Suplemento de la Revista Mexicana de Física **3** 0308070 (2022) 1–6

Multi-meson model applied to $D \to hhh$

P. C. Magalhães

Instituto Tecnológico de Aeronáutica,
12.228-900 São José dos Campos, SP, Brazil; H.H. Wills Physics Laboratory,
University of Bristol, Bristol, BS8 1TLB, United Kingdom

A.C. dos Reis

Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, Brazil.

M.R. Robilotta

Instituto de Física, Universidade de São Paulo, São Paulo, Brazil.

Received 4 January 2022; accepted 11 February 2022

In this work, we discuss a new phenomenological model suited to all SU(3) mesonic two-body final-state interactions up to energies around 2 GeV to replace the standard isobar model. We show that the new model provides a clear indication of the mechanism responsible for the sharp rise observed in the $\pi\pi$ phase around 1 GeV. The phenomenological amplitudes proposed here are suited to any number of resonances, incorporate chiral symmetry at low energies, include coupled channels, and respect unitarity.

Keywords: Meson-meson interaction; three-body decay; hadronic decays; unitary amplitudes.

DOI: https://doi.org/10.31349/SuplRevMexFis.3.0308070

1. Introduction

The current situation in three-body hadronic decay is progressing on the experimental side. More comprehensive investigations can be done nowadays, using the very large and pure samples provided by the LHC experiments, and still more data is expected in the near future, including neutral particles, with Belle II, BES III and LHCb (Run 2) experiments. These decays involve two distinct sets of interactions. They begin with a primary vertex, in which the light $SU(3)$ quarks produced in the weak reaction disturb the surrounding QCD vacuum and give rise to an initial set of mesons. This state then evolves by means of purely hadronic final state interactions (FSIs), whereby mesons rescatter many times before being detected. There are, however, many challenges on the theoretical side to fully describe the dynamics of those decays, in particular, the role of hadronic final state interactions and their rich structure. In a model for $D \to KKK$ [1] we developed the first seed of the present model where the meson-meson amplitudes where derived in the K-Matrix approximation and resulted in a good fit to LHCb data [2]. A considered extension of these ideas results in an SU(3) meson-meson tool kit [3], where we explore the main characteristics of a two-body interaction in a coupled-channel and multi-resonance description to be used in amplitude analyses of hadron decays.

The main model used to analyse data and characterize resonances is the standard isobar model. Its basic assumption is that a decay amplitude can be represented by a coherent sum of both non-resonant and two-body resonant contributions, with the latter described by a Briet-Wigner function depending on resonance mass m_k and a width Γ_k, given by

$$[\text{line shape}]_k \quad \to \quad [\text{BW}]_k = \frac{1}{[s - m_k^2 + i\, m_k\, \Gamma_k]}. \qquad (1)$$

There are, however, some limitations in this approach that result in serious flaws of the data analysis models already rather clear, such as: it violates two-body unitarity when there is more than one resonance with the same quantum numbers, it does not incorporate isospin and, especially important, it is totally unsuited for dealing with coupled channels. In the $SU(3)$ sector, scattering amplitudes for pions, kaons and etas are strongly coupled and cannot be represented as sums of individual contributions. At present, as one knows, QCD cannot be directly applied to heavy meson decays, but their effective counterparts can. Effective lagrangians rely just on hadron masses and coupling constants, ensuring that the physical meaning of parameters is preserved from process to process. This differs from the standard isobar model where the physical meaning of parameters it yields from data fits can be distorted due to the above mentioned conceptual gaps.

In this presentation we explore the two-body scattering amplitudes, departing from effective lagrangians aiming at constructing guess functions for heavy-meson decays. The meson-meson interaction amplitudes are directly associated with observed quantities and also important substructures of hadronic decay amplitudes.

2. Scattering amplitude

Generally, any three-body heavy meson decay can be represented by the series of diagrams given in Fig. 1, where

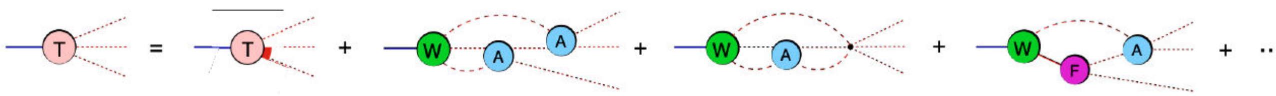

FIGURE 1. Full three-body decay series.

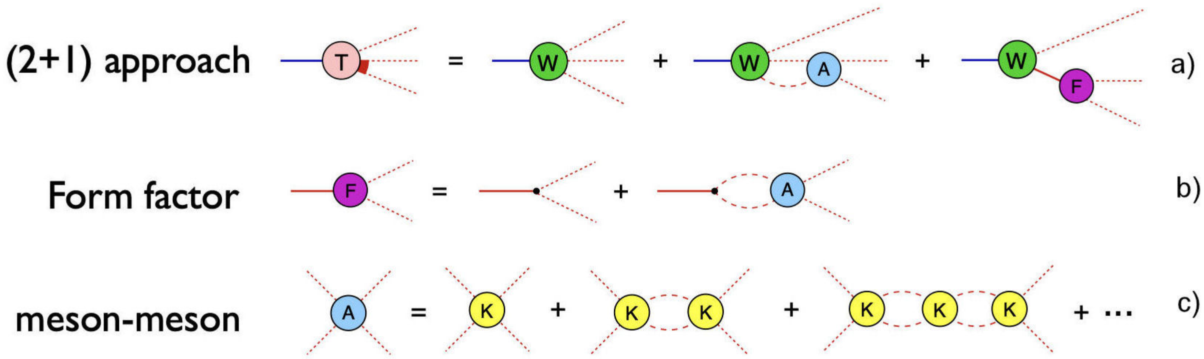

FIGURE 2. a) Decay amplitude in the $2+1$ approximation; b) the resonance form factor; c) scattering amplitude A represented by the iteraction of the Kernel K with the loop of mesons.

W represents the weak vertex topologies, A is the two-body interaction amplitude, F the resonance form-factor production (Fig. 2b)) and the first diagram represents the (2+1) approximation where the interaction with the third particle is neglected, as given by diagrams in Fig. 2a). Independently of the weak interaction topologies and the approximation used to describe the three-body dynamics, the meson-meson interaction amplitude A is a main building block.

In this presentation I showed the main characteristics of the amplitude A and how to include multiple resonances without disturbing unitarity. For details and comparison with the standard isobar model we point the reader to Ref. [3].

The scattering amplitude $A_{(k\ell|ab)}^{(J,I)}$ for the process $P_k\, P_\ell \;\rightarrow\; P_a\, P_b$ in a channel with spin J and isospin I is described by the perturbative series in Fig. 2c) and defined as:

$$\mathcal{A} = \mathcal{K} \times (1 + [\text{loop} \times \mathcal{K}] + [\text{loop} \times \mathcal{K}]^2 + [\text{loop} \times \mathcal{K}]^3 + \cdots) \,, \tag{2}$$

$$\text{loop} = \varnothing^R + i\, \varnothing^I \;. \tag{3}$$

The geometric series in Eq. (2) can be summed and one has

$$\mathcal{A} = \frac{\mathcal{K}}{D} \,, \qquad D = 1 - (\text{loop} \times \mathcal{K}) \;. \tag{4}$$

A very important feature of this result is that the amplitude $\mathcal{A}$ is unitary, provided $\mathcal{K}$ is real. This property is quite general and derives from the structure of the denominator D, which is suitably complex owing to the well defined imaginary function $i\, \varnothing^I$ in Eq. (3). The dynamical content of meson-meson (PP) interactions is incorporated into the kernels $\mathcal{K}_{(kl|ab)}^{(J,I)}$, which are real functions of masses and coupling constants based on ChPT lagrangians [4–6]. All kernels are written as sums of a leading-order (LO) chiral polynomial [4] and next-to-leading-order (NLO) resonance contributions [6]. They include crossed amplitudes at tree level, but no loops in t- and u-channels. To summarize, the amplitude includes four kinds of ingredients, namely:

- **a. Coupled channels -** this sector of the problem is rather standard and model independent. In our notation, the coupling among the various channels is implemented by the mixing matrices $M_{ab}^{(J,I)}$ as in Eq. (15).
- **b. Multi-resonance dynamics -** We add as many resonances per channel needed. While in kernels, resonances have no widths and are characterized just by their naked poles. The width will rise from the unitarization procedure. The inclusion of several resonances is performed by adding these poles in the unitarization scheme.
- **c. Unitarization -** We neglect four-meson intermediate states and the unitarization of amplitudes is directly associated with the s-channel two-meson propagators $\varnothing$ that occur in the full scattering amplitude. These functions contain real and imaginary parts: $\varnothing = \varnothing^R + i\, \varnothing^I$. The latter, given by Eq. (17), are free from ambiguities and constitute the only source of imaginary terms in the amplitudes $A_{(k\ell|ab)}^{(J,I)}$. In particular, resonance widths are necessarily proportional to $\varnothing^I$. The real component, $\varnothing^R$, has infinite components which are replaced by renormalization constants. The form of this component in the case of several resonances is given by Eq. (19).
- **d. Free parameters -** The parameters entering our amplitudes consist basically of masses and coupling constants and, in principle, are completely free. Thus, our

amplitudes are guess functions with open parameters, to be determined by fits do data. Most of the symbols used to label these parameters were borrowed from chiral perturbation theory, especially Ref. [6]. Their numerical meanings, however, are not exactly the same. In chiral perturbation theory, the values of parameters are extracted by comparing results from calculations performed to a given order with observables. As loops are divergent and need to be renormalized, values for parameters quoted in the literature also depend on renormalization scales. This kind of procedure is theoretically consistent and yields a precise description of low-energy phenomena.

3. $\pi\pi$ S-wave example

To illustrate the model we took $\pi\pi$ scattering amplitude S-wave as an example. With $(J, I) = (0, 0)$, $\pi\pi$ can couple to $K\bar{K}$ and to $\eta\eta$, the four pion channel was neglected at this point. In the scalar-isoscalar sector, $SU(3)$ gives rise to octet and singlet states So and $S1$, which can be combinations of the observed resonances $f_a = f(1370)$ and $f_b = f(980)$. All the functions needed for the amplitude are described in the next section.

To show the importance of going beyond the standard resonance Breit-Wigner description, in Fig. 3a), we neglect $K\bar{K}$ and $\eta\eta$ couplings and compare results from two versions of Eq. (6). One of them keeps just its third term, representing an octet resonance (R), and the other also includes the first term, describing a contact chiral interaction (C+R), which is one of the signatures of post QCD physics. One notes that the contact term is rather important and the dominance of the resonance is restricted to a narrow band around its mass m_{fb}.

The opening of the $K\bar{K}$ channel is studied in Fig. 3b), for the same C+R case considered before, keeping the resonance mass fixed at $m_{fb} = 0.98$ GeV, while adopting two fake values for M_K, namely 0.48 and 0.50 GeV, so as to have the

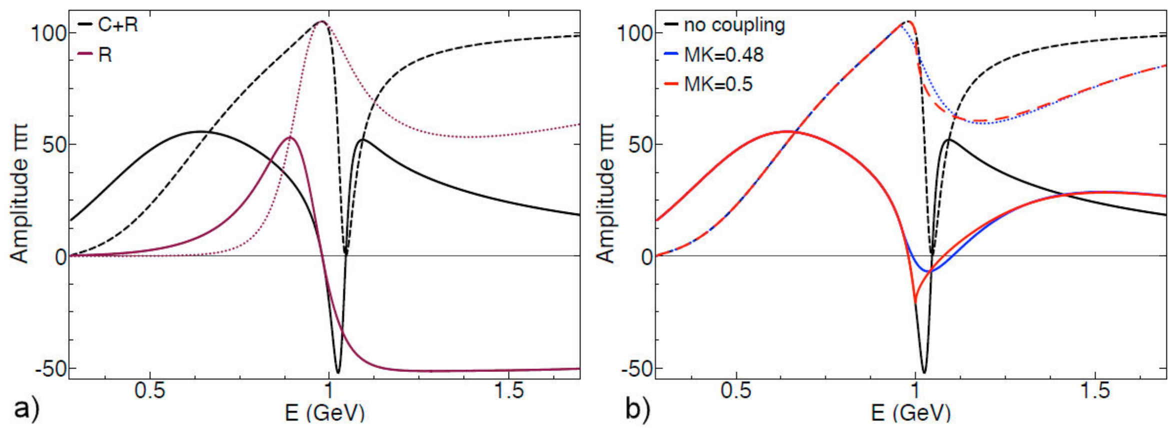

FIGURE 3. a) predictions for real (full curves) and imaginary (dashed curves) parts of the scalar-isoscalar $\pi\pi$ amplitude a) for a single resonance (R) and the same resonance superimposed to a chiral contact term (C+R), and b) for a single resonance superimposed to a non-resonant background (NR+R) for no coupled channels (black) and a coupled $K\bar{K}$ channel with threshold below (blue) and above (red) the resonance mass.

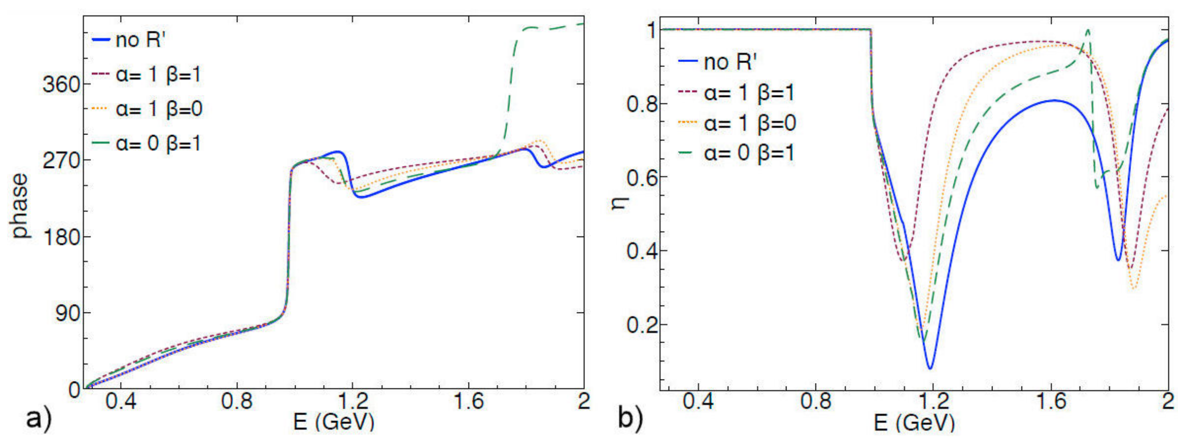

FIGURE 4. a) Predictions for phase shifts and b) inelasticity parameters for the scalar $\pi\pi$ amplitude with an extra resonance of mass $m_{R'} = m_{f_0} = 1.7$ GeV.

$K\bar{K}$ threshold both below and above it. As expected, all curves coincide below the thresholds. Above them, however, one learns that the impact of the coupling is important, since the previous C+R form provides a very poor representation for the new results, irrespective of the value of M_K chosen.

The model proposed here allows for the inclusion of any number of resonances. In order to illustrate this procedure, we consider the case of an extra resonance R' and begin by resorting to Eq. (5). New resonances mean, of course, new masses and coupling constants and, as the number of channels is large, one could have, in principle, too many new degrees of freedom to be fitted by data. In order to be conservative, we suggest that the same forms displayed after the arrows in Eqs. (9)-(14) be used, with $([c_d \,\text{or}\, \tilde{c}_d]\,[s-\text{mass}^2]+c_{(R|ab)}) \rightarrow\ [c_d \,\text{or}\, \tilde{c}_d]\,)\alpha\,(s-\text{mass}^2)+\beta_{R'}\,\mu^2)$. In the case of the s-dependent couplings, this preserves the $SU(3)$ structure, with a scale given by chiral perturbation theory [6], $c_d = 0.032$ GeV and $\tilde{c}_d\ =\ 0.018$ GeV, whereas $\mu = 1$ GeV is just a scale. These choices allow both α and β to be dimensionless free parameters and one may guess that their values will be not far from $-1 \leq \alpha, \beta \leq 1$.

In Fig. 4 we present the phase shifts and inelasticity parameters for $\pi\pi$ for a choice of values of α and β. The high energy region is more sensitive to the inclusion of the extra resonance and, its shape is strongly affected by a background due to channel-coupling.

The unitarization proceedure of adding new resonances can be done sistematically. Owing to renormalization, the real parts of the functions Ω must be supplemented by arbitrary constants, to be fixed by experiment and that is why a model dependence comes in. In our model, we fix the subtracted constant by forcing the real part of Ω to be zero at all the resonance masses and introduce a form factor to ensure that loop corrections do not spoil chiral symmetry results at low energies. The final renormalization constant for three-resonances is given in Eq. (19).

4. Relevant functions

Functions needed to describe $\pi\pi \rightarrow \pi\pi$ scattering amplitude, with $(J,I)=(0,0)$ in a three-coupled channel example.

$$\begin{aligned} A^{(0,0)}_{(\pi\pi|\pi\pi)} = \frac{1}{D^{(0,0)}}\Bigg(& \left[\left\{1-M^{(0,0)}_{22}\right\}\left\{1-M^{(0,0)}_{33}\right\} - M^{(0,0)}_{23}M^{(0,0)}_{32}\right]\mathcal{K}^{(0,0)}_{(\pi\pi|\pi\pi)} \\ & + \left[M^{(0,0)}_{12}\ \left\{1-M^{(0,0)}_{33}\right\} + M^{(0,0)}_{13}\ M^{(0,0)}_{32}\right]\mathcal{K}^{(0,0)}_{(KK|\pi\pi)} \\ & + \left[M^{(0,0)}_{13}\ \left\{1-M^{(0,0)}_{22}\right\} + M^{(0,0)}_{12}\ M^{(0,0)}_{23}\right]\ \mathcal{K}^{(00)}_{(88|\pi\pi)}\Bigg), \end{aligned} \tag{5}$$

$$\mathcal{K}^{(\prime,\prime)}_{(\pi\pi|\pi\pi)} = \frac{(2s-M^2_\pi)}{F^2} - \frac{G_{(f_a|\pi\pi|\pi\pi)}}{s-m^2_{f_a}} - \frac{G_{(f_b|\pi\pi|\pi\pi)}}{s-m^2_{f_b}} - \frac{G^2_{(f'|\pi\pi)}}{s-m^2_{f'}}, \tag{6}$$

$$\mathcal{K}^{(\prime,\prime)}_{(\pi\pi|\mathcal{KK})} = \frac{\sqrt{3}s}{2F^2} - \frac{G_{(f_a|\pi\pi|KK)}}{s-m^2_{f_a}} - \frac{G_{(f_b|\pi\pi|KK)}}{s-m^2_{f_b}} - \frac{G_{(f'|\pi\pi)}G_{(f'|KK)}}{s-m^2_{f'}}, \tag{7}$$

$$\mathcal{K}^{(\prime,\prime)}_{(\pi\pi|\forall\forall)} = \frac{\sqrt{3}M^2_\pi}{3F^2} - \frac{G_{(f_a|\pi\pi|88)}}{s-m^2_{f_a}} - \frac{G_{(f_b|\pi\pi|88)}}{s-m^2_{f_b}} - \frac{G_{(f'|\pi\pi)}G_{(f'|88)}}{s-m^2_{f'}}, \tag{8}$$

$$G_{(So|\pi\pi)} = -\frac{\sqrt{2}}{F^2}\,(c_d\,s - [c_d - c_m]\,2M^2_\pi) \rightarrow -\frac{\sqrt{2}}{F^2}(c_d\,[s-2M^2_\pi] + c_{(So|\pi\pi)}), \tag{9}$$

$$G_{(So|KK)} = \frac{\sqrt{6}}{3\,F^2}(c_d s - [c_d - c_m]2M^2_K) \rightarrow \frac{\sqrt{6}}{3\,F^2}(c_d[s-2M^2_K] + c_{(So|KK)}), \tag{10}$$

$$G_{(So|88)} = \frac{\sqrt{6}}{3\,F^2}\,(c_d[s-2M^2_8) + c_m[16M^2_K - 10M^2_\pi]/3) \rightarrow \frac{\sqrt{6}}{3\,F^2}(c_d\,[s-2M^2_8] + c_{(So|88)}), \tag{11}$$

$$G_{(S1|\pi\pi)} = \frac{2\sqrt{3}}{F^2}(\bar{c}_d\,s - [\bar{c}_d - \bar{c}_m]2M^2_\pi) \rightarrow \frac{2\sqrt{3}}{F^2}(\bar{c}_d\,[s-2M^2_\pi] + c_{(S1|\pi\pi)}), \tag{12}$$

$$G_{(S1|KK)} = \frac{4}{F^2}(\bar{c}_d\,s - [\bar{c}_d - \bar{c}_m]2M^2_K) \rightarrow \frac{4}{F^2}(\bar{c}_d\,[s-2M^2_K] + c_{(S1|KK)}), \tag{13}$$

$$G_{(S1|88)} = \frac{2}{F^2}(\bar{c}_d s - [\bar{c}_d - \bar{c}_m]2M^2_8) \rightarrow \frac{2}{F^2}(\bar{c}_d[s-2M^2_8] + c_{(S1|88)}). \tag{14}$$

In RChPT [6], one has $|c_d| = 0.032$ MeV, $|c_m| = 0.042$ MeV, $|\tilde{c}_d| = |c_d|/\sqrt{3}$ and $|\tilde{c}_m| = |c_m|/\sqrt{3}$.

$$M_{11}^{(0,0)} = -\mathcal{K}_{(\pi\pi|\pi\pi)}^{(0,0)} [\Omega_{\pi\pi}^S/2] \,, \qquad M_{12}^{(0,0)} = -\mathcal{K}_{(\pi\pi|KK)}^{(0,0)} [\Omega_{KK}^S/2] \,, \qquad M_{13}^{(0,0)} = -\mathcal{K}_{(\pi\pi|88)}^{(0,0)} [\Omega_{88}^S/2] \,,$$
$$M_{21}^{(0,0)} = -\mathcal{K}_{(\pi\pi|KK)}^{(0,0)} [\Omega_{\pi\pi}^S/2] \,, \qquad M_{22}^{(0,0)} = -\mathcal{K}_{(KK|KK)}^{(0,0)} [\Omega_{KK}^S/2] \,, \qquad M_{23}^{(0,0)} = -\mathcal{K}_{(KK|88)}^{(0,0)} [\Omega_{88}^S/2] \,,$$
$$M_{31}^{(0,0)} = -\mathcal{K}_{(\pi\pi|i88)}^{(0,0)} [\Omega_{\pi\pi}^S/2] \,, \qquad M_{32}^{(0,0)} = -\mathcal{K}_{(KK|88)}^{(0,0)} [\Omega_{KK}^S/2] \,, \qquad M_{33}^{(0,0)} = -\mathcal{K}_{(88|88)}^{(0,0)} [\Omega_{88}^S/2], \tag{15}$$

where

$$\Omega_{ab}^S = -\frac{\Pi_{ab}(s)}{16\pi^2} \,, \tag{16}$$

and $\Pi_{ab}(s)$ represents the regular parts of loop integrals defined in detail in the Appendix of Ref. [3]. The imaginary component is model independent and reads

$$[\Omega_{ab}^S]^I = -\frac{1}{8\pi}\, \frac{Q_{ab}}{\sqrt{s}}\, \theta(s - [M_a + M_b]^2) \,, \tag{17}$$

$$Q_{ab} = \frac{\sqrt{l}}{2\sqrt{s}} = \frac{1}{2}\sqrt{s - 2(M_a^2 + M_b^2) + (M_a^2 - M_b^2)^2/s} \,, \tag{18}$$

where θ is the Heaviside step function. The real part includes the renormalization constant for three resonances and the final propagator is given by:

$$\Omega_{ab}^S(s) \to \frac{1}{16\pi^2}\Bigg(F_x(s)\, \frac{[s-m_y^2]\,[s-m_z^2]}{[m_x^2-m_y^2]\,[m_x^2-m_z^2]}\, \Pi_{ab}^R(m_x^2) + F_y(s)\, \frac{[m_x^2-s]\,[s-m_z^2]}{[m_x^2-m_y^2]\,[m_y^2-m_z^2]}\, \Pi_{ab}^R[m_y^2]$$
$$+ F_z(s)\, \frac{[m_x^2-s][m_y^2-s]}{[m_x^2-m_z^2][m_y^2-m_z^2]}\, \Pi_{ab}^R[m_z^2] - \Pi_{ab}(s) \Bigg). \tag{19}$$

5. Final remarks

The standard isobar model (SIM) was produced more than 50 years ago and is still widely used, in spite of its many limitations. In the case of heavy-meson decays into three mesons, the model relies on the (2+1) approximation, whereby strong final state interactions involve just a two-body interacting system in the presence of a spectator. The assumption that meson-meson amplitudes are strongly dominated by resonances is essential to the model. We argue that QCD has a strong impact on this picture and that the SIM may be reliable for vector mesons in uncoupled channels but is not suited for scalar mesons. Nowadays, a proper description of low-energy meson-meson interactions requires contact with chiral perturbation theory, which implements QCD by means of effective lagrangians. Although originally developed for low-energy processes, this theory can be reliably extended through the inclusion of resonances and unitarization techniques. Another problem of the SIM concerns the coupling of channels. This effect is compulsory and we have shown that resonances cannot be considered as dynamically isolated objects beyond coupling thresholds. This happens because pole dominance in a given channel is contaminated by background effects occurring elsewhere.

We present an alternative phenomenological meson-meson amplitudes to the $SU(3)$ sector, which is suitable for amplitude analyses of heavy-meson decays. Their main features include:

- **a. Unitarization -** All amplitudes are automatically unitary for energies below the first coupling threshold.
- **b. Coupled channels -** The treatment of coupled channels is standard and gives rise to the expected inelasticities.
- **c. Dynamics -** Interactions are described by chiral lagrangians, which include both pure pseudoscalar vertices and bare resonances, with free masses and coupling constants. This ensures that chiral symmetry is obeyed at low-energies and also gives rise to fitting parameters with well defined physical meaning.
- **d. Model for meson loops -** Two-meson loops are an important component of scattering amplitudes. In the s-channel, they are given by real functions below threshold and acquire an imaginary part above it. The latter is fully determined by theory whereas the former involve unknown renormalization constants. We propose a model for these real parts, which comply with chiral symmetry and can accommodate any number of resonances.

e. Systematic inclusion of resonances - The model can accommodate any number of resonances in each given channel.

f. Free parameters have physical meaning - The free parameters of the model are resonance masses and constants describing their couplings to pseudoscalar mesons. Thus, their conceptual meaning is both rather conventional and process independent, whereas their empirical values can be extracted from different reactions. This allows one to envisage a situation in which one could compare various sets of values for the *same* parameters as determined, for instance, from chiral perturbation theory, meson-meson scattering up to 2 GeV, $D \to \pi\pi\pi$, $D \to \pi\pi K$ and other processes. This would definitely promote understanding and, hopefully, much needed progress.

In this constructive approach, all imaginary terms in the amplitudes can be traced back to loops, which are also responsible for the finite widths of resonances. The parameters to be fitted are just resonance masses and coupling constants, which have a rather transparent physical meaning. As examples, we have discussed scalar amplitudes, phase shifts and inelasticity parameters for $\pi\pi$ scatterings, employing the low-energy parameters given in Ref. [6]. One notices that the main differences occur close to the first inelastic threshold, showing that the new model provides a clear indication for the mechanism responsible for the sharp rise observed in the $\pi\pi$ phase around 1 GeV.

Acknowledgements

PCM work was supported by Marie Currie (MSCA) grant no. 799974, and CAPES - PRINT grant 88887.580984/2020-00.

1. R. Aoude, P. Magalhães, A. Dos Reis and M. Robilotta, *Multimeson model for the $D^+ \to K^+K^-K^+$ decay amplitude*, Phys. Rev. D **98** (2018) 056021,
2. R. Aaij *et al.* [LHCb],*Dalitz plot analysis of the $D^+ \to K^-K^+K^+$ decay*, JHEP **04** (2019) 063,
3. P. C. Magalhães, A. C. dos Reis and M. R. Robilotta, *Multibody decay analyses: A new phenomenological model for meson-meson subamplitudes*, Phys. Rev. D **102** (2020) 076012,
4. J. Gasser and H. Leutwyler,*Chiral Perturbation Theory to One Loop*, Ann. Phys. **158** (1984) 142,
5. J. Gasser and H. Leutwyler, *Chiral Perturbation Theory: Expansions in the Mass of the Strange Quark*, Nucl. Phys. B **250** (1985) 465,
6. G. Ecker, J. Gasser, A. Pich and E. De Rafael, *The Role of Resonances in Chiral Perturbation Theory*, Nucl. Phys. B **321** (1989) 311,