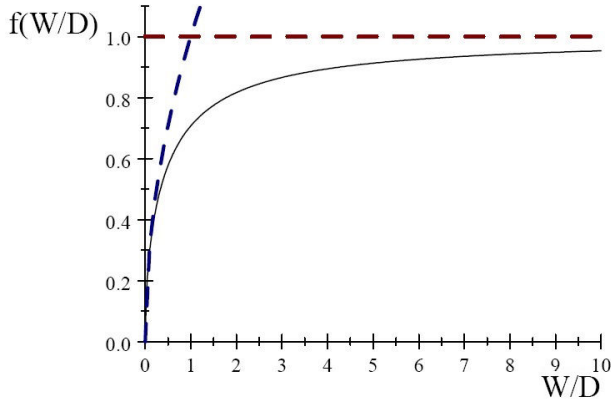


FIGURE 3. Plot of  $f(W/D)$  given by Eq. (7), continuous curve. The red dashed curve is an asymptotic limit of  $f$  when  $W/D \gg 1$ . The blue dashed curve behaves as  $(W/D)^{1/2}$  and fit very well the curve  $f(W/D)$  when  $W/D \ll 1$ .

In agreement with Davis and Foye [7] the discharge rate correlation for rectangular slots, of sides  $D, W \gg d$  (i.e., leaving aside the effect of the grain size), obtained from Eq. (4), is

$$\dot{m} = c\rho g^{1/2}WD \left( \frac{2DW}{D+W} \right)^{1/2}. \quad (5)$$

In experiments Davis and Foye [7] spanned the parametric interval  $0.04 \leq W/D \leq 3.45$ ; thus they also made experiments in the limit of vertically elongated slots, where  $W/D \ll 1$ . Their data fit very well the experimental measurements.

We can rewrite Eq. (5) in terms of the ratio  $W/D$ , it yields to the relation

$$\begin{aligned} \dot{m} &= \sqrt{2}c\rho g^{1/2}WD^{3/2} \left( \frac{W/D}{1+W/D} \right)^{1/2} \\ &= \sqrt{2}c\rho g^{1/2}WD^{3/2} f(W/D), \end{aligned} \quad (6)$$

where

$$f(W/D) = \left( \frac{W/D}{1+W/D} \right)^{1/2}. \quad (7)$$

In Fig. 3 we show the plot of  $f(W/D)$ , through it we found graphically that  $f(W/D) \rightarrow 1$  when  $W/D \gg 1$  (dashed red line) and, consequently, using this result, Eq. (5) transforms into

$$\dot{m} \approx \sqrt{2}c\rho g^{1/2}WD^{3/2}, \quad (8)$$

for horizontally elongated slots.

$$\dot{m} \approx \sqrt{2}c\rho g^{1/2}DW^{3/2}, \quad (9)$$

corresponding to vertically elongated slots.

Asymptotic relationships (8) and (9) apparently represents very different limits but both are embraced by the single Eq. (5) of Brown and Richards [10].

## 4. Further analysis

In a recent study, Zhou *et al.* [14] made experiments with slot-like orifices in bins with square and circular cross-sectional areas of bins (with negligible sidewall thickness) and they have reported the parametric region  $0.1 \leq W/D \leq 14$ . Through experimental data, they reported the existence of two regimes, one obeying

$$\dot{m} = c\rho_b g^{1/2}DW^{3/2}, \text{ if } W/D \ll 1, \quad (10)$$

and other one where

$$\dot{m} = c\rho_b g^{1/2}WD^{3/2}, \text{ if } W/D \gg 1. \quad (11)$$

Thus, the latest correlations confirm the existence of the asymptotic limits (8) and (9); this and the good agreement of experimental data by Davis and Foye [7] appears to confirm the general validity of the Brown and Richards formula for slots.

In another experimental study, Medina *et al.* [15] performed the discharge of grains through rectangular slots made in vertical thick sidewalls of bins for cases where  $W \sim D$ , more specifically  $W = 1$  cm and  $D = 1.5$  cm, 2 cm, 2.5 cm, and 3 cm, therefore  $0.33 \leq W/D \leq 0.66$ . In spite of the influence of the sidewall thickness, the valid relationship for the mass flow rate, in this case, has the form

$$\dot{m} = c\rho_b g^{1/2}D_H^{5/2}(\alpha - \theta_r), \quad (12)$$

where  $\alpha$  is the *angle of wall* (the angle among the vertical line and the straight line segment whose endpoints both lie on the lower outer rim and the inner upper rim of the slot) and  $\theta_r$  is the angle of repose of the granular material, thus the factor  $(\alpha - \theta_r)$  is related to the change in the discharge rate due to the wall thickness [15]. In the limit of zero wall thickness  $(\alpha - \theta_r) \rightarrow (\pi/2 - \theta_r) \sim 1$ , for many materials [6]. Thus, in silos with vertical very thin sidewalls and slot-like orifices, the Eq. (12) will be given by the approximate form

$$\dot{m} \simeq c\rho_b g^{1/2}D_H^{5/2}. \quad (13)$$

For rectangular slots Eq. (13) is now

$$\dot{m} \simeq c\rho_b g^{1/2} \left( \frac{2DW}{D+W} \right)^{5/2}. \quad (14)$$

It is important to know how much predictions of Eqs. (5) and (14) deviates among them. In the case of Eq. (5), it can be modified by assuming that  $D = W + \varepsilon$ , where  $\varepsilon$  in experiments of Medina *et al.* takes values between  $0.5 \leq \varepsilon \leq 2.0$  cm. In terms of  $W$  and  $\varepsilon$  the Eq. (5) yields the approximate formula

$$\dot{m} \approx c\rho g^{1/2}W^{5/2}G\left(\frac{\varepsilon}{W}\right), \quad (15)$$

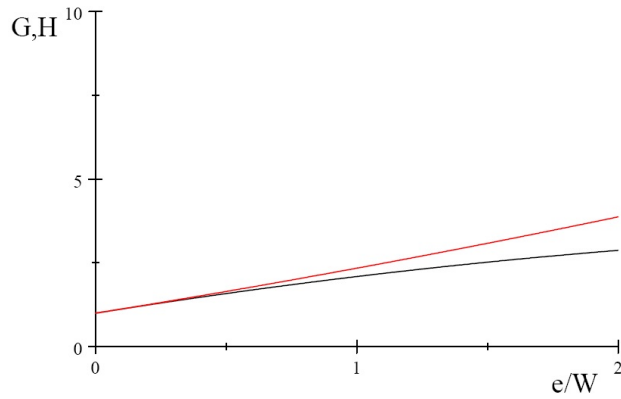


FIGURE 4. Plots of  $G(\varepsilon/W)$  (Eq. (16)), red curve, and  $H(\varepsilon/W)$  (Eq. (18)), black curve, for experimental values of  $\varepsilon/W$  given by Medina *et al.*, [15].

where

$$G\left(\frac{\varepsilon}{W}\right) = \left[1 + \frac{5}{4} \frac{\varepsilon}{W} + \frac{3}{32} \left(\frac{\varepsilon}{W}\right)^2 + O\left(\left(\frac{\varepsilon}{W}\right)^3\right)\right], \quad (16)$$

meanwhile the Eq. (14) is now

$$\dot{m} \simeq c\rho_b g^{1/2} W^{5/2} H\left(\frac{\varepsilon}{W}\right), \quad (17)$$

whence

$$H\left(\frac{\varepsilon}{W}\right) = \left[1 + \frac{5}{4} \frac{\varepsilon}{W} - \frac{5}{32} \left(\frac{\varepsilon}{W}\right)^2 + O\left(\left(\frac{\varepsilon}{W}\right)^3\right)\right], \quad (18)$$

up to order  $(\varepsilon/W)^2$  the difference among  $G$  and  $H$  is of order  $(\varepsilon/W)^2/4$ . Moreover, Fig. 4 shows the plots of  $G(\varepsilon/W)$

(red curve) and  $H(\varepsilon/W)$  (black curve) which are near the same for  $\varepsilon/W < 1$  and separates for  $\varepsilon/W > 1$ .

The comparison between plots of  $G$  and  $H$  was done because of such factors in Eqs. (15) and (17) represents the substantial difference among both formulas. We presented this analysis to show the Brown and Richards formula (Eq. (5)) and that of Medina *et al.* (Eq. (14)) produce comparable results in the limit of thin sidewalls, however further studies will be necessary to get a definitive formula for slots and thick sidewalls.

## 5. Conclusions

In this work we have analyzed the Brown and Richards formula, Eq. (5) for the discharge of non-cohesive granular materials through rectangular slots in vertical sidewalls of negligible thickness. By rewritten such a formula in terms of the ratio  $W/D$ , we have shown that there are two different limits: one corresponding to vertically elongated slots and another one for horizontally elongated slots. However, both asymptotic limits are contained in the Brown and Richards formula. We also compared the Brown and Richards formula with that of Medina *et al.*, Eq. (14), for non-elongated slots and we found that both produce discharge rates of the same order of magnitude. More work is necessary to prove if the Brown and Richards formula is valid for elongated slots in thick sidewalls.

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