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Ramanujan summation and the Casimir effect

Wolfgang Bietenholz
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México,
Apartado Postal 70-543, 04510 Ciudad de México, México

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Srinivasa Ramanujan was a great self-taught Indian mathematician, who died a century ago, at the age of only 32, one year after returning from England. Among his numerous achievements is the assignment of sensible, finite values to divergent series, which correspond to Riemann's ζ-function with negative integer arguments. He hardly left any explanation about it, but following the few hints that he gave, we construct a direct justification for the best known example, based on analytic continuation. As a physical application of Ramanujan summation we discuss the Casimir effect, where this way of removing a divergent term corresponds to the renormalization of the vacuum energy density, in particular of the photon field. This leads to the prediction of the Casimir force between conducting plates, which has now been accurately confirmed by experiments. Finally, we review the discussion about the meaning and interpretation of the Casimir effect. This takes us to the mystery surrounding the magnitude of Dark Energy.

Keywords: Ramanujan summation; Casimir effect; renormalization; ζ-function; dark energy.

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We present some remarks about Ramanujan summation, the Riemann ζ-function and its application to the Casimir effect, following the lines of our detailed review [1].

1. Ramanujan summation

Srinivasa Ramanujan was a great Indian genius of mathematics, an autodidact, who discovered numerous amazing formulae. He was born in 1887 in Erode and grew up in Kumbakonam, two towns in Eastern India[i].

After finishing High School he moved to Madras (today Chennai), where he lived in extreme poverty. He was not admitted to university, but he started to elaborate stunning mathematical formulae, often dealing with series. Impressed by his achievements, the prominent British mathematician Godfrey Hardy invited him to Cambridge UK from 1914–19. Hardy described Ramanujan's discoveries as a "process of mingled argument, and intuition".

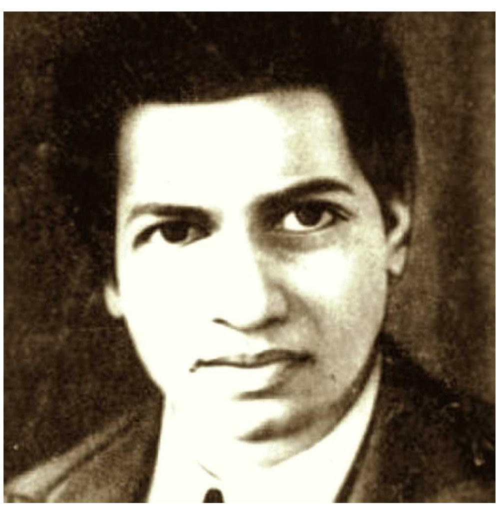

FIGURE 1. Srinivasa Ramanujan (1887–1920).

An example is the following sequence of approximations to the number π,

$$\frac{1}{\pi_N} = \frac{\sqrt{8}}{99^2} \sum_{n=0}^{N} \frac{(4n)!}{(4^n n!)^4} \, \frac{1103 + 26390n}{99^{4n}} \, , \tag{1}$$

with an extremely fast convergence: for $N = 0, 1, 2 \dots$ one obtains $|\pi - \pi_N| = \mathcal{O}(10^{-8(N+1)})$.

One of Ramanujan's famous achievements, which caused — and still causes — confusion, was the assignment of finite values to divergent series. Here are three examples, which he wrote down in a notebook [2], and also in the first letter that he sent to Hardy when he still lived in Madras[ii],

$$\sum_{n=1}^{\infty} 1 \,\hat{=}\, -\frac{1}{2} \, , \quad \mathcal{R} := \sum_{n=1}^{\infty} n \,\hat{=}\, -\frac{1}{12} \, , \qquad \sum_{n=1}^{\infty} n^3 \,\hat{=}\, \frac{1}{120} \, . \tag{2}$$

This may look strange, but Hardy recognized the values of Riemann's ζ-function. For $\mathrm{Re}\, z > 1$ it is given by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \, . \tag{3}$$

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i I thank T.R. Govindarajan for precise information; this was not accurate in Ref. [1].

ii We write $\hat{=}$ for "associated with", whereas Ramanujan shocked people by writing a straight equal sign.

Back in the 19th century, Bernhard Riemann had performed the analytic continuation to $z \in \mathbb{C} - \{1\}$ [3]. It can be summarized with the identity

$$\zeta(z) = \frac{(2\pi)^z}{\pi} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z) \ , \tag{4}$$

which implies in particular

$$\zeta(-k) = -\frac{B_{k+1}}{k+1} \ , \quad k \in \mathbb{N} \ , \tag{5}$$

which coincides with Eqs. (2) for $k = 0, 1, 2$. B_{k+1} are the Bernoulli numbers (with the convention $B_1 = 1/2$).

Best known is the case of $\mathcal{R} \,\hat{=}\, \zeta(-1)$, where Ramanujan only documented two intermediate steps [2],

$$\begin{aligned} \mathcal{E} &:= 1 - 2 + 3 - 4 + 5 \ldots \,\hat{=}\, \frac{1}{4} \\ \mathcal{R} - \mathcal{E} &\,\hat{=}\, 4 + 8 + 12 + \ldots \,\hat{=}\, 4\mathcal{R} \ \Rightarrow\ \mathcal{R} \,\hat{=}\, -\frac{1}{12}. \end{aligned} \tag{6}$$

This looks like uncontrolled operations on divergent series, but the sensible results — in three cases, cf. Eqs. (2) — cannot be by accident. Ramanujan mostly worked on a slate, he wrote down only little on paper, which was expensive. We can only speculate about his undocumented intermediate steps. Here we present a reasoning, which follows the lines of Eqs. (6) and clarifies the meaning of $\mathcal{R}$. It invokes analytic continuation, which is the only valid justification.

Let us consider $|z| < 1$, and the limit $z \to -1$, for

$$\begin{aligned} G(z) &:= \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \overbrace{\longrightarrow}^{z\to -1} \overbrace{1 - 1 + 1 - 1 + \ldots}^{\text{Grandi's series}} \,\hat{=}\, \frac{1}{2}, \\ G'(z) &= \sum_{n=0}^{\infty} n z^{n-1} = \frac{1}{(1-z)^2} \overbrace{\longrightarrow}^{z\to -1} \mathcal{E} \,\hat{=}\, \frac{1}{4} \ . \end{aligned} \tag{7}$$

The original series (on the left) only converge for $|z| < 1$, but we arrive at holomorphic (complex analytic) functions, with unique analytic continuations to $\mathbb{C} - \{1\}$. $G(z)$ is the familiar geometrical series, and in its derivative $G'(z)$ the series $\mathcal{R}$ requires $z \to 1$, which is singular. We regularize $\mathcal{R}$ in two more ways, again at $|z| < 1$,

$$\begin{aligned} R_1(z) &= 1 - 2z + 3z^2 - 4z^3 \cdots = \frac{1}{(1+z)^2} \ , \\ R_2(z) &= 1 + 2z^2 + 3z^4 + 4z^6 \cdots = \frac{1}{(1-z^2)^2} \ , \end{aligned}$$

with divergent limits $R_{1,2}(z \to -1)$. However, in the linear combination

$$R_1(z) + 4z R_2(z) = G'(z)$$

the poles in the Laurent series of $z = -1 + \varepsilon$ cancel. We obtain $(1/4) + \mathcal{O}(\varepsilon)$, so now the analytic continuation to $z = -1$ works,

$$\mathcal{R} - 4\mathcal{R} \,\hat{=}\, \frac{1}{4} \quad \Rightarrow \quad \mathcal{R} \,\hat{=}\, -\frac{1}{12} \ . \tag{8}$$

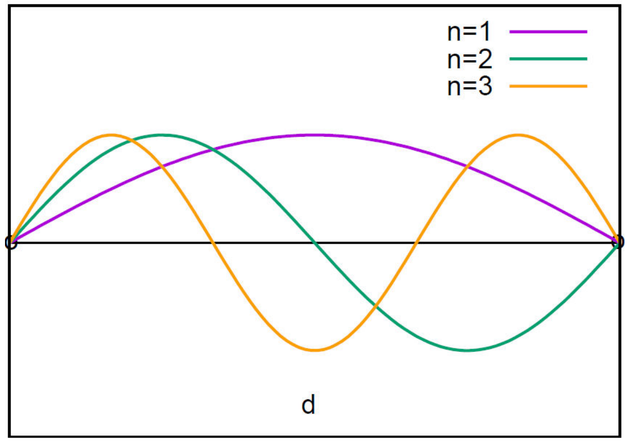

FIGURE 2. A symbolic illustration of the leading standing waves between two Dirichlet boundaries in one spatial dimension, separated by some distance d.

In Ref. [1] we also derived $\zeta(0) = -1/2$ in a similar way. For a general and systematic discussion of Ramanujan summation we refer to Ref. [4].

Is this just mathematical entertainment? No, it applies to quantum field theory: for a suitable system, Ramanujan summation removes a counter-term in a physically sensible manner, and provides renormalized results. In particular, it predicts a force, which has now been experimentally measured, but let us begin with a toy model.

2. The 1d Casimir effect

The Casimir effect was first predicted in Ref. [5], but here we follow the point of view which Hendrik Casimir (1909–2000) expressed a little later [6], after a discussion with Niels Bohr. For comprehensive overviews, we refer to Refs. [7, 8].

As a toy model, we first consider a free, massless, neutral scalar field in one spatial dimension, $\phi(t,x) \in \mathbb{R}$. We impose Dirichlet boundaries at $x = 0$ and d, which enforce $\phi(t,0) = \phi(t,d) = 0$. Inside this interval, the configurations can be expanded in terms of standing waves, as symbolically illustrated in Fig. 2, with wave numbers $k_n = n\pi/d$, $n = 1, 2, 3 \ldots$, and ground state energies $E_n = (1/2)k_n$ (in natural units, $\hbar = c = 1$).

In the interval $[\,0,\ d\,]$ we obtain the bare (ρ) and the renormalized ($\rho_{\rm r}$) vacuum energy density,

$$\rho(d) = \frac{1}{2d} \sum_{n\geq 1} k_n = \frac{\pi}{2d^2} \sum_{n \geq 1} n \,\hat{=}\, -\frac{\pi}{24 d^2} = \rho_{\rm r}(d) \ . \tag{9}$$

Hence, the renormalized energy between the boundaries amounts to $E_{\rm r}(d) = d\,\rho_{\rm r}(d)$. If we consider the boundary at d as flexible, we obtain the force $F_{\rm r}(d)$ acting on it,

$$F_{\rm r}(d) = -E'_{\rm r}(d) = -\frac{\pi}{24 d^2} \ . \tag{10}$$

We see that the boundaries are *attracted* to each other.

In the last step of Eq. (9) we have applied Ramanujan summation: it amounts to subtracting the *counter-term* $\rho(\infty)$, which provides the *renormalized* energy density and force.

To confirm this property, notice that for $\rho(\infty)$ the sum over the modes k_n turns into an integral. Hence the renormalized energy density $\rho_{\rm r}(d)$ emerges as the difference between a sum and an integral, which can be expanded with the Euler-Maclaurin formula, see *e.g.* Ref. [10]. Ref. [1] discusses in detail how this leads to the result in Eq. (9) — in agreement with Ramanujan summation, in particular with $\mathcal{R} \,\hat{=}\, -1/12$, the famous result that we re-derived in Sec. 1.

The application of the Euler-Maclaurin formula requires some regularization (with several ingredients), but the final result confirms Eq. (5), and therefore also Eqs. (2) and (9), for any choice of the regularization (if certain conditions are fulfilled); this is a generic feature of renormalization[iii].

If we want to safely exclude effects from the vacuum energy *outside* the given interval, we can assume three Dirichlet boundaries, where only the central one (the "piston") is flexible. This is also discussed in Ref. [1], and it justifies the form of the renormalized force in Eq. (10).

3. 3d Casimir effect of a photon field

Let us proceed to a realistic setting in 3 spatial dimensions: it consists of two parallel plates, which are (perfectly or at least well) conducting, both with area A, separated by a short distance $d \ll \sqrt{A}$, as sketched in Fig. 3.

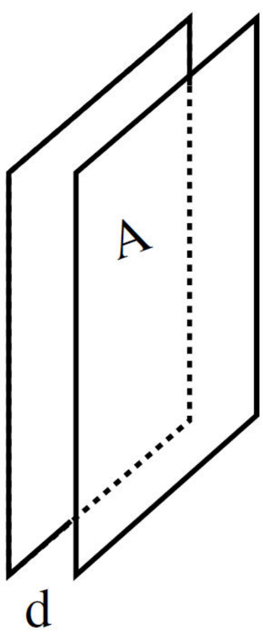

FIGURE 3. Set-up for the phenomenological Casimir effect between two conducting plates. For this setting, the Casimir force was first successfully measured in an experiment in Padua [11].

It is a good approximation to consider only the vacuum energy *between* the plates, $E(d) = A d \rho(d)$, obtained from the photon ground state energy density $\rho(d)$. In the corresponding formula, we treat the momentum components parallel to the plates, k_1, k_2, as continuous, while the vertical component is discrete, as in the 1d case of Sec. 2,

$$E(d) = \frac{A}{(2\pi)^2} \int dk_1 dk_2 \sum_{n=0}^{\infty} \sqrt{k_1^2 + k_2^2 + (\tfrac{\pi n}{d})^2} = \frac{A}{6\pi} \sum_{n=0}^{\infty} \Big[K^2 + (\tfrac{\pi n}{d})^2 \Big]^{3/2} \Big|_{K=0}^{\infty} \qquad (11)$$

(a factor of 2 accounts for the photon polarizations). Infinitesimally short waves are not sensitive to the presence of the plates (at finite separation d). Therefore only the lower bound, $K = 0$, contributes to the difference $\rho_{\rm r}(d) = \rho(d) - \rho(\infty)$, which leads to

$$\frac{E_{\rm r}(d)}{A} \,\hat{=}\, -\frac{1}{6\pi}\frac{\pi^3}{d^3}\zeta(-3) = -\frac{\pi^2}{720 d^3} \; ,$$
$$F_{\rm r}(d) = -E'_{\rm r}(d) = -\frac{\pi^2 A}{240 d^4} \simeq -1.3 \cdot 10^{-7}\,{\rm N} \left(\frac{\mu{\rm m}}{d}\right)^4 \frac{A}{{\rm cm}^2} \; , \qquad (12)$$

where we applied the last Ramanujan sum of Eq. (2). To arrive at a force in terms of Newton (N), we had to insert a factor $\hbar c$, which indicates that we are dealing with a relativistic quantum effect.

Again, Ramanujan summation provides a renormalized result, and again the force is attractive, although the signs of $\zeta(-1)$ and $\zeta(-3)$ are opposite (the lower integral bound at $K = 0$ flips the sign once more).

This force was conclusively measured, first in 1997/8, with about 5% accuracy [12, 13]. In these experiments, the geometry was a plate and a sphere, because it is very difficult to keep two plates exactly parallel. The first successful experiment with parallel plates, as in Fig. 3, was performed by a collaboration in Padua, Italy [11]. They used parallel silicon stripes with area $A = 1.9 \times 1.2\,{\rm mm}$, and their separation varied from $d = 0.5\,\mu{\rm m}$ to $3\,\mu{\rm m}$. According to Eq. (12) this implies an attractive Casimir force of strength $F_{\rm r} = -4.7 \cdot 10^{-7}$ N to $-3.7 \cdot 10^{-10}$ N. This force could be measured by using a fiber-optic interferometer, and monitoring the shift in the resonator frequency.

iii Of course, in the case of interacting fields, renormalization takes more than subtracting a divergent term, see *e.g.* Ref. [9]. In general one assigns renormalized, energy-dependent values to the fields and their couplings.

4. Interpretation of the Casimir Effect

What kind of force is this? At first sight, it does not seem to be part of the interactions described by the Standard Model. However, since it refers to the photon field, this effect must be electromagnetic, *i.e.* a facet of Quantum Electrodynamics (QED)^{iv}. What seems strange, however, is that the coupling α did not appear in the considerations of Sec. 3. We will come back to this puzzle in Subsec. 4.1.

The Casimir force was derived from the QED vacuum energy density, a quantity, which is not directly observable. So does its experimental confirmation imply indirect evidence for the physical existence of the QED vacuum energy? This is a wide-spread point of view, which is expressed *e.g.* in the books [7, 8], and the review [14] even writes that the existence of the vacuum energy density of the photon field "has been spectacularly demonstrated by the Casimir effect".

Actually, a direct manifestation of the vacuum energy density in the Universe does exist. Here we refer to Dark Energy, which (essentially) corresponds to the Cosmological Constant. It provides the most natural explanation for the accelerated expansion of the Universe, which was observed at the end of the 20th century, and which corresponds to $\rho_{\rm DE} \approx 2 \cdot 10^{-3}\ {\rm eV}^4$.

One is tempted to relate it to $\rho(\infty)$, with a cutoff, most naturally at the Planck scale, $E_{\rm Planck} \simeq 1.2 \cdot 10^{28}\ {\rm eV}$, but the density obtained is this manner is *much* larger than the Dark Energy density. If we truncate the momentum integral at the Planck scale,

$$\rho = \frac{1}{(2\pi)^3} \int_{|k| \leq E_{\rm Planck}} d^3k\, k = \frac{1}{8\pi^2} E_{\rm Planck}^4 \ , \qquad (13)$$

we obtain a prediction for the vacuum energy, which is about 121 orders of magnitude too large — perhaps the worst prediction in the history of science — and the reason for this fiasco is still not well understood (a pedagogical discussion is given in the appendix of Ref. [15]).

In a world with unbroken supersymmetry, the (positive) bosonic and (negative) fermionic vacuum energy cancel. However, even if one still assumes supersymmetry to exist, it has to be broken in our low-energy world, such that this cancellation takes place only in part. One still obtains a vacuum energy, which is at least 60 orders of magnitude too large [16], so supersymmetry does not solve the problem.

Just for fun, let us consider the Dark Energy density, which corresponds to the observation, $\rho_{\rm DE}$, and ask what cavity between two conducting plates it would take to obtain the same renormalized vacuum energy density, according to Eq. (12). Actually the sign is different, so let us consider the absolute value. The condition $\rho_{\rm DE} = |\rho_{\rm r}(d)| = \pi^2/(720 d^4)$ requires the distance $d \approx 0.3\,\mu{\rm m}$, which happens to be close to the minimal separation in the Padua experiment.

4.1. Where does the electromagnetic coupling α appear?

Now let us address the rôle of the electromagnetic coupling α, as we promised. It was analyzed in particular by Jaffe *et al.*; their point of view is summarized in Ref. [17]. They insist that the original picture by Casimir and Polder [5] was the correct one, *i.e.* they consider the Casimir force as a pure *van der Waals force.*^{v} Jaffe concludes that "Casimir forces can be computed without reference to zero-point energy", which would mean that their observation does *not* imply the physical existence of the QED vacuum energy.

Jaffe *et al.* do obtain an α-dependent Casimir force, $F_{\rm r}(\alpha)$, with $F_{\rm r}(\alpha = 0) = 0$. This seems trivial, but their result at $\alpha \to \infty$ is amazing: they derive a finite Casimir force in this limit, which coincides with $F_{\rm r}$ in Eq. (12), as obtained from the vacuum energy consideration. This explains why α did not appear there, but then we are left with the question why this result matches the observations so well. It turns out that in an experimentally realistic setting, the exact force is close to this approximation [17],

$$F_{\rm r}(\alpha \gg 10^{-5}) \simeq F_{\rm r}(\alpha = \infty) = -\frac{\pi^2}{240 d^4} \ , \qquad (14)$$

which easily holds for $\alpha \simeq 1/137$. This picture seems consistent, so is the effect de-mystified?

4.2. Are there repulsive Casimir forces?

There are objections against the latter point of view, which are expressed *e.g.* by Lamoreaux [12]. He insists that Casimir force and van der Waals force are conceptually different, because "the van der Waals force is always attractive, whereas the sign of the Casimir force is geometry dependent."

Indeed, numerous theoretical studies predict a repulsive Casimir force for certain geometric settings [18], such as specific parallelepipeds [19]. Jaffe *et al.* reject the repulsive scenario, and here the discussion enters subtle details [20]. Ref. [7] approves the equivalence of van der Waals and Casimir forces, but insists that they can still be repulsive.

A repulsive Casimir force was actually measured in 2009 [21], in agreement with a historic prediction in Ref. [22], but for materials immersed in a fluid. Hence the physical existence of $\rho_{\rm QED}$ — in vacuum — remains an exciting, open question.

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iv In principle it should exist for other fields as well, but if they are massive or self-interacting, the corresponding Casimir effect in a realistic setting tends to be strongly suppressed or over-shadowed.

v We mean the van der Waals — or London-van der Waals— force in the narrow sense: a collective, induced, (usually) attractive multi-pole interaction between (electrically neutral) molecules or atoms.

Since particle physics is very well described by Quantum Field Theory, one might argue that quantum fields — and therefore also the QED vacuum — are practically inevitable. However, the Casimir force was also derived with a Green's function technique, in the framework of QED, *without* referring to the QED vacuum energy density [23].

Of course, the question whether or not $\rho_{\rm QED}$ exists is usually irrelevant — in general we only care about energy *differences*, so an additive constant does not matter. An exception is the expansion of the Universe, and *perhaps* — depending on its interpretation — the Casimir effect.

Appendix

A. Photon in a Casimir cavity

An interesting prediction is the *Scharnhorst effect:* it states that in a Casimir cavity between parallel plates, the speed of light in vacuum — for photons travelling perpendicular to the plates — should increase [24]. The predicted effect is so tiny that it cannot be experimentally tested — for instance for a plate separation of 1 μm the relative increase would be of $\mathcal{O}(10^{-32})$. Still it led to a discussion if this could — in principle — lead to a causality paradox. This issue is reviewed and extensively discussed in Ref. [25].

We end with a simpler question: does a photon, which passes through a Casimir cavity (for instance parallel to the plates) change its energy, in the spirit of Bernoulli's Principle in fluid dynamics?

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