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Internal structure of pion and kaon: an algebraic model and its implications

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We describe an algebraic approach to address aspects of the structure of pseudo-scalar mesons through their corresponding generalized parton distributions, from which parton distribution functions and electromagnetic form factors can be derived. A direct relation between such distributions and the so-called parton distribution amplitudes can be established since the generalized parton distributions are obtained from the overlap of light-front wave functions.

Keywords: QCD; Schwinger-Dyson equations; Bethe-Salpeter amplitude; algebraic model; hadronic physics; GPDs.

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1. Introduction

One of the ways to address several aspects of the internal structure of hadron is by extracting the corresponding generalized parton distrubutions (GPDs) from hard scattering processes [1-5]. GPDs provide a three-dimensional picture of the hadron through their explicit dependence on the momentum fraction x [6] that carries a quark within the hadron, the longitudinal momentum fraction transfer (ξ) and the momentum transferred to the hadron (t). Thus, a more complete picture of the mass and momentum distribution within the hadron can be obtained. Additionally, GPDs are directly related to the electromagnetic form factors (EFFs) [7] which are extracted from elastic scattering processes and the parton distribution functions (PDFs) [3] which are connected to the deep inelastic scattering processes.

In principle, GPDs can be written in terms of propagators and amplitudes, obtained within the Schwinger-Dyson equations (SDEs) formalism [8-10]. However, a fully numerical approach becomes a strenuous task; so, for practical purposes, models that mimic the numerical solutions of these equations can be satisfactorily employed instead. In this work, we present an algebraic model that enables us to develop several functions of interest in an analytical manner while preserving close connection with quantum chromodynamics (QCD). The starting point is the Bethe-Salpeter wave function (BSWF) expressed by means of the *Anstäze* written in terms of this novel algebraic model (AM). Subsequently, projecting onto the light cone, the light-front wave function (LFWF) is readily obtained [11]. Its integration on the transverse momentum squared ($k_\perp$) allows us to access the parton distribution amplitude (PDA) [12]. The GPDs are then derived from the overlap representation [9,10,13]. Our work is primarily focused on pions and kaons which would be of central focus in the Electron-Ion Collider. It is augmented by the fact that we have fairly reliable information of their PDA, [9,14,15]. Our AM is naturally suited for its extension to other pseudoscalar mesons such as η_c and η_b, and then hopefully to η and η'.

2. Algebraic Model

The function that describes the bound states of a quark and an antiquark in QCD is the Bethe-Salpeter (BS) wave function (WF), which is defined as the BS amplitude ($\Gamma_{\rm M}$) sandwiched between the corresponding quark and antiquark propagators ($S_{q,(\bar{h})}$) as follows

$$\chi_{\rm M}\left(k_-, P\right) = S_q(k)\Gamma_{\rm M}\left(k_-, P\right) S_{\bar{h}}\left(k-P\right) , \qquad (1)$$

where $k_- = k - P/2$, $P^2 = -m_{\rm M}^2$ is the mass of a meson $\rm M$ $= |q\,\bar{q}\rangle$ and the valence quark and antiquark flavors are denoted by the labels q and $\bar{h}$.

The quark propagator and the BS amplitude (BSA) can be obtained from solutions of the corresponding SDEs. However, one can construct simpler yet effective models which capture the important non-perturbative essence of QCD and at the same time allow for a less cumbersome description of the internal structure of the mesons of our interest. However, instead of coining an algebraic model for each individual meson, we propose one which has the potential to provide a unified description of all pseudoscalar mesons:

$$S_{q(\bar{h})}(k) = \left[-i\gamma\cdot k + M_{q(\bar{h})}\right]\Delta\left(k^2, M^2_{q(\bar{h})}\right), \quad (2)$$

$$n_{\rm M}\Gamma_{\rm M}(k,P) = i\gamma_5\int_{-1}^{1} dw\,\rho_{\rm M}(w)\left[\hat{\Delta}\left(k_w^2,\Lambda_w^2\right)\right]^{\nu}, \quad (3)$$

$\Delta(s,t) = (s+t)^{-1}$, $\hat{\Delta}(s,t) = t\Delta(s,t), k_w = k + (w/2)P$.

Here, $M_{q(\bar{h})}$ corresponds to the constituent mass of the quark (antiquark) $q(\bar{h})$, $n_{\rm M}$ is a normalization constant, and $\rho_{\rm M}(w)$ is the spectral density, whose shape determines the pointwise behavior of the BSA. Moreover, the parameter $\nu > -1$ controls the asymptotic behavior of the BSA. And finally, $\Lambda_w^2 \equiv \Lambda^2(w)$ is defined as follows:

$$\Lambda^2(w) = M_q^2 - \frac{1}{4}\left(1-w^2\right)m_{\rm M}^2 + \frac{1}{2}(1-w)\left(M_{\bar{h}}^2 - M_q^2\right). \quad (4)$$

The explicit dependence on w on one hand helps us simplify the algebra and obtain closed expressions between different key functions, and on the other, introduce a convenient generality. Using Eqs. (1)-(3), the BSWF becomes

$$n_{\rm M}\chi_{\rm M}(k_-,P)=\mathcal{M}_{q,\bar{h}}(k,P)\int_{-1}^{1} dw\,\tilde{\rho}_{\rm M}^{\nu}(w)\mathcal{D}^{\nu}_{q,\bar{h}}(k,P), \quad (5)$$

where $\mathcal{M}_{q,\bar{h}}(k=p+P,P)$ is the tensor structure that characterizes $\chi_M(k,P)$ and the function $\mathcal{D}^{\nu}_{q,\bar{q}}(k,P)$ is a product of quadratic denominators. Without loss of generality, $\tilde{\rho}_{\rm M}^{\nu}(w)$ has been defined in terms of the spectral density as

$$\tilde{\rho}_{\rm M}^{\nu}(w) \equiv \rho_{\rm M}(w)\Lambda_w^{2\nu}. \quad (6)$$

Introducing Feynman parameterization in Eq. (5), it is possible to combine the denominators of $\mathcal{D}^{\nu}_{q,\bar{q}}(k,P)$. Thanks to an ingenious change of variables and reorganization of the order of integration, it is possible to integrate $k_\perp$ as well as one of the Feynman parameters, resulting in the following form of the BSWF:

$$n_{\rm M}\chi_{\rm M}(k_-,P) = \mathcal{M}_{q,\bar{h}}(k,P)\int 01 d\alpha\mathcal{F}_{\rm M}(\alpha,\sigma^{\nu+2}), \quad (7)$$

$$\mathcal{F}_{\rm M}(\alpha,\sigma^{\nu+2}) = 2^{\nu}(\nu+1)\left[\int_{-1}^{1-2\alpha} dw\left(\frac{\alpha}{1-w}\right)^{\nu} + \int_{1-2\alpha}^{1} dw\left(\frac{1-\alpha}{1+w}\right)^{\nu}\right]\frac{\tilde{\rho}_{\rm M}^{\nu}(w)}{\sigma^{\nu+2}}, \quad (8)$$

where $\sigma = (k-\alpha P)^2 + \Lambda^2_{1-2\alpha}$, and α is a Feynman parameter. This expression allows us to obtain key functions directly and analytically as we will shortly see.

3. Light front wave functions

The leading-twist light-front wave function for a pseudoscalar meson is obtained by projecting the BSWF onto the light front:

$$\psi^q_{\rm M}\left(x,k_\perp^2\right) = \text{tr}\int_{dk_\parallel}\delta_n^x(k_{\rm M})\gamma_5\gamma\cdot n\,\chi_{\rm M}(k_-,P), \quad (9)$$

where $\delta_n^x(k_{\rm M}) = \delta(n\cdot k - x\,n\cdot P)$, n is the light-like four-vector, so it must satisfy that $n^2 = 0$ and $n\cdot P = -m_{\rm M}$. As usual, x corresponds to the momentum fraction of the quark. The trace is taken on the color and the Dirac indices. We employ the notation $\int_{dk_\parallel} \equiv \int d^2k_\parallel/\pi$.

On the other hand, the distribution of Mellin moments for the LFWF allows us to write:

$$\langle x^m\rangle_{\psi^q_{\rm M}} = \int_0^1 dx\,x^m\,\psi^q_{\rm M}\left(x,k_\perp^2\right). \quad (10)$$

Then, from the Eqs. (7)-(10), one readily arrives at

$$\langle x^m\rangle^q_{\psi_{\rm M}} = \text{tr}\,\frac{1}{n\cdot P}\int_{dk_\parallel}\left[\frac{n\cdot k}{n\cdot P}\right]^m\gamma_5\gamma\cdot n\chi_{\rm M}(k_-,P) = \int_0^1 d\alpha\alpha^m\left[\frac{12}{n_{\rm M}}\frac{\mathcal{Y}_{\rm M}(\alpha,\sigma_\perp^{\nu+1})}{\nu+1}\right], \quad (11)$$

where

$$\mathcal{Y}_{\rm M}(\alpha,\sigma_\perp^{\nu+1}) = \mathcal{F}_{\rm M}(\alpha,\sigma_\perp^{\nu+1})(\alpha M_{\bar{h}} + (1-\alpha)M_q), \quad (12)$$

with $\sigma_\perp = k_\perp^2 + \Lambda^2_{1-2\alpha}$. Comparing Eqs. (10)-(11) we can observe the peculiarity that by identifying the Feynman parameter α with the momentum fraction x, and as a consequence of the uniqueness of the Mellin moments, the LFWF reads:

$$\psi^q_{\rm M}(x,k_\perp^2) = \left[\frac{12}{n_{\rm M}}\frac{\mathcal{Y}_{\rm M}(x,\sigma_\perp^{\nu+1})}{\nu+1}\right]. \quad (13)$$

The form of this equation will help us in the following algebraic manipulations. However, what will allow further development and obtaining direct relationship between all distribution functions with the PDAs is the ingenuous dependence of Λ_w on w, as we explain in next sections.

4. Parton distribution amplitudes

The immediate connection between the LFWF and PDA is revealed by integration of the $k_\perp$ degrees of freedom of $\psi_{\rm M}^q(x,k_\perp)$, namely

$$f_{\rm M}\phi_{\rm M}^q(x) = \frac{1}{16\pi^3}\int d^2k_\perp \psi_{\rm M}^q\left(x,k_\perp^2\right) \ , \qquad (14)$$

where $f_{\rm M}$ is the leptonic decay constant corresponding to the meson M. To calculate the integral over $k_\perp$ we observe that in the Eqs. (8) and (13), only $1/\sigma_\perp^{\nu+1}$ depends on $k_\perp$.

Integrating and rearranging terms, the following direct relationship between the LFWF and the PDA is obtained:

$$\psi_{\rm M}^q(x,k_\perp^2) = 16\pi^2 f_{\rm M}\frac{\nu\Lambda_{1-2x}^{2\nu}}{(k_\perp^2+\Lambda_{1-2x}^2)^{\nu+1}}\phi_{\rm M}^q(x) \ , \qquad (15)$$

where, it must be true that $\int_0^1 dx\phi_{\rm M}^q(x) = 1$. The above expression is a special merit of the AM that we have put forward. It expresses the fact that given a PDA it is trivial to obtain its corresponding LFWF, without the need to find or propose a spectral density $\tilde{\rho}^{\,\nu}(w)$.

The LFWF and the PDA depend on an intrinsic mass scale ζ, where

$$\phi_{\rm M}^q(x;\zeta) = \phi_{\rm M}^{\bar{h}}(1-x;\zeta) \ . \qquad (16)$$

In the next section, we will be able to exploit this result, Eq. (15), to calculate the GPDs by means of the overlap representation of the LFWF.

5. Generalized parton distributions

The valence quark GPD at the hadronic scale ζ_H, can be calculated through the overlap representation of the LFWF as:

$$H_{\rm M}^q(x,\xi,t) = \int\frac{d^2k_\perp}{16\pi^3}\psi_{\rm M}^{q*}\left(x^-,(\mathbf{k}_\perp^-)^2\right)\psi_{\rm M}^q\left(x^+,(\mathbf{k}_\perp^+)^2\right) \ ,$$
$$x^\pm = \frac{x\pm\xi}{1\pm\xi}, \mathbf{k}_\perp^\pm = k_\perp \mp \frac{\Delta}{2}\frac{1-x}{1\pm\xi} \ , \qquad (17)$$

where p denotes the initial momentum of the meson, while p' the final momentum. Note that $P=(p+p')/2$. The quantity $-t=\Delta^2=(pp')^2$ is the momentum transferred to the meson and $\xi=[-n\cdot\Delta]/[2n\cdot P]$ is the longitudinal momentum fraction transfer also known as the skewness variable. This overlap representation is only valid in the DGLAP region, $|x|>\xi$. Moreover, this function encodes all the properties of the meson on a ζ_H scale with the quark and antiquark fully dressed. Expression (17) can be explicitly evaluated by making use of Eq. (15):

$$H_{\rm M}^q(x,\xi,t) = (16\pi^2 f_{\rm M}\nu)^2\phi_{\rm M}^q(x^+)\phi_{\rm M}^q(x^-)\Lambda_{1-2x^+}^{2\nu}\Lambda_{1-2x^-}^{2\nu}$$
$$\times\int\frac{d^2k_\perp}{16\pi^3}\frac{1}{[((\mathbf{k}_\perp^-)^2+\Lambda_{1-2x^-}^2)((\mathbf{k}_\perp^+)^2+\Lambda_{1-2x^+}^2)]^{\nu+1}} \ . \qquad (18)$$

Once more, to integrate $k_\perp$ we introduce Feynman parameterization and perform a change of variable. Once the integration is done, the GPD can be rewritten as:

$$H_{\rm M}^q(x,\xi,t) = \mathcal{N}\phi_{\rm M}^q(x^+)\phi_{\rm M}^q(x^-)\Lambda_{1-2x^+}^{2\nu}\Lambda_{1-2x^-}^{2\nu}$$
$$\times\frac{\Gamma(2\nu+2)}{\Gamma^2(\nu+1)}\int_0^1 du\frac{u^\nu(1-u)^\nu}{[\mathbb{M}^2(u)]^{2\nu+1}} \ , \qquad (19)$$

where $\mathbb{M}^2(u) = c_2u^2+c_1u+c_0$, with

$$c_2 = \frac{(1-x)^2}{(1-\xi^2)^2}t,$$
$$c_1 = -\frac{(1-x)^2}{(1-\xi^2)^2}t+\Lambda_{1-2x^+}^2-\Lambda_{1-2x^-}^2,$$
$$c_0 = \Lambda_{1-2x^-}^2 \ . \qquad (20)$$

The integration over du in Eq. (19) is doable for values of $\nu>-1$. Note that if we take $\xi=0$, we can obtain an algebraic solution for the GPD by doing an expansion around $-t\approx 0$, which yields

$$H_{\rm M}^q(x,0,t) \overset{t\to 0}{\approx} \mathcal{N}\frac{\phi_{\rm M}^{q2}}{\Lambda_{1-2x}^2}\left[1-c_\nu^{(1)}(1-x)^2\left(\frac{-t}{\Lambda_{1-2x}^2}\right)+...\right],$$
$$c_\nu^{(1)} = \frac{(1+\nu)(1+2\nu)}{2(3+2\nu)} \ . \qquad (21)$$

These results will be useful for the following sections.

6. Parton distribution functions

The PDF of the valence quark is defined as the *forward limit* of the GPD ($t=0$, $\xi=0$). It is thus the first term of the expansion in Eq. (21)

$$q_{\rm M}(x) \equiv H_{\rm M}^q(x,0,0) = \mathcal{N}\frac{\phi_{\rm M}^{q2}(x)}{\Lambda_{1-2x}^2} \ , \qquad (22)$$

where $\mathcal{N} = \left(\int_0^1 dx\,[\phi_{\rm M}^2(x)/\Lambda_{1-2x}^2]\right)^{-1}$ is the normalization constant of the PDF. It should be noted that in the chiral limit on the LFWF, Eq. (15), the dependence on $k_\perp$ and x is factorized. It leads to the following expression for the PDF

$$q_{\rm M}(x;\zeta_H) = \frac{\phi_{\rm M}^{q2}(x;\zeta_H)}{\int_0^1 dx\ \phi_{\rm M}^{q2}(x;\zeta_H)}. \qquad (23)$$

In this way, we observe that the degree of factorizability of the AM depends entirely on Λ^2_{1-2x}. Moreover, for the case where there is isospin symmetry, Λ^2_{1-2x} becomes only a constant and we obtain a factorized LFWF that will provide sensible results. However, we focus on the general case.

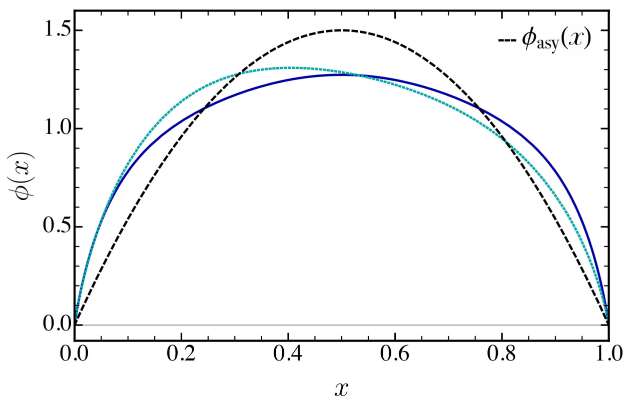

FIGURE 1. Parton distribution functions corresponding to pion and kaon. The solid (blue) curve is the PDA of the pion and the dotted (cyan) line is the PDA of the kaon [12]. The dashed (black) line is the PDA in the asymptotic limit.

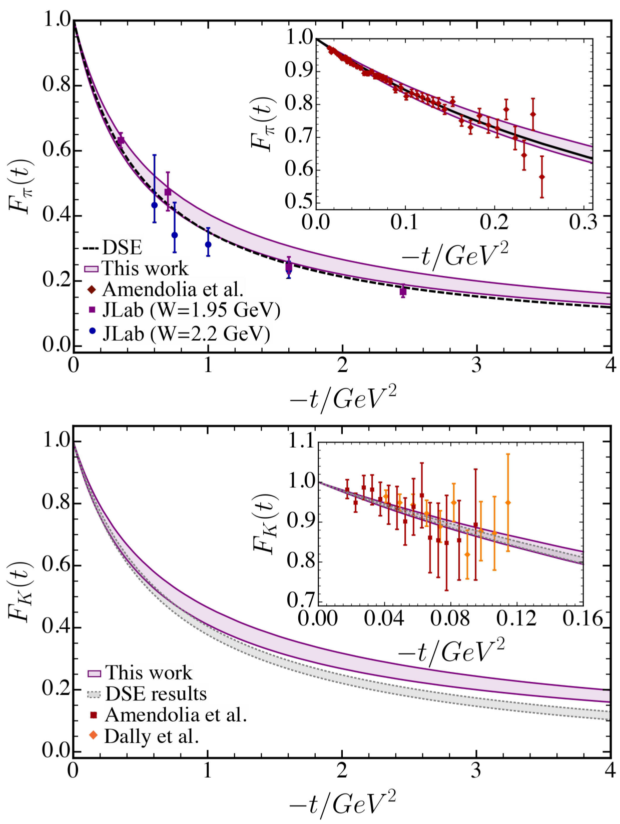

FIGURE 2. Pion and kaon elastic form factors obtained from Eqs. (19), (24) and (26). In both cases the purple band between solid lines represents to our results of the AM, wich corresponds to the pion and kaon respectively, where M_u = 316.8 MeV, M_s/M_u = 1.813 and ν=1 with a 5% of variation around the charge radius (r_π = 0.659 fm and r_K= 0.6 fm). Furthermore, in the upper panel circles, squars [16] and diamonds are experimental data results and the dashed (black) line represents SDE results [17], while in the lower panel diamonds [18] and squars [19] are experimental data results and the gray band correspond to SDE results [20].

7. Electromagnetic form factor

The EFF corresponding to the quark q of the meson, is related to the GPD as follows:

$$F^q_{\rm M}(t) = \int_{-1}^{1} dx\ H^q_{\rm M}(x,\xi,t). \tag{24}$$

Thus, for the complete meson, the form factor would be

$$F_{\rm M}(t) = e_q F^q_{\rm M}(t) + e_{\bar{h}} F^{\bar{h}}_{\rm M}(t)\ , \tag{25}$$

where $e_{q,\bar{h}}$ are the electric charges of the constituent valence quarks that are in units of positron charge. On the other hand, the GPD has polynomial properties that make the EFF independent of ξ. We can thus take $\xi \to 0$.

$$F^q_{\rm M}(t) = \int_{0}^{1} dx\ H^q_{\rm M}(x,0,t)\ . \tag{26}$$

Furthermore, in the symmetric limit of isospin, $F_{\rm M}(t) = F^q_{\rm M}(t)$. Using a Taylor expansion of the EFF, the charge radius ($r^2_{\rm M}$) can be expressed as

$$r^2_{\rm M} = -6\frac{dF(t)}{dt}\bigg|_{t=0}\ . \tag{27}$$

From this expression, and considering Eqs. (21), (25) and (26), we can get the following expression for the charge radius:

$$r^2_{\rm M}{=}6c^{(1)}_\nu \int_0^1 dx\ \left[e_q\frac{q_{\rm M}(x)}{\Lambda^2_{1-2x}}{+}e_{\bar{h}}\frac{q_{\rm M}(1-x)}{\Lambda^2_{2x-1}}\right](1{-}x)^2\ . \tag{28}$$

As we have mentioned earlier, in the symmetric limit of isospin Λ^2_{1-2x} becomes a constant. The charge radius would then be directly related to the PDF and the PDA.

8. Conclusions

We have developed an SDE-inspired algebraic model that ingenuously allows us to obtain expressions relating the pseudoscalar mesons LFWFs with their corresponding PDAs. We subsequently derive GPDs, PDFs and EFFs in a completely analytical manner. We use parameterized expressions of the

PDAs corresponding to the pion and the kaon which are shown in Fig. 1 to construct GDPs and then extract EFF displayed in Fig. 2. We compare our results with the experimental data as well as with results obtained by means of the SDE [16-20]. One can immediately observe excellent agreement. Within the uncertainty band, our results coincide with almost all the experimental data as well as with the results obtained from the full SDE treatment for values of t less than 2 GeV. For higher values of t, it only differs minimally. However, it has the same behavior, while for the kaon all the experimental data coincide with our band. The results from the direct SDE again coincide with ours for small values of t. For values greater than 1 GeV, it follows the same qualitative behavior although it differs slightly as compared to the results enclosed within our band. Although we have focused entirely on pion and kaon, our model allows, with prior knowledge of PDA, to systematically derive GPDs and other functions of interest for all pseudoscalar mesons.

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