Suplemento de la Revista Mexicana de Física **3** 0308060 (2022) 1–5

Amplitude analyses of multi-body hadronic $D^+_{(s)}$ decays at BESIII

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Received 15 January 2022; accepted 15 February 2022

Using e^+e^- annihilation data corresponding to a total integrated luminosity of 6.32 fb^{-1} and 2.93 fb^{-1} collected at the center-of-mass energies 4.178-4.226 GeV and 3.773 GeV with the BESIII detector, we have performed amplitude analyses of the decays $D^+_s \to K^-K^+\pi^+$, $D^+_s \to K^0_S\pi^+\pi^0$, $D^+_s \to K^0_S K^-\pi^+\pi^+$, $D^+_s \to K^+K^-\pi^+\pi^0$, $D^+_s \to \eta\pi^+\pi^+\pi^-$, and $D^+ \to K^0_S K^+\pi^0$. We present the results based on these amplitude analyses where rich structures have been observed. In addition, we also report observations of some new hadronic $D^+_{(s)}$ decay modes $D^+_s \to K^0\rho(770)^+$, $D^+_s \to K^*(892)^0\pi^+$, $D^+_s \to K^*(892)^+\pi^0$, and $D^+_{(s)} \to a_0(980)^+\rho^0$ and the determinations of their decay branching fractions which are $5.46 \pm 0.84_{\rm stat.} \pm 0.44_{\rm syst.} \times 10^{-3}$, $2.71 \pm 0.72_{\rm stat.} \pm 0.30_{\rm syst.} \times 10^{-3}$, $0.75 \pm 0.24_{\rm stat.} \pm 0.06_{\rm syst.} \times 10^{-3}$, and $0.21 \pm 0.08_{\rm stat.} \pm 0.05_{\rm syst.}\%$

Keywords: BESIII; $D^+_{(s)}$ meson; amplitude analysis; multi-body decay.

DOI: https://doi.org/10.31349/SuplRevMexFis.3.0308060

1. Introduction

The hadronic decays of $D^+_{(s)}$ mesons are dominated by quasi two-body processes [1], such as $D^+_{(s)} \to PP$, $D^+_{(s)} \to VP$, $D^+_{(s)} \to VS$, $D^+_{(s)} \to VV$, $D^+_{(s)} \to AP$, and $D^+_{(s)} \to AV$, where P, V, S, and A denote pseudo-scalar, vector, scalar, and axial-vector mesons, respectively. Most of their decay branching fractions (BFs) can be predicted theoretically [2–4], while some non-perturbative contributions, such as final-state interactions, make some of them hard to predict. Therefore, measurements of the quasi two-body decay branching fractions are important to test the theoretical calculations and can help the understanding of $D^+_{(s)}$ meson decay mechanisms.

The BESIII detector [5] records symmetric e^+e^- collisions provided by the Beijing Electron Position Collider (BEPCII) storage ring [6], which operates with a peak luminosity of 1×10^{33} cm^{-2}s^{-1} at the center-of-mass energy ($\sqrt{s}$) range from 2.0 to 4.9 GeV. The threshold energy of $D_sD^*_s$ and $D\bar{D}$ pairs are produced at 4.178-4.226 GeV [7] and 3.773 GeV [8], respectively, corresponding to a total integrated luminosity of 6.32 fb^{-1} and 2.93 fb^{-1}. Hadronic decays of charmed hadron can be studied with almost free of background based on these data samples. Because the double-tag (DT) [9] reconstructed candidate has both $D^+_sD^-_s$ or D^+D^- mesons. In the paper, charge conjugate states are implied.

2. Strategy

The relative magnitudes and phases of the partial waves and the masses and widths of intermediate-resonant contribution in these decays are determined by an un-binned-maximum-likelihood fit. A probability density function (PDF) constructs the likelihood function, which depends on the momenta of the final daughter particles. $\mathcal{A}_n$ is the amplitude of the n^{th} intermediate state defined by

$$\begin{aligned}\mathcal{A}_n(p_j) &= P^1_n(m_1)P^2_n(m_2)S_n(p_j)\\ &\quad\times F^1_n(p_j)F^2_n(p_j)F^{D^+_{(s)}}_n(p_j),\end{aligned} \tag{1}$$

where $F^{1,2}_n(p_j)$ and $F^{D^+_{(s)}}_n(p_j)$ are the Blatt-Weisskopf barrier factors for the intermediate resonances 1, 2, and $D^+_{(s)}$, the $P^1_n(m_1)$ and $P^2_n(m_2)$ are the propagator function, and the spin factor is described by function $S_n(p_j)$.

The coherent sum of these amplitudes of intermediate processes is described as the total decay amplitude $\mathcal{M}(p_j)$, which is $\mathcal{M}(p_j) = \sum c_n\mathcal{A}_n(p_j)$, where c_n is the corresponding complex coefficient $\rho_n e^{i\phi_n}$. The magnitude ρ_n and phase ϕ_n are determined by the amplitude analysis. The $f_S(p_j)$ is the signal PDF written as

$$f_S(p_j) = \frac{\epsilon(p_j)\left|\mathcal{M}(p_j)\right|^2 R(p_j)}{\int \epsilon(p_j)\left|\mathcal{M}(p_j)\right|^2 R(p_j)\,dp_j}\,, \tag{2}$$

where the final four-momentum p_j parameterizes the detection efficiency ($\epsilon(p_j)$), and the standard element of multi-body phase space is described by $R(p_j)$. In the Eq. 2, the fitted variables $\epsilon(p_j)$ and $R(p_j)$ terms are independent, so they are regarded as constant terms in the fit. The normalization integrals are determined by an MC integration,

$$\begin{aligned}\int \epsilon(p_j)\left|\mathcal{M}(p_j)\right|^2 R(p_j)\,dp_j &\approx \frac{1}{N_{\rm MC}}\\ &\quad\times\sum_{k_{\rm MC}}^{N_{\rm MC}} \frac{\left|\mathcal{M}(p_j^{k_{\rm MC}})\right|^2}{\left|\mathcal{M}^{\rm gen}(p_j^{k_{\rm MC}})\right|^2}\,,\end{aligned} \tag{3}$$

where the number of the selected MC events is described by $N_{\rm MC}$, the index of the $k^{th}_{\rm MC}$ event is written as $k_{\rm MC}$, and PDF

$\mathcal{M}^{\text{gen}}(p_j)$ is used to generate the MC samples of MC integration. Due to the differences from the PID and tracking between data and simulation, we determine the effect by

$$\gamma_\epsilon(p_j) = \prod_i \frac{\epsilon_{i,\text{data}}(p_j)}{\epsilon_{i,\text{MC}}(p_j)}, \tag{4}$$

where i refers to tracking or PID, $\epsilon_{i,\text{data}}(p_j)$ and $\epsilon_{i,\text{MC}}(p_j)$ is the tracking or PID efficiency as a function of the momenta of the daughter particles for data and MC. After weighting each signal MC event with γ_ϵ, MC integration is modeled as

$$\int \epsilon(p_j)\left|\mathcal{M}(p_j)\right|^2 R(p_j)\,dp_j \approx \frac{1}{N_{\text{MC}}} \times \sum_k^{N_{\text{MC}}} \frac{\left|\mathcal{M}(p_j^{k_{\text{MC}}})\right|^2 \gamma_\epsilon(p_j^{k_{\text{MC}}})}{\left|\mathcal{M}^{\text{gen}}(p_j^{k_{\text{MC}}})\right|^2}. \tag{5}$$

In these amplitude analyses, the background contribution is described by the background PDF:

$$f_{\mathcal{B}}(p_j) = \frac{\mathcal{B}(p_j)R(p_j)}{\int \mathcal{B}(p_j)R(p_j)\,dp_j}. \tag{6}$$

According to the background events from the inclusive MC sample in the signal region, we can model the corresponding background shape $\mathcal{B}(p_j)$ in data, and background shape $\mathcal{B}(p_j)$ is derived using RooNDKeysPdf [10], which is a kernal estimation method [11] implemented in RooFit [10]. A superposition of Gaussian kernels (RooFit) models the distribution of an input dataset. After Adding the background PDF to the signal PDF incoherently, the combined PDF is written as

$$\begin{aligned} w_{\text{sig}} f_S(p_j) &+ (1-w_{\text{sig}}) f_{\mathcal{B}}(p_j) \\ &= w_{\text{sig}} \frac{\epsilon(p_j)\left|\mathcal{M}(p_j)\right|^2 R(p_j)}{\int \epsilon(p_j)\left|\mathcal{M}(p_j)\right|^2 R(p_j)\,dp_j} \\ &+ (1-w_{\text{sig}}) \frac{\mathcal{B}(p_j)R(p_j)}{\int \mathcal{B}(p_j)R(p_j)\,dp_j}, \end{aligned} \tag{7}$$

where w_{sig} is the purity of the signal. We factorize the $\epsilon(p_j)$ term out from the combined PDF by $\mathcal{B}_\epsilon(p_j) \equiv \mathcal{B}(p_j)/\epsilon(p_j)$. Its contribution enters into the normalization and the background PDF. As a consequence, the combined PDF are described by

$$\begin{aligned} w_{\text{sig}} f_S(p_j) &+ (1-w_{\text{sig}}) f_{\mathcal{B}}(p_j) \\ &= \epsilon(p_j)R(p_j)\left[\frac{w_{\text{sig}}\left|\mathcal{M}(p_j)\right|^2}{\int \epsilon(p_j)\left|\mathcal{M}(p_j)\right|^2 R(p_j)\,dp_j}\right. \\ &+ \left.\frac{(1-w_{\text{sig}})B_\epsilon(p_j)}{\int \epsilon(p_j)B_\epsilon(p_j)R(p_j)\,dp_j}\right]. \end{aligned} \tag{8}$$

The integration in the denominator of the background term can also be handled by the MC integration,

$$\int \epsilon(p_j)B_\epsilon(p_j)R(p_j)\,dp_j \approx \frac{1}{N_{\text{MC}}} \times \sum_k^{N_{\text{MC}}} \frac{B_\epsilon(p_j^k)}{\left|\mathcal{M}^{\text{gen}}(p_j^k)\right|^2}. \tag{9}$$

In the end, the log-likelihood function is described as

$$\ln \mathcal{L} = \sum_k^{N_D} \ln\left[w_{\text{sig}} f_S(p_j^k) + (1-w_{\text{sig}}) f_{\mathcal{B}}(p_j)\right], \tag{10}$$

where N_D is used for candidate events in data. For $D_s^+ \to K^-K^+\pi^+$ and $D^+ \to K_s^0 K^+\pi^0$, w_{sig} is equal to 1.

2.1. Blatt-Weisskopf barriers

For a decay process $A \to BC$, the Blatt-Weisskopf barriers [12] depend on the angular momenta $L = 0, 1, 2$ and the momentum q of the final-state particle B or C in the rest system of A. They are defined by

$$\begin{aligned} X_{L=0}(q) &= 1, \\ X_{L=1}(q) &= \sqrt{\frac{z_0^2+1}{z^2+1}}, \\ X_{L=2}(q) &= \sqrt{\frac{z_0^4+3z_0^2+9}{z^4+3z^2+9}}, \end{aligned} \tag{11}$$

with $z_0 = q_0 R$ and $z = qR$. The momentum q is given by

$$q = \sqrt{\frac{(s_A+s_B-s_C)^2}{4s_A} - s_B}, \tag{12}$$

where s_A, s_B and s_C refer to the squared invariant masses of particles A, B, and C, respectively. The value of q_0 is that of q when $s_A = m_A^2$. For the D_s^+ meson and intermediate resonances, the effective radii of barrier R are fixed to be 5.0 GeV^{-1} and 3.0 GeV^{-1}, respectively. Especially, the effective radii for the D^+ meson and intermediate resonances is fixed to be 5.0 GeV^{-1} and 1.5 GeV^{-1} for $D^+ \to K^+K_S^0\pi^0$.

2.2. Propagator

The intermediate resonances $\phi(1020)$, $K^*(892)$, $\bar{K}_1^0(1270)$, and $a_1(1260)^+$ are parameterized with the relativistic Breit-Wigner (RBW) formula,

$$P(m) = \frac{1}{(m_0^2-m^2) - im_0\Gamma(m)}, \tag{13}$$

$$\Gamma(m) = \Gamma_0 \left(\frac{q}{q_0}\right)^{2L+1} \left(\frac{m_0}{m}\right) \left(\frac{X_L(q)}{X_L(q_0)}\right)^2, \tag{14}$$

where m_0 and Γ_0 are the nominal masses and widths of the intermediate resonances, respectively. The value of q_0 in Eq. (14) is that of q when $s_a = m_0^2$.

The intermediate resonance $\rho(770)$ is modeled as the Gounaris-Sakurai (GS) formula [13],

$$P_{\rm GS}(m) = \frac{1 + d\frac{\Gamma_0}{m_0}}{(m_0^2 - m^2) + f(m) - im_0\Gamma(m)}, \qquad (15)$$

$$f(m) = \Gamma_0 \frac{m_0^2}{q_0^3}\Bigg(q^2[h(m) - h(m_0)] + (m_0^2 - m^2)q_0^2 \frac{dh}{d(m^2)}\Big|_{m^2=m_0^2}\Bigg), \qquad (16)$$

with

$$h(m) = \frac{2}{\pi}\frac{q}{m}\ln\left(\frac{m+2q}{2m_\pi}\right), \qquad (17)$$

and the function $dh/d(m^2)$ is defined as

$$\frac{dh}{d(m^2)}\Big|_{m^2=m_0^2} = h(m_0)[(8q_0^2)^{-1} - (2m_0^2)^{-1}] + (2\pi m_0^2)^{-1}, \qquad (18)$$

the parameter $d = f(0)/\Gamma_0 m_0$ is fixed by the normalization condition at $P_{\rm GS}(0)$. Hence, the parameter d is

$$d = \frac{3}{\pi}\frac{m_\pi^2}{q_0^2} ln\left(\frac{m_0 + 2q_0}{2m_\pi}\right) + \frac{m_0}{2\pi q_0} - \frac{m_\pi^2 m_0}{\pi q_0^3}. \qquad (19)$$

The intermediate resonance $a_0(980)$ is modeled as Flatté formula:

$$P_{a_0(980)} = \frac{1}{M^2 - s - i(g_{\eta\pi}\rho_{\eta\pi}(s) + g_{K\bar{K}}\rho_{K\bar{K}}(s))}, \quad (20)$$

where s is the $\pi\eta$ invariant mass squared, the Lorentz invariant PHSP factors $\rho_{\eta\pi}(s)$ and $\rho_{K\bar{K}}(s)$ are $2q/\sqrt{s_a}$, and the $g^2_{\eta\pi} = 0.341 \pm 0.004$ GeV^2/c^4 and $g^2_{K\bar{K}} = (0.892 \pm 0.022)g^2_{\eta\pi}$ are constants, reported in Ref [14].

2.3. Spin factors

As the limit of PHSP, we only consider the states with angular momenta up to two. For a process $A \to BC$, we define the spin projection operator according to the discussion in Ref. [15], $P^{(S)}_{\mu_1\cdots\mu_S\nu_1\cdots\nu_S}$ is

$$P^{(1)}_{\mu\nu}(A) = -g_{\mu\nu} + \frac{p_{A\mu}p_{A\nu}}{p_A^2}, \qquad (21)$$

$$P^{(2)}_{\mu_1\mu_2\nu_1\nu_2}(A) = \frac{1}{2}(P^{(1)}_{\mu_1\nu_1}(A)P^{(1)}_{\mu_2\nu_2}(A) + P^{(1)}_{\mu_1\nu_2}(A)P^{(1)}_{\mu_2\nu_1}(A)) - \frac{1}{3}P^{(1)}_{\mu_1\mu_2}(A)P^{(1)}_{\nu_1\nu_2}(A), \qquad (22)$$

where the quantities p_A, p_B, and p_C denote the momenta of particles A, B, and C, respectively, and $r_A = p_B - p_C$. The covariant tensors are constructed from the corresponding momenta p_A, p_B, and p_C,

$$\tilde{t}^{(1)}_\mu(A) = -P^{(1)}_{\mu\mu'}(A)r_A^{\mu'}, \qquad (23)$$

$$\tilde{t}^{(2)}_{\mu\nu}(A) = P^{(2)}_{\mu\nu\mu'\nu'}(A)r_A^{\mu'}r_A^{\nu'}. \qquad (24)$$

3. Amplitude analyses and branching fraction measurements

3.1. Amplitude analysis of $D_s^+ \to K^-K^+\pi^+$

Based on the data of 4399 signal candidates with 99.6% purity, the amplitude analysis of the decay $D_s^+ \to K^-K^+\pi^+$ has been performed by BESIII [12] at $\sqrt{s} = 4.178$ GeV. We perform a model-independent partial wave analysis to extract the S-wave line-shape in K^+K^- low-mass resonance, due to the large overlap of $a_0(980) \to K^+K^-$ and $f_0(980) \to K^+K^-$ and their common J^{PC}, it is difficult to distinguish between $a_0(980)$ and $f_0(980)$. They are considered as a combined state $S(980)$. According to the detection efficiency of the results in the amplitude analysis, we obtain $\mathcal{B}(D_s^+ \to K^-K^+\pi^+) = (5.47 \pm 0.08_{\rm stat.} \pm 0.13_{\rm syst.})\%$ with much more precision. The BFs of intermediate processes $D_s^+ \to \phi\pi^+$ and $D_s^+ \to K^+\bar{K}^*(892)^0$ are 4.60 ± 0.17 and 3.94 ± 0.12 which are consistent with corresponding theory predictions [2].

3.2. Amplitude analysis of $D_s^+ \to K_S^0\pi^+\pi^0$

Based on the data of 1385 signal candidates with around 94% purity, the amplitude analysis of the decay $D_s^+ \to K_S^0\pi^+\pi^0$ has been performed by BESIII for the first time [16]. According to the detection efficiency of the results in the amplitude analysis, we obtain $\mathcal{B}(D_s^+ \to K_S^0\pi^+\pi^0) = (5.43 \pm 0.30_{\rm stat.} \pm 0.15_{\rm syst.}) \times 10^{-3}$ which is improved by about a factor of 3 compared to the PDG value [1]. The BFs of intermediate processes $D_s^+ \to K^0\rho(770)^+$, $D_s^+ \to K^*(892)^0\pi^+$, and $D_s^+ \to K^*(892)^+\pi^0$ are valuable for understanding of quark flavor SU(3) symmetry, and other related theoretical issues. The A_{cp} for the channels $D_s^+ \to K_S^0\pi^+\pi^0$ and $D_s^- \to K_S^0\pi^-\pi^0$ is calculated to be $(2.7 \pm 5.5_{\rm stat.} \pm 0.9_{\rm syst.})\%$, and no evidence for CP violation was observed.

3.3. Amplitude analysis of $D_s^+ \to K_S^0K^-\pi^+\pi^+$

Based on the data of 609 signal candidates with about 85.4% purity, the amplitude analysis of the decay $D_s^+ \to K_S^0K^-\pi^+\pi^+$ has been performed by BESIII for the first time [17]. According to the detection efficiency of the results in the amplitude analysis, we obtain $\mathcal{B}(D_s^+ \to K_S^0K^-\pi^+\pi^+) = (1.46 \pm 0.05_{\rm stat.} \pm 0.05_{\rm syst.})\%$ compared with previous experiment with much more precision. The

BF of dominant process $D_s^+ \to K^*(892)^+\bar{K}(892)^0$ is calculated to be $(5.93 \pm 0.47_{\rm stat.} \pm 0.74_{\rm syst.})\%$.

3.4. Amplitude analysis of $D_s^+ \to K^-K^+\pi^+\pi^0$

Based on the data of 3088 signal candidates with 97.5% purity, the amplitude analysis of the decay $D_s^+ \to K^-K^+\pi^+\pi^0$ has been performed by BESIII for the first time [18]. The BFs of dominant processes $D_s^+ \to \phi\rho(770)^+$ and $D_s^+ \to \bar{K}^{*0}K^{*+}$ are observed to be $(2.75 \pm 0.07_{\rm stat.} \pm 0.15_{\rm syst.})\%$ and $(1.25 \pm 0.05_{\rm stat.} \pm 0.06_{\rm syst.})\%$, respectively, with much better precision. Besides, the three body resonances $K_1(1270)^0$ and $K_1(1400)^0$ were found to contribute in the decay amplitude. According to the detection efficiency of the results in the amplitude analysis, we obtain $\mathcal{B}(D_s^+ \to K^+K^-\pi^+\pi^0) = (5.42 \pm 0.10_{\rm stat.} \pm 0.17_{\rm syst.})\%$ compared with previous experiment with much more precision. The ratio $R_{K_1(1270)^0} = \mathcal{B}_{K_1(1270)^0 \to K^*(892)\pi}/\mathcal{B}_{K_1(1270)^0 \to K\rho(770)}$ is determined to be $0.99 \pm 0.15_{\rm stat.} \pm 0.18_{\rm syst.}$, which is agree with other experiments [19, 20].

3.5. Amplitude analysis of $D_s^+ \to \eta\pi^+\pi^+\pi^-$

Based on the data of 1306 signal candidates with larger than 85% purity, the amplitude analysis of the decay $D_s^+ \to \eta\pi^+\pi^+\pi^-$ has been performed by BESIII for the first time [21]. The BF of dominant process $D_s^+ \to \eta a_1(1260)^+, a_1(1260)^+ \to \rho(770)^0\pi^+$ is obtained to be $(1.73 \pm 0.14_{\rm stat.} \pm 0.08_{\rm syst.})\%$. According to the detection efficiency of the results in the amplitude analysis, we obtain $\mathcal{B}(D_s^+ \to \eta\pi^+\pi^+\pi^-) = (3.12 \pm 0.13_{\rm stat.} \pm 0.09_{\rm syst.})\%$ compared with previous experiment with much more precision. Besides, we observe the W-annihilation (WA) process $D_s^+ \to a_0(980)^+\rho(770)^0, a_0(980)^+ \to \eta\pi^+$, whose absolute BF is determined to be $(0.21 \pm 0.08_{\rm stat.} \pm 0.05_{\rm syst.})\%$, which is significantly larger than the BFs of other pure WA decays $D_s^+ \to \rho(770)^0\pi^+$ and $D_s^+ \to \pi^0\pi^+$. The measurement provides a good opportunity to distinguish various WA mechanisms and understand underlying nature of the resonance $a_0(980)^+$.

3.6. Amplitude analysis of $D^+ \to K^+K_S^0\pi^0$

Based on the data of 692 signal candidates With 97.4% purity, the amplitude analysis of the singly Cabibbo suppressed decay $D^+ \to K^+K_S^0\pi^0$ has been performed by BESIII for the first time [22]. The BF of dominant process $D^+ \to K^*(892)^+K_S^0$ is obtained to be $(8.69 \pm 0.40_{\rm stat.} \pm 0.64_{\rm syst.} \pm 0.51_{\rm Br.}) \times 10^{-3}$, the result is in agreement with the previous results [1, 23] with much more precision. And the BF of intermediate process $D^+ \to \bar{K}^*(892)^0K^+$ is obtained to be $(3.10 \pm 0.46_{\rm stat.} \pm 0.68_{\rm syst.} \pm 0.18_{\rm Br.)}) \times 10^{-3}$, the isospin symmetric process result is agree with the previous result [1] and theoretical predictions [4, 24].

4. Summary and Outlook

Based on e^+e^- annihilation data corresponding to a total integrated luminosity of 6.32 fb^{-1} and 2.93 fb^{-1} collected at the $\sqrt{s} = 4.178 - 4.226$ GeV and $\sqrt{s} = 3.773$ GeV with the BESIII detector, we report amplitude analyses and the BF measurements for $D_{(s)}^+$ decay modes. Many structures were observed in these decays and the results are the most precise up to date. According to these results, we can check CP violation, deeply understand the weak annihilation process mechanisms, and test SU(3)$_F$ symmetry. For the near future in hadronic charm meson decays, BESIII will produce more results at $\sqrt{s} = 3.773$ GeV corresponding to a total integrated luminosity of 17 fb^{-1} in the future [7].

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