# Cluster reducibility of multiquark operators and tetraquark-adequate QCD sum rules

W. Lucha

*Institute for High Energy Physics, Austrian Academy of Sciences,*
*Nikolsdorfergasse 18, A-1050 Vienna, Austria.*
*e-mail: Wolfgang.Lucha@oeaw.ac.at.*

D. Melikhov

*D. V. Skobeltsyn Institute of Nuclear Physics, M. V. Lomonosov Moscow State University, 119991 Moscow, Russia.*
*Joint Institute for Nuclear Research, 141980 Dubna, Russia.*
*Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria.*
*e-mail: dmitri_melikhov@gmx.de.*

H. Sazdjian

*Université Paris-Saclay, CNRS/IN2P3, IJCLab, 91405 Orsay, France.*
*e-mail: sazdjian@ijclab.in2p3.fr.*

If connecting properties of a multiquark hadron with those exhibited by its constituents, quantum chromodynamics sum rules inferred along the routes of traditional wisdom necessarily involve contributions not related at all to and thus not presenting information about multiquarks. Realizing this deficiency, we propose to increase, for the example of tetraquarks, the predictive power of the quantum chromodynamics sum-rule formalism by disposal of all of the unwanted contributions from the very beginning; this move is easily accomplished by subjecting the contributions to our perspicuous selection criterion.

## 1. Bound states induced by strong interactions

The most basic description of all strong interactions observed in elementary particle physics is (at least, at present) provided by *quantum chromodynamics* (QCD), a quantum field theory, exhibiting invariance under local transformations forming the non-Abelian gauge group SU(3). Its basic degrees of freedom encompass the gluons (vector bosons transforming according to the eight-dimensional adjoint representation of SU(3)) plus a set of flavoured quarks (fermions transforming according to the fundamental, three-dimensional representation of SU(3)). For both quarks and gluons, the degree of freedom induced by this local SU(3) gauge invariance is referred to as their colour.

Viewed from the angle of QCD, a hadron is perceived as a bound state of the quark and gluon degrees of freedom. Thus, for any hadron $H$ an interpolating operator $\mathcal{O}$ of $H$ is a (local) *gauge-invariant* operator composed of quark and gluon fields that has a nonzero overlap with the hadron's state $|H\rangle$, that is, enjoys a nonvanishing matrix element if sandwiched between the QCD vacuum $|0\rangle$ and the hadron state $|H\rangle$: $\langle 0|\mathcal{O}|H\rangle \neq 0$.

Recently, we worked out [1–11] for a popular approach to hadrons called QCD sum rules a, from our point of view, kind of improvement tailored to the peculiarities of *exotic* hadrons.

*QCD sum rules* [12] relate observable features of hadrons to the degrees of freedom (quarks and gluons) and parameters of the quantum field theory governing the strong interactions, quantum chromodynamics; they may be easily established by evaluation of suitable $n$-point correlation functions of hadron interpolating operators (defined by quarks and gluons) at both hadron level (by inserting a complete set of hadron states) and QCD level (by adopting the operator product expansion [13]). By construction, their distinctive characteristic is to provide a *nonperturbative* approach by means of analytic relationships.

All QCD-controlled colour-singlet bound states of quarks and gluons may be categorized into two disjoint sets, ordinary hadrons comprising quark–antiquark mesons and three-quark baryons, and *exotic hadrons*, with all *multiquarks* (containing as constituents more quarks than merely one quark–antiquark pair or just three quarks) as a prominent set of representatives.

The most distinctive feature common to all multiquarks is their ability for clustering [7]: in contrast to ordinary hadrons, every multiquark may decompose into clusters of, eventually, *ordinary* hadrons. As a consequence, any multiquark must be viewed simultaneously as a strongly bound compact state and as a rather loosely bound aggregate of these ordinary hadrons.

For the sake of clarity, our ideas are best illustrated for the least complex systems presumably realized by multiquarks of smallest possible number of constituents: bound states of two quarks and two antiquarks, forming the subset of *tetraquarks*.

## 2. Multiquark-hadron interpolating operators

The starting point of such line of argument is the specification of any adopted multiquark interpolating operator with respect to its for our analysis most useful internal Lorentz, colour and flavour structure in terms of quark fields $q_a^\alpha(x)$ endowed with colour indices $(\alpha, \beta, \dots)$ as well as flavour indices $(a, b, \dots)$.

Concerning notations, we find it preferable to skip all (for all following considerations irrelevant) references to parity or spin degrees of freedom, and thus suppress all Dirac matrices.

### 2.1. Tetraquark interpolating operators: colour singlets

All hadrons are colour singlets, vulgo colourless. Thus, every admissible hadron interpolating operator $\mathcal{O}$ has to be likewise an overall colour singlet. This constraint, however, does in no way predetermine the internal colour structure of the operator $\mathcal{O}$. For tetraquark interpolating operators $\theta$, this trivial insight entails that their colour composition can be chosen to be, *e.g.*, of the singlet–singlet nature $\theta \sim (\overline{q}_\alpha\, q^\alpha)\,(\overline{q}_\beta\, q^\beta)$, or, in terms of the Levi-Civita symbol $\epsilon_{\alpha\beta\gamma}$, of the antitriplet–triplet form $\theta \sim \epsilon_{\alpha\beta\gamma}\,(q^\beta\, q^\gamma)\,\epsilon^{\alpha\delta\varepsilon}\,(\overline{q}_\delta\, \overline{q}_\varepsilon)$, or, adopting all generators $T^A$, $A = 1, 2, \dots, 8$, of the gauge group SU(3) in the fundamental representation, of the octet–octet type $\theta \sim (\overline{q}\, T^A\, q)\,(\overline{q}\, T^A\, q)$. By means of a Fierz transformation [14], however, any colour structure of tetraquark interpolating operators can be cast into a sum of products of two colour-singlet quark bilinears; these findings clearly relativize the significance of an "appropriate" choice of interpolating operator. That is to say, all choices are equivalent, it suffices to consider the singlet–singlet structure.

### 2.2. Tetraquark interpolating operators: quark flavours

In principle, as a consequence of the clustering property of all multiquarks, the flavour composition of these is subject to the same constraints as those of ordinary hadrons, that is, to none. Table I of Ref. [4] provides an exhaustive classification of the potential open or hidden quark-flavour content of tetraquarks.

For tetraquarks, upon taking advantage of the opportunity to perform Fierz transformations, it clearly suffices to confine all investigations to generic tetraquark interpolating operators

$$\theta_{\bar{a}b\bar{c}d}(x) \equiv j_{\bar{a}b}(x)\, j_{\bar{c}d}(x), \tag{1}$$

defined by a product of two colourless quark-bilinear currents

$$j_{\bar{a}b}(x) \equiv \overline{q}_{a\alpha}(x)\, q_b^\alpha(x)\,. \tag{2}$$

An evidently hindering implication [15,16] of multiquark clustering is that two-point correlation functions of tetraquark interpolating operators (1) receive not merely nonfactorizable (NF) contributions, that potentially convey information about tetraquarks, but also factorizable terms, that definitely do not:

$$\langle \mathcal{T}(\theta_{\bar{a}b\bar{c}d}(x)\, \theta_{\bar{a}b\bar{c}d}^\dagger(0))\rangle = \langle \mathcal{T}(\theta_{\bar{a}b\bar{c}d}(x)\, \theta_{\bar{a}b\bar{c}d}^\dagger(0))\rangle_{\mathrm{NF}}$$

$$+\, \langle \mathcal{T}(j_{\bar{a}b}(x)\, j_{\bar{a}b}^\dagger(0))\rangle\, \langle \mathcal{T}(j_{\bar{c}d}(x)\, j_{\bar{c}d}^\dagger(0))\rangle\,.$$

With respect to the tetraquark flavour degrees of freedom, in the following we shall focus, for definiteness, to the case of *flavour-exotic* tetraquark hadrons: bound states of two quarks and two antiquarks that carry four mutually different flavours, $a \neq b \neq c \neq d$. For these, precisely two linearly independent interpolating operators do exist, in representation (1) given by

$$\theta_{\bar{a}b\bar{c}d}^{(1)}(x) \equiv j_{\bar{a}b}(x)\, j_{\bar{c}d}(x)\,, \qquad \theta_{\bar{a}d\bar{c}b}^{(2)}(x) \equiv j_{\bar{a}d}(x)\, j_{\bar{c}b}(x)\,.$$

## 3. Correlation function of four quark bilinears

A very promising approach fully in line with Eq. (1) is to reap all the information sought from tetraquark intermediate states of correlation functions of four suitably chosen colour-singlet quark-bilinear currents (interpolating only *ordinary* mesons).

In our search for *multiquark adequacy* of any tool utilized for the investigation of exotic hadrons, an (evidently) decisive move is to find a means both to discard all those contributions that definitely won't have an impact on the multiquarks under study and to identify and retain exactly all those contributions — labelled *multiquark-phile* — that *might* be of relevance for the description of any multiquarks in the focus of our interest. For the particular case of tetraquarks, we formulated, in terms of Feynman diagrams, a criterion [1] enabling us to distil only the *tetraquark-phile* [2,5] among the entirety of contributions to a four-point correlation function of quark-bilinear currents:

> In terms of the Mandelstam variable $s$ defined by the four relevant external momenta $p_1, p_2, p_3, p_4$ on equal footing via $s \equiv (p_1{+}p_2)^2 = (p_3{+}p_4)^2$, a *tetraquark-phile* [2,5] Feynman diagram has to
> • exhibit a non-polynomial dependence on $s$ and
> • develop a branch cut, defined by a branch point $\hat{s}$ governed by the masses $m_a, m_b, m_c, m_d$ of the (anti)quarks $\overline{q}_a, q_b, \overline{q}_c, q_d$ forming the *tetraquark hadron* according to $\hat{s} = (m_a{+}m_b{+}m_c{+}m_d)^2$.

The (really pivotal) question whether some Feynman diagram under consideration is capable of developing such four-quark singularities may be unambiguously decided, by means of the Landau equations [17] (for explicitly worked out examples of the latters' application to *tetraquarks*, consult Refs. [3,8,11]). The fulfilment of this necessary but not sufficient prerequisite guarantees straightforwardly a Feynman diagram's *suitability* to contribute to the formation of the suspected tetraquark pole (located, for a generic tetraquark $T$ of mass $M_T$, at $s = M_T^2$).

Flavour exoticism characterizing the subset of tetraquarks presently in our focus prompts us to categorize the considered four-point correlation functions of quark-bilinear currents (2) into *flavour-preserving* (occasionally dubbed direct) ones and *flavour-reordering* (occasionally called recombination) ones:
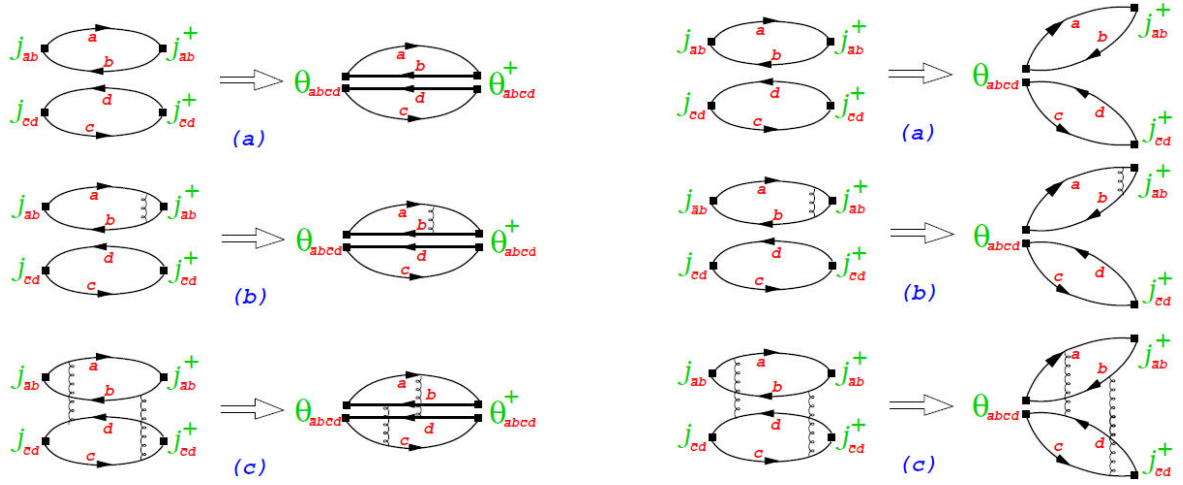
FIGURE 1. Exemplary perturbative contributions of lowest conceivable strong-coupling orders $\alpha_s^0$ (a), $\alpha_s$ (b), and $\alpha_s^2$ (c) to *flavour-preserving correlation functions* of four quark-bilinear currents $j$ and the latters' configuration-space vertex contractions to correlation functions of either *two* tetraquark interpolating operators $\theta^{(1)}$ (left) [6,9], or *one* tetraquark interpolating operator $\theta^{(1)}$ plus two quark-bilinear currents (right) [6].
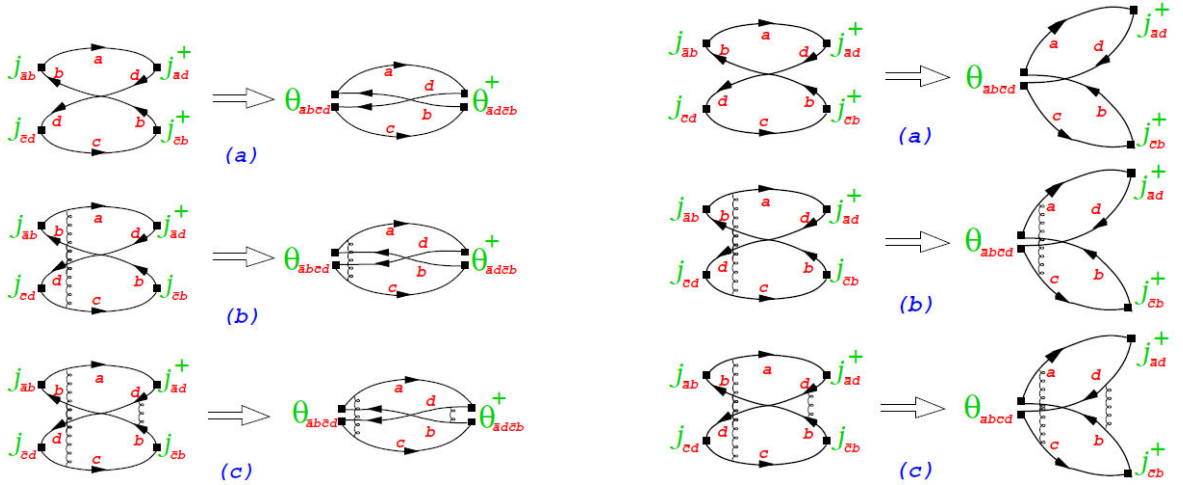


FIGURE 2. Exemplary perturbative contributions of lowest conceivable strong-coupling orders $\alpha_s^0$ (a), $\alpha_s$ (b), and $\alpha_s^2$ (c) to *flavour-reordering correlation functions* of four quark-bilinear currents, and configuration-space vertex contractions thereof, to correlation functions of either *two* tetraquark interpolating operators $\theta^{(1)}$, $\theta^{(2)}$ (left) [6,8] or *one* tetraquark interpolating operator $\theta^{(1)}$ and two quark-bilinear currents (right) [8].

- flavour-preserving correlation functions are of the form

$$\langle \mathcal{T}(j_{\bar{a}b}\, j_{\bar{c}d}\, j_{\bar{a}b}^\dagger\, j_{\bar{c}d}^\dagger) \rangle \,, \qquad \langle \mathcal{T}(j_{\bar{a}d}\, j_{\bar{c}b}\, j_{\bar{a}d}^\dagger\, j_{\bar{c}b}^\dagger) \rangle \,; \quad (3)$$

- flavour-reordering correlation functions are of the form

$$\langle \mathcal{T}(j_{\bar{a}d}\, j_{\bar{c}b}\, j_{\bar{a}b}^\dagger\, j_{\bar{c}d}^\dagger) \rangle \,. \qquad (4)$$

For such categories, Figs. 1 and 2, respectively, exemplify the perturbative contributions of lowest orders in the coupling $\alpha_s$, related to the fundamental coupling parameter of QCD, $g_s$, by

$$\alpha_s \equiv \frac{g_s^2}{4\pi} \,.$$

Nonperturbative contributions involving vacuum condensates — whose presence is required by these correlation functions'

operator product expansions and below implicitly understood — may be and have been investigated on equal footing [8,10].

Finally, pairwise configuration-space coordinate merging of quark-bilinear operators reduces their four-point functions,

- upon a twofold contraction, to the correlation functions of two tetraquark interpolating operators (Figs. 1 and 2, left), among others yielding the tetraquark masses $M_T$;

- upon single contractions, to the correlation functions of just a sole tetraquark interpolating operator and the two unaffected quark-bilinear currents (Figs. 1 and 2, right) that govern, for instance, the tetraquarks' decay widths.
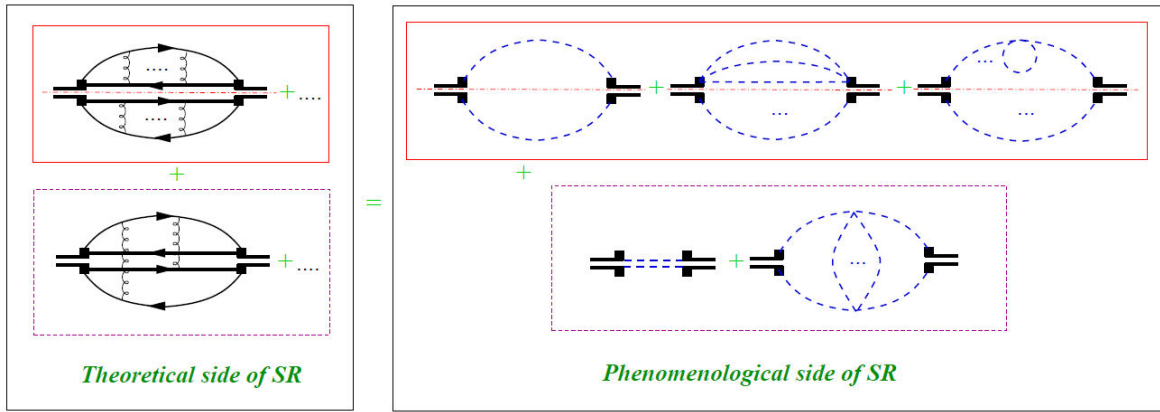
FIGURE 3. Disentanglement of the QCD sum-rule output of *flavour-preserving* correlation functions of two tetraquark interpolating operators $\theta$, generating pairs of *ordinary-meson* QCD sum rules reflecting the nature of the *factorizable* contributions (top row, separated by dash-dotted lines) as well as the requested tetraquark-adequate QCD sum rules embodying the totality of *nonfactorizable* contributions (bottom row) [6,9].
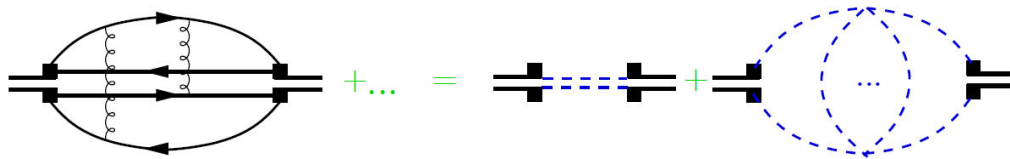


FIGURE 4. Symbolic depiction of the tetraquark-adequate QCD sum rules [6] as deduced from the *flavour-preserving* correlation functions of two tetraquark interpolating operators $\theta$, relating exclusively *tetraquark-phile* QCD-level contributions [involving two or more internal gluons (curly lines)] to hadron-level contributions of tetraquark poles (dashed double lines) and nonfactorizable mesonic contributions (dashed lines).
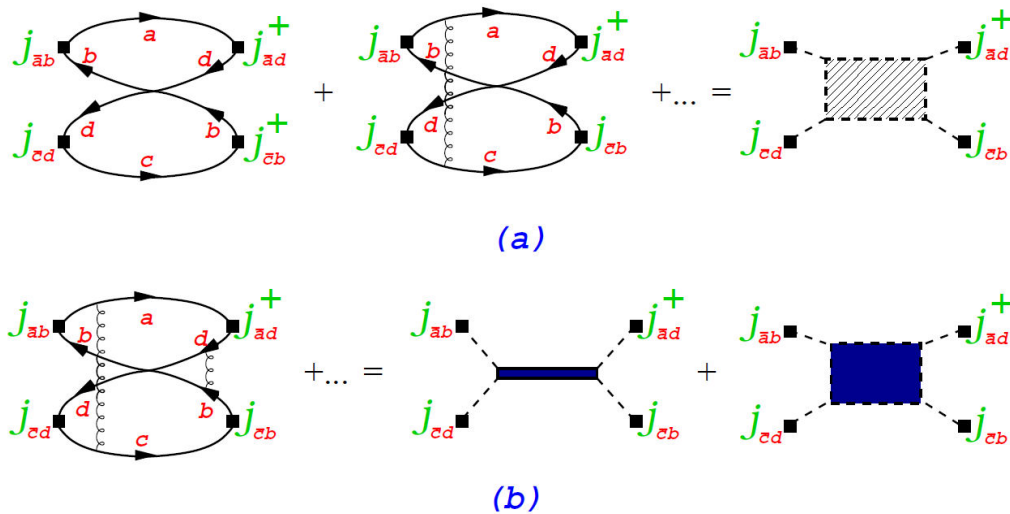


FIGURE 5. Disjointedness of the two interrelationships inferable from *flavour-reordering* correlation functions of four quark-bilinear currents [8]: (a) duality of QCD-level contributions found to be *not* tetraquark-phile and hadron-level contributions not involving two-meson $s$-channel cuts (hatched rectangle), on the one hand, and (b) tetraquark-adequate QCD sum rules relating all *tetraquark-phile* QCD-level contributions to hadron-level contributions exhibiting either tetraquark poles (horizontal bar) or two-meson $s$-channel cuts (filled rectangle), on the other hand.

Application of our criterion [1] shows (after unfolding the Feynman diagrams in the flavour-reordering case (4)) without any difficulty that, *exclusively*, contributions of, at least, order $\alpha_{\rm s}^2$ (Figs. 1c) and 2c)) may support four-quark singularities, contributing to the eventual development of a tetraquark pole.

## 4. Yield: multiquark-adequate QCD sum rule

Ultimately, allowing insights, at QCD level acquired, into the separation of contributions to four-point correlation functions of quark-bilinear currents into those that are multiquark-phile and those that undoubtedly are not to enter the QCD sum-rule machinery implicates *QCD–hadron interrelations* [6,8] more adequate to the needs of multiquarks than the traditional ones.

In the flavour-preserving case, this necessitates to identify and discard a pair of ordinary-meson QCD sum rules (Fig. 3), to be left with *tetraquark-focused* QCD sum rules [6] (Fig. 4).

In the flavour-reordering case, things are not that easy [8]. Jettisoning all clearly multiquark-irrelevant ballast (Fig. 5a)) demands case-by-case analysis of the QCD–hadron relations' singularity structure [8]; the reward for the peeling efforts is a QCD sum rule closer to one's targeted tetraquarks (Fig. 5b)).

In retrospect, for both possible quark-flavour distributions it is achievable to devise formulations of QCD sum rules [6,8] that enable users to zoom in on (peculiarities of) multiquarks.

## 5. Concise summary, conclusion, and prospect

We showed, for flavour-exotic tetraquarks, how to construct a variant of QCD sum rule for multiquarks not burdened by any contribution bearing no relationship to multiquarks [6,8], and we don't expect to encounter unsurmountable obstacles in the course of extending this concept to other multiquark varieties.

## Acknowledgements

1. W. Lucha, D. Melikhov, and H. Sazdjian, Narrow Exotic Tetraquark Mesons in Large-$N_c$ QCD, *Phys. Rev. D* **96** (2017) 014022, https://doi.org/10.1103/PhysRevD.96.014022.

2. W. Lucha, D. Melikhov, and H. Sazdjian, Exotic States and Their Properties from Large-$N_c$ QCD, *EPS-HEP 2017* **314** (2017) 390, https://doi.org/10.22323/1.314.0390.

3. W. Lucha, D. Melikhov, and H. Sazdjian, Tetraquark and Two-Meson States at Large $N_c$, *Eur. Phys. J. C* **77** (2017) 866, https://doi.org/10.1140/epjc/s10052-017-5437-x.

4. W. Lucha, D. Melikhov, and H. Sazdjian, Exotic Tetraquark Mesons in Large-$N_c$ Limit: an Unexpected Great Surprise, *EPJ Web of Conferences* **192** (2018) 00044, https://doi.org/10.1051/epjconf/201819200044.

5. W. Lucha, D. Melikhov, and H. Sazdjian, Are There Narrow Flavour-Exotic Tetraquarks in Large-$N_c$ QCD?, *Phys. Rev. D* **98** (2018) 094011, https://doi.org/10.1103/PhysRevD.98.094011

6. W. Lucha, D. Melikhov, and H. Sazdjian, Tetraquark-Adequate Formulation of QCD Sum Rules, *Phys. Rev. D* **100** (2019) 014010, https://doi.org/10.1103/PhysRevD.100.014010.

7. W. Lucha, D. Melikhov, and H. Sazdjian, Cluster Reducibility of Multiquark Operators, *Phys. Rev. D* **100** (2019) 094017, https://doi.org/10.1103/PhysRevD.100.094017.

8. W. Lucha, D. Melikhov, and H. Sazdjian, Tetraquark-Adequate QCD Sum Rules for Quark-Exchange Processes, *Phys. Rev. D* **100** (2019) 074029, https://doi.org/10.1103/PhysRevD.100.074029.

9. W. Lucha, D. Melikhov, and H. Sazdjian, Multiquark-Adequate QCD Sum Rules: the Case of Flavour-Exotic Tetraquarks, *EPJ Web of Conferences* **222** (2019) 03016, https://doi.org/10.1051/epjconf/201922203016.

10. W. Lucha, D. Melikhov, and H. Sazdjian, OPE and Quark-Hadron Duality for Two-Point Functions of Tetraquark Currents in the $1/N_c$ Expansion, *Phys. Rev. D* **103** (2021) 014012, https://doi.org/10.1103/PhysRevD.103.014012.

11. W. Lucha, D. Melikhov, and H. Sazdjian, Tetraquarks in Large-$N_c$ QCD, *Prog. Part. Nucl. Phys.* **120** (2021) 103867, https://doi.org/10.1016/j.ppnp.2021.103867.

12. M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, QCD and Resonance Physics. Theoretical Foundations, *Nucl. Phys. B* **147** (1979) 385, https://doi.org/10.1016/0550-3213(79)90022-1.

13. K. G. Wilson, Non-Lagrangian Models of Current Algebra, *Phys. Rev.* **179** (1969) 1499, https://doi.org/10.1103/PhysRev.179.1499.

14. M. Fierz, Zur Fermischen Theorie des $\beta$-Zerfalls, *Z. Phys.* **104** (1937) 553, https://doi-org.ezproxy.cern.ch/10.1007/BF01330070.

15. S. Coleman, *Aspects of Symmetry-Selected Erice Lectures* (Cambridge University Press, Cambridge, UK, 1985), Chap. 8, https://doi.org/10.1017/CBO9780511565045.

16. S. Weinberg, Tetraquark Mesons in Large-$N$ Quantum Chromodynamics, *Phys. Rev. Lett.* **110** (2013) 261601, https://doi.org/10.1103/PhysRevLett.110.261601.

17. L. D. Landau, On Analytic Properties of Vertex Parts in Quantum Field Theory, *Nucl. Phys.* **13** (1959) 181, https://doi.org/10.1016/0029-5582(59)90154-3.