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Minimal complete sets for two-pseudoscalar-meson photoproduction

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Within the present work, the main points of the talk and paper shall be recapped, which are amongst others the foundations of Moravcsik's Theorem, the definition of the polarization observables, the extraction of complete sets as well as the implication for experimentalists.

Keywords: Complete experiment; two-pseudoscalar-meson photoproduction; polarization observables; graph theory.

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1. Introduction

The superior goal is the determination of the matrix elements of the transition operator $\mathcal{T}$ for photoproduction reactions with two pseudoscalar mesons in the final state. Therefore, polarization observables have to be measured and hereinafter analysed via a partial wave analysis in order to extract electromagnetic multipoles $\mathcal{M}$, which determine $\mathcal{T}$ [2].

However, as the polarization observables are mathematically described by a bilinear form of the complex amplitudes t_i [3–5], mathematical ambiguities arise [6]. In order to avoid these, one has to perform a complete-experiment analysis [7]. In the case of two-pseudoscalar-meson photoproduction, this is a highly non-trivial task as eight complex amplitudes [3] are needed to describe this reaction.

Although the analytical approach is possible, it has not been done yet as it is an extremly hard task. For this reason alternative approaches should be explored, especially with regard to even more challenging reactions like vector-meson photoproduction.

2. Moravcsik's theorem

The paper by Moravcsik was published in 1985 [8]. Since then it has not yet attracted much attention. However, its graph theoretical approach is appealing in several ways, compared to the analytical method:

1. As it relies on simple concepts of graph theory, the theorem is easy to understand and implement.
2. The visualization with the help of graphs is intuitive, yet an abstract representation of the problem.
3. It allows for an easier access of complete sets for photoproduction reactions, especially the ones which are described by more than four complex amplitudes.

The underlying assumption of the theorem is the *a priori* knowledge about the moduli of the N complex amplitudes t_i as well as the real and imaginary parts of the bilinear products $t_i^* t_j$ [8, 10].

Lets turn to the actual formulation of the problem using graph theory: A graph consists of nodes connected via edges.

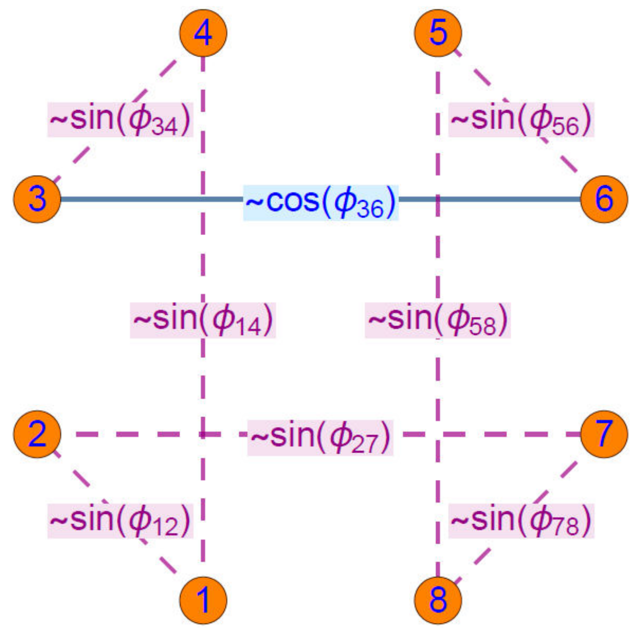


FIGURE 1. An example for an edge configuration of a cycle graph with eight nodes (enumerated points) is shown. The edges indicate whether the complex amplitudes are connected via the real (solid line) or imaginary (dashed line) part of the corresponding bilinear product. The respective correlation to the relative phase ϕ_{ij} is indicated [1].

Each node corresponds to a complex amplitude t_i and each edge to the real/imaginary part of the corresponding bilinear product $t_i^* t_j$ [8]. An example is shown in Fig. 1.

A graph is said to represent a complete set of polarization observables if it fulfils two requirements. On the one hand, it has to be a connected graph, which is equivalent to the validity of a consistency relation between the relative phases, *e.g.*: $\phi_{12} + \phi_{23} + \cdots + \phi_{n1} = 0$. On the other hand, the graph has to consist of an odd number of edges which correspond to the imaginary part of a bilinear product. For an in depth explanation on the underlying details the authors refer to the main paper [1].

The following analysis focuses on cycle graphs, as these allow to use the minimal number of bilinear products in order to resolve the discrete mathematical ambiguities [8].

3. Polarization observables

The photoproduction reaction of two-pseudoscalar-mesons can be fully described by eight complex amplitudes. Hence, one deals with 64 polarization observables. Their mathematical description was first performed by Roberts and Oed [3]. There are single-, double- and triple polarization observables involved. Especially the measurement of the latter ones is an experimental challenge as a recoil polarimeter is needed [1]. Such experimental difficulties are taken into account when talking about advantageous future measurements in Sec. 6.

For the purpose of the following analysis the observables $\mathcal{O}$ are rewritten in terms of bilinear products $\mathcal{O}^{\alpha} = \sum_{i,j=1}^{8} t_i^* \Gamma_{ij}^{\alpha} t_j$, with $\alpha \in [1, \ldots, 64]$ and complex matrices Γ^{α}. These matrices can be grouped into eight distinct sets according to their shape. The 64 Γ^{α} matrices can be expressed as Kronecker products of Pauli and identity matrices, see Table I.

4. Determination of complete experiments

The following procedure is based on combinatorial and numerical methods. Initially one constructs all 2520 unique graph topologies with eight nodes, *i.e.* the number of complex amplitudes necessary to describe the reaction. Three distinct graph topologies can be seen in Fig. 2.

In a second step, all possible edge configurations, involving an odd number of edges which correspond to the imaginary part of the bilinear product, are constructed for each of the 2520 graphs. An example is shown in Fig. 1. This yields in total $2520 \cdot 128 = 322560$ graphs, which correspond to a complete set of observables [1].

The next step involves the mapping of the bilinear forms, *i.e.* the edges of a graph, to observables via the formula $t_j^* t_i = (1/8) \sum_{\alpha=1}^{64} \Gamma_{ij}^{\alpha} \mathcal{O}^{\alpha}$ [1]. In other words, one determines which of the 64 polarization observables are needed in order to construct the graph under consideration. This is done for each of the 322560 graphs. After filtering out duplicates, 5964 unique complete sets of observables of different lengths remain.

In view of a future experimental verification, one is naturally interested in complete sets of minimal length. The current level of knowledge [9, 10] suggest that the minimal length of a complete set of observables is equal to $2N$, where N is the number of complex amplitudes in the given problem. Thus the subsequent analysis focuses on the subgroup of 392 distinct sets of length 24, which then are reduced to length 16 while retaining the completeness. This reduction is done via the numerical solution of polynomial systems, for details see [1].

So far, 4185 unique truly minimal sets of length $2N = 16$ have been found. Of special interest are a subgroup of 69 sets which involve only one triple polarization observable, see Table V in Ref. [1].

In addition, an analytic derivation based on the phase-fixing approach of Ref. [11] is shown for a specific combination of observables, see Table VII in Ref. [1].

5. Pool of measurements

It remains the task to give an advice for experimentalists concerning future measurements. Therefore an extensive list of previously performed measurements regarding the photoproduction of two pseudoscalar mesons has been collected in Table IV of the paper [1]. This list includes more than 55 measurements in the period of time from 1976 to 2020. Different facilities like ELSA, MAMI, JLAB and GRAAL contributed to these measurements. However, one has to keep in mind that due to this large timespan especially the earlier measurements with lower-performing setups suffer with regard to the statistical quality of the data sets.

The reaction $\gamma p \rightarrow \pi^0 \pi^0 p$ is very well suited to study baryon resonances decaying into $\Delta(1232)\pi$ [12]. This might be the reason why by far most of the measurements were performed for this reaction.

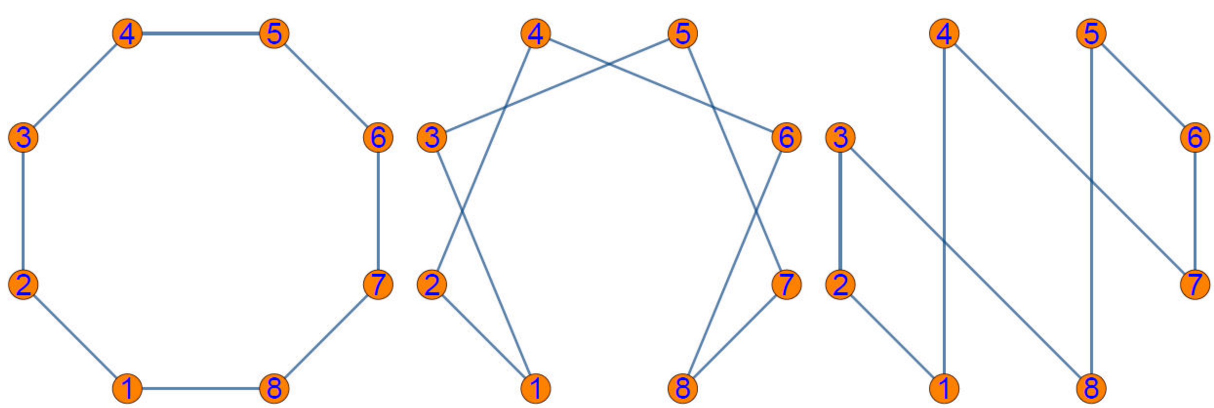
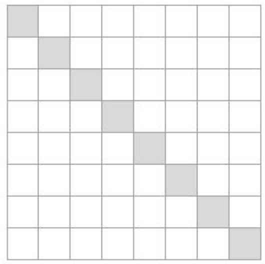
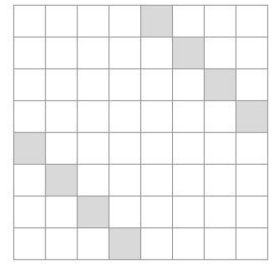
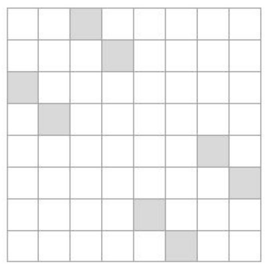
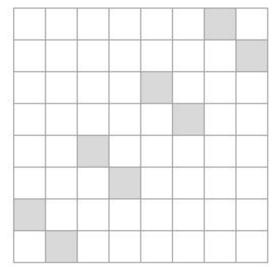
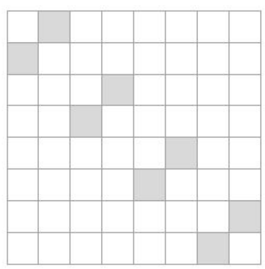
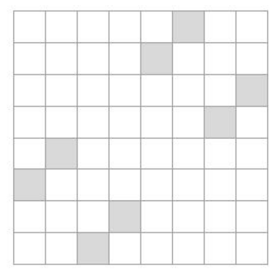
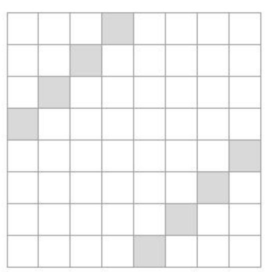
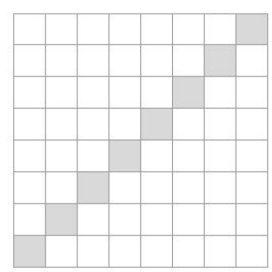


FIGURE 2. Three out of 2520 unique cycle graph topologies with eight nodes are shown [1].

TABLE I. Using the well known Pauli matrices in combination with the Kronecker product, the 64 polarization observables of two-pseudoscalar-meson photoproduction are defined. The non-zero matrix elements are indicated via gray shaded cells [1].

Γ-matrices	Definition	Shape-class	Γ-matrices	Definition	Shape-class
Γ^{I}_{1}	$\sigma^3 \otimes I_2 \otimes I_2$		Γ^{V}_{c1}	$\sigma^1 \otimes I_2 \otimes I_2$	
Γ^{I}_{2}	$I_2 \otimes \sigma^3 \otimes I_2$		Γ^{V}_{c2}	$\sigma^1 \otimes \sigma^3 \otimes I_2$	
Γ^{I}_{3}	$I_2 \otimes I_2 \otimes \sigma^3$		Γ^{V}_{c3}	$\sigma^1 \otimes I_2 \otimes \sigma^3$	
Γ^{I}_{4}	$\sigma^3 \otimes \sigma^3 \otimes \sigma^3$		Γ^{V}_{c4}	$\sigma^1 \otimes \sigma^3 \otimes \sigma^3$	
Γ^{I}_{5}	$I_2 \otimes \sigma^3 \otimes \sigma^3$		Γ^{V}_{s1}	$-\sigma^2 \otimes I_2 \otimes I_2$	
Γ^{I}_{6}	$\sigma^3 \otimes I_2 \otimes \sigma^3$		Γ^{V}_{s2}	$-\sigma^2 \otimes \sigma^3 \otimes I_2$	
Γ^{I}_{7}	$\sigma^3 \otimes \sigma^3 \otimes I_2$		Γ^{V}_{s3}	$-\sigma^2 \otimes I_2 \otimes \sigma^3$	
Γ^{I}_{8}	$I_2 \otimes I_2 \otimes I_2$		Γ^{V}_{s4}	$-\sigma^2 \otimes \sigma^3 \otimes \sigma^3$	
$\Gamma^{\mathrm{II}}_{c1}$	$I_2 \otimes \sigma^1 \otimes I_2$		$\Gamma^{\mathrm{VI}}_{c1}$	$\sigma^1 \otimes \sigma^1 \otimes I_2$	
$\Gamma^{\mathrm{II}}_{c2}$	$\sigma^3 \otimes \sigma^1 \otimes I_2$		$\Gamma^{\mathrm{VI}}_{c2}$	$-\sigma^2 \otimes \sigma^2 \otimes I_2$	
$\Gamma^{\mathrm{II}}_{c3}$	$I_2 \otimes \sigma^1 \otimes \sigma^3$		$\Gamma^{\mathrm{VI}}_{c3}$	$\sigma^1 \otimes \sigma^1 \otimes \sigma^3$	
$\Gamma^{\mathrm{II}}_{c4}$	$\sigma^3 \otimes \sigma^1 \otimes \sigma^3$		$\Gamma^{\mathrm{VI}}_{c4}$	$-\sigma^2 \otimes \sigma^2 \otimes \sigma^3$	
$\Gamma^{\mathrm{II}}_{s1}$	$-I_2 \otimes \sigma^2 \otimes I_2$		$\Gamma^{\mathrm{VI}}_{s1}$	$-\sigma^2 \otimes \sigma^1 \otimes I_2$	
$\Gamma^{\mathrm{II}}_{s2}$	$-\sigma^3 \otimes \sigma^2 \otimes I_2$		$\Gamma^{\mathrm{VI}}_{s2}$	$-\sigma^1 \otimes \sigma^2 \otimes I_2$	
$\Gamma^{\mathrm{II}}_{s3}$	$-I_2 \otimes \sigma^2 \otimes \sigma^3$		$\Gamma^{\mathrm{VI}}_{s3}$	$-\sigma^2 \otimes \sigma^1 \otimes \sigma^3$	
$\Gamma^{\mathrm{II}}_{s4}$	$-\sigma^3 \otimes \sigma^2 \otimes \sigma^3$		$\Gamma^{\mathrm{VI}}_{s4}$	$-\sigma^1 \otimes \sigma^2 \otimes \sigma^3$	
$\Gamma^{\mathrm{III}}_{c1}$	$I_2 \otimes I_2 \otimes \sigma^1$		$\Gamma^{\mathrm{VII}}_{c1}$	$\sigma^1 \otimes I_2 \otimes \sigma^1$	
$\Gamma^{\mathrm{III}}_{c2}$	$\sigma^3 \otimes I_2 \otimes \sigma^1$		$\Gamma^{\mathrm{VII}}_{c2}$	$\sigma^1 \otimes \sigma^3 \otimes \sigma^1$	
$\Gamma^{\mathrm{III}}_{c3}$	$I_2 \otimes \sigma^3 \otimes \sigma^1$		$\Gamma^{\mathrm{VII}}_{c3}$	$-\sigma^2 \otimes I_2 \otimes \sigma^2$	
$\Gamma^{\mathrm{III}}_{c4}$	$\sigma^3 \otimes \sigma^3 \otimes \sigma^1$		$\Gamma^{\mathrm{VII}}_{c4}$	$-\sigma^2 \otimes \sigma^3 \otimes \sigma^2$	
$\Gamma^{\mathrm{III}}_{s1}$	$-I_2 \otimes I_2 \otimes \sigma^2$		$\Gamma^{\mathrm{VII}}_{s1}$	$-\sigma^2 \otimes I_2 \otimes \sigma^1$	
$\Gamma^{\mathrm{III}}_{s2}$	$-\sigma^3 \otimes I_2 \otimes \sigma^2$		$\Gamma^{\mathrm{VII}}_{s2}$	$-\sigma^2 \otimes \sigma^3 \otimes \sigma^1$	
$\Gamma^{\mathrm{III}}_{s3}$	$-I_2 \otimes \sigma^3 \otimes \sigma^2$		$\Gamma^{\mathrm{VII}}_{s3}$	$-\sigma^1 \otimes I_2 \otimes \sigma^2$	
$\Gamma^{\mathrm{III}}_{s4}$	$-\sigma^3 \otimes \sigma^3 \otimes \sigma^2$		$\Gamma^{\mathrm{VII}}_{s4}$	$-\sigma^1 \otimes \sigma^3 \otimes \sigma^2$	
$\Gamma^{\mathrm{IV}}_{c1}$	$I_2 \otimes \sigma^1 \otimes \sigma^1$		$\Gamma^{\mathrm{VIII}}_{c1}$	$\sigma^1 \otimes \sigma^1 \otimes \sigma^1$	
$\Gamma^{\mathrm{IV}}_{c2}$	$\sigma^3 \otimes \sigma^1 \otimes \sigma^1$		$\Gamma^{\mathrm{VIII}}_{c2}$	$-\sigma^2 \otimes \sigma^2 \otimes \sigma^1$	
$\Gamma^{\mathrm{IV}}_{c3}$	$-I_2 \otimes \sigma^2 \otimes \sigma^2$		$\Gamma^{\mathrm{VIII}}_{c3}$	$-\sigma^2 \otimes \sigma^1 \otimes \sigma^2$	
$\Gamma^{\mathrm{IV}}_{c4}$	$-\sigma^3 \otimes \sigma^2 \otimes \sigma^2$		$\Gamma^{\mathrm{VIII}}_{c4}$	$-\sigma^1 \otimes \sigma^2 \otimes \sigma^2$	
$\Gamma^{\mathrm{IV}}_{s1}$	$-I_2 \otimes \sigma^2 \otimes \sigma^1$		$\Gamma^{\mathrm{VIII}}_{s1}$	$-\sigma^2 \otimes \sigma^1 \otimes \sigma^1$	
$\Gamma^{\mathrm{IV}}_{s2}$	$-\sigma^3 \otimes \sigma^2 \otimes \sigma^1$		$\Gamma^{\mathrm{VIII}}_{s2}$	$-\sigma^1 \otimes \sigma^2 \otimes \sigma^1$	
$\Gamma^{\mathrm{IV}}_{s3}$	$-I_2 \otimes \sigma^1 \otimes \sigma^2$		$\Gamma^{\mathrm{VIII}}_{s3}$	$-\sigma^1 \otimes \sigma^1 \otimes \sigma^2$	
$\Gamma^{\mathrm{IV}}_{s4}$	$-\sigma^3 \otimes \sigma^1 \otimes \sigma^2$		$\Gamma^{\mathrm{VIII}}_{s4}$	$\sigma^2 \otimes \sigma^2 \otimes \sigma^2$	

6. Implication for experimentalists

In consideration of the information gathered and the results gained, the most promising set for future measurements with regard to an unambiguous complete-experiment analysis is presented:

$$\{\mathrm{I}^{\odot}, \mathrm{P_y}, \mathrm{P_{y'}}, \mathcal{O}^{\odot}_{\mathrm{yy'}}, \mathcal{O}_{\mathrm{yy'}}, \mathrm{P}^{\odot}_{\mathrm{y'}}, \mathrm{P}^{\odot}_{\mathrm{y}}, \mathrm{I}_0, \mathrm{P_x}, \mathrm{P_z}, \mathrm{P_{x'}}, \mathrm{P^s_x}, \mathrm{P}^{\odot}_{\mathrm{x}}, \mathrm{P^c_z}, \mathrm{P}^{\odot}_{\mathrm{z}}, \mathrm{P}^{\odot}_{\mathrm{x'}}\}. \quad (1)$$

Already eight of these observables were measured within the reaction $\gamma p \to \pi^0\pi^0 p$, namely I_0, $\mathrm{I}^{\odot}$, $\mathrm{P_x}$, $\mathrm{P_y}$, $\mathrm{P_{y'}}$, $\mathcal{O}_{\mathrm{yy'}}(= -\mathrm{I^c})$, $\mathrm{P^s_x}$ and $\mathrm{P}^{\odot}_{\mathrm{z}}$, although the energy and angular ranges do not have a perfect overlap between the different measurements. The remaining eight observables could be measured within three different experiments, for details see [1].

7. Conclusion

Moravcsik's theorem, *i.e.* a graph theoretical approach, was adapted in order to derive complete sets of observables for the photoproduction reaction of two pseudoscalar mesons. Incorporating numerical methods, 4185 unique truly minimal sets of length $2N = 16$ have been found so far. 69 of these sets contain only one triple polarization observable. With the help of these results and a summary on previously performed measurements, the most promising minimal complete set for future measurements was selected. For a more detailed treatment of the topic, the authors refer to the paper [1].

In addition, a further article concerning the determination of complete sets of observables was published recently [13]. Via an improved graphical criterion, the author argues that in the case of two-pseudoscalar-meson photoproduction the new procedure gives complete sets of observables of length 20, in contrast to the procedure presented in this paper which yields 24. Hence, this would greatly shrink the numerical effort when reducing these sets to truly minimal complete sets of length 16. To be precise, it would shrink the number of truly minimal sets of length 16 which have to be tested for completeness from 735471 to 4845 which is a factor of 151.8.

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