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Transverse momentum dependent parton distribution functions and QCD evolution equations

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We provide an overview over Transverse Momentum Dependent (TMD) Parton Distribution Functions (PDFs). While we will also comment on TMD PDFs in general, we will focus on their use for the description of hadronic reactions in the so-called low x limit. Here $x = M^2/s$ and M is the hard scale of the process, while $\sqrt{s}$ is the center of mass energy of the reaction. We will explain why this are interesting quantities whose exploration serves a manifold purpose. In particular we will explain why these are interesting quantities both for the accurate description of LHC data and why exploration of such quantities is a central goal of the future Electron Ion Collider. In a second part of this talk we will then discuss how perturbative QCD allows us to formulate and solve differential equations, which describe the dependence of this TMD PDFs on various kinematic variables.

Keywords: Quantum chromodynamics; high energy factorization; transverse momentum dependent factorization.

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1. Introduction

Understanding the physics of dense systems provides a plethora of challenges to physicists. This is mainly due since their description intrinsically requires to deal with many-particle states and is obviously further complicated if the interaction between particles is not weak. A scenario where this is realized are dense systems subject to the strong nuclear force with a prominent collider physics example provided by heavy ion collisions. The latter serve to a large extend for the study of the Quark Gluon Plasma (QGP), a state of matter which is believed to have existed the last time in the early universe. While some of the QGP properties are well understood, the initial state leading to its formation poses still open questions [1, 2]. A closely related question is the formation of an over occupied system of gluons, which eventually leads to saturation of gluon densities [3]. Such high gluon densities have been argued to lead to the emergence of semi-hard scale, the so-called saturation scale Q_s [4, 5], which can be interpreted as the inverse correlation length of color charges in the dense medium. This saturation scale has an interesting dependence on the center of mass energy of the collision, or to be more precise on the variable x which denotes the ratio $x = M^2/s$ where M is a certain characteristic (hard) scale of the process and $\sqrt{s}$ the center of mass energy. Since particles fluctuations are time dilated at high center of mass energies or low x, such fluctuations can generate themselves new particles and they therefore drive the system into a high density regime. This is particularly true for the case of a non-Abelian gauge theory – such as Quantum Chromodynamics (QCD) – where gluons can emit further gluons. One therefore finds a correlation length which is shrinking with center of mass energy and correspondingly a saturation scale $Q_s(x)$, which is growing with center of mass energy. If densities are high enough, this saturation scale can eventually reach values of a few GeV and $Q_s(x) \gg \Lambda_{\rm QCD}$ with $\Lambda_{\rm QCD}$ the QCD characteristic scale of the order of a few hundred MeV and the problem might be possibly studied using weak coupling techniques, albeit in the presence of high gluon densities. Understanding the emergence of this saturation scale and characterizing its growth with energy as well as its relevance for the determination of the value of the QCD strong coupling constant α_s is one of the central physics goals of the future Electron Ion Collider [6]. The latter constitutes a new collider project, which is foreseen to be constructed within the next 10 years at Brookhaven National Laboratory (USA) as a successor to the Relativistic Heavy Ion Collider (RHIC). In such electron ion collisions, the high energy ion provides an ensemble of dense color charges, which – in the electron rest frame – are further concentrated due to Lorentz contraction. The electron provides on the other hand – through the emission of photons which then interact with the colored states of matter – a dilute and point-like probe. The point-like nature is here of particular interest, since the electron possesses no substructure; it therefore does not break up in the collision with the ion. Instead it can be observed in the final state and allows for a precise reconstruction of the kinematics of a certain event.

A process of particular interest for the exploration of these high density effects is given by the almost back-to-back production of two hadrons, *e.g.* pions, where both pions carry large transverse momentum with respect to the collision axis. At high center of mass energies, an intuitive understanding can be gained through the process depicted in Fig. 1. This process is characterized by the presence of a certain hierarchy of scales. The transverse momenta $\mathbf{p}_i, i = q, \bar{q}$ of the produced hadrons constitute in this case the hard scale (which allows to make use of QCD weak coupling techniques), which is much bigger than the momentum imbalance $\mathbf{q}$ of both

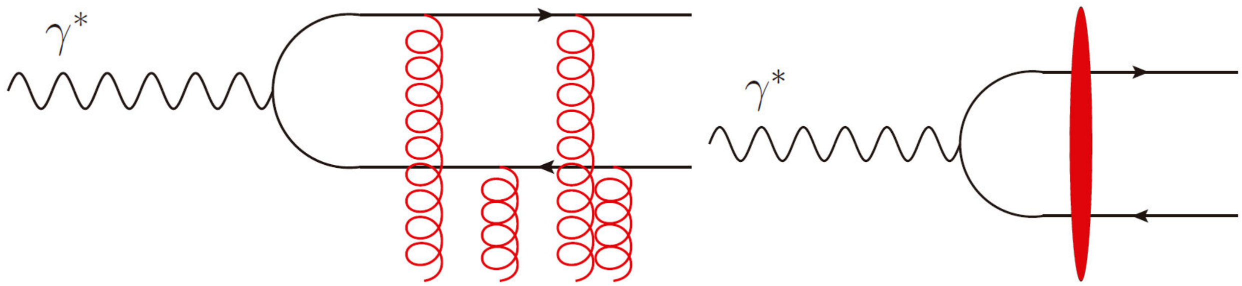

FIGURE 1. A virtual photon, emitted from an electron, interacts with dense gluonic field of the target, *i.e.* the ion. While in terms of Feynman diagrams, this gluonic field is correctly thought of as multiple gluon exchange, the actual interaction rather takes place with classical field, as appropriate for high occupation numbers. To probe the saturation scale generated by the dense gluonic field, one considers the scenario when both hadrons (here quarks) part away in opposite (transverse) directions. The sum of both transverse momenta $\mathbf{p}_i$, $i = q, \bar{q}$ therefore adds up to almost zero, with a small imbalance $\mathbf{q} = \mathbf{p}_q + \mathbf{p}_{\bar{q}}$. This momentum q is then sensitive to potential presence of a dynamically generated saturation scale; in particular the resulting momentum distribution is expected to peak at this scale Q_s.

hadrons, $\mathbf{q} = \mathbf{p}_q + \mathbf{p}_{\bar{q}}$, $|\mathbf{p}_i| \gg |\mathbf{q}| \sim \mathbf{Q_s}$, see also the discussion of Fig. 1. As a consequence one expects important corrections to this observables due to evolution from an initial renormalization scale μ_i (of the order of $|\mathbf{q}|$) up to μ_f (of the order of the hadron transverse momentum). These effects have been studied first in Ref. [7], based on a framework developed in Ref. [8]. While the authors of [7] conclude that a study of the saturation scale at the EIC will be possible, using the observable of di-hadron de-correlation, they also find that the above mentioned evolution – encoded in a so-called Sudakov form factor – provide an important correction to the precise description and analysis of this observable. While [7] only include a resummation of double-logarithms at fixed running coupling, more refined frameworks, have been presented in Refs. [9, 10]. They are all based in the discussion of so-called Transverse Momentum Dependent (TMD) Parton Distribution Functions (PDFs); in particular the TMD gluon distribution. The latter is then studied in the presence of a dense gluonic field, where coupling to the field is treated within the framework of high energy factorization. While the framework of [9, 10] is able to include important running coupling corrections, their anomalous dimension – which describes the evolution of the TMD PDF operator – misses an important contribution related to the resummation of single logarithms. Such a contribution has been found in a study based on collinear factorization [11]. Since this resummation relies on the renormalization of ultra-violet divergences of the previously mentioned TMD PDF operator, the corresponding anomalous dimension must be universal and cannot depend on the use of collinear vs. high energy factorization. In Ref. [12] this calculation has been repeated but using a different framework for the calculation of high energy factorized matrix elements at next-to-leading order in the strong coupling constant, *i.e.* Lipatov's high energy effective action has employed for that purpose, [13, 14], see also [15] for a recent review. Unlike [10], our calculation does not address the complete dense limit, but is instead restricted to the dilute approximation where the gluon field is not dense. Nevertheless we find that within this framework it is possible to obtain the correct anomalous dimension also within high energy factorization and therefore the consistency of this framework. In the following we will summarize a few aspects of this work. For full details we refer the interested reader to [12].

2. The gluon TMD within high energy factorization

To be specific we consider scattering of two hadrons with light-like momenta p_A and p_B which serve to define the light-cone directions

$$(n^\pm)^\mu = \frac{2}{\sqrt{s}} p^\mu_{A,B}, \qquad s = 2 p_A \cdot p_B, \tag{1}$$

which yields the following Sudakov decomposition of a generic four-momentum,

$$\begin{aligned} k &= k^+ \frac{n^-}{2} + k^- \frac{n^+}{2} + k_T, \\ k^\pm &= k \cdot n^\pm, \qquad n^\pm \cdot k_T = 0. \end{aligned} \tag{2}$$

Here, k_T is the embedding of the Euclidean vector $\mathbf{k}$ into Minkowski space, so $k_T^2 = -\mathbf{k}^2$. Following [11, 16] the TMD PDFΓ^{ij}, $i, j = 1, 2$ is then given by

$$\begin{aligned} \Gamma^{ij}_{g/B}(x_B, \zeta_B; \mathbf{q}, \mu) &= -\frac{\delta^{ij}}{2} f_{g/B}(x_B, \zeta_B; \mathbf{q}_b, \mu) \\ &+ \left(\frac{\delta^{ij}}{2} + \frac{\mathbf{q}^i \mathbf{q}^j}{\mathbf{q}^2} \right) h_{g/B}(x_B, \zeta_B; \mathbf{q}_b, \mu). \end{aligned} \tag{3}$$

In terms of QCD fields [11, 17]:

$$x\Gamma^{ij}_{g/B}(x_B,\zeta_B;\mathbf{q},\mu) = \lim_{\sigma,y_n\to\infty}\int\frac{d\xi^+d^2\xi}{2(2\pi)^3p_B^-}e^{i(x_Bp_B^-\xi^+/2-\mathbf{q}\cdot\xi)}\,\tilde{\mathcal{S}}(2y_c,\sigma;\mu,\xi)$$
$$\times\left\langle h(p_B)\left|\mathrm{Tr}\left[\left(\mathcal{W}^{n(\sigma)}_\xi G^{-i}(\xi)\right)^\dagger\,\mathcal{W}^{n(\sigma)}_0 G^{-j}(0)\right]\right|h(p_B)\right\rangle\Big|_{\xi^-=0}. \quad (4)$$

Here $\tilde{\mathcal{S}}(2y_c,\sigma;\mu,\xi)$ denotes the soft factor and $n(\sigma)\simeq n^-$, with $\sigma\to\infty$ a suitable regulator whose precise implementation can be found in Ref. [12]. Gauge links are in general given as a combination of a longitudinal and a transverse gauge link [18], where the transverse gauge link is placed at light-cone infinity. Working in covariant gauge, the gauge field at infinity vanishes and the transverse gauge link therefore equals one. We will therefore in the following not consider the transverse gauge link. The longitudinal gauge link is on the other hand given by

$$\mathcal{W}^n_\xi = \mathrm{P}\exp\left(-\frac{g}{2}\int\limits_{-\infty}^{0}d\lambda n\cdot v(\lambda n+\xi)\right), \quad (5)$$

where $v_\mu(x) = -it^a v^a_\mu(x)$ denotes the gluonic field and

$$D_\mu = \partial_\mu + g v_\mu, \qquad G^{\mu\nu} = \frac{1}{g}\left[D^\mu, D^\nu\right] = -it^a G^{\mu\nu}_a. \quad (6)$$

For the soft factor there exists various prescriptions in the literature; in Ref. [12] the most general soft factor introduced in Ref. [18] has been used,

$$\tilde{\mathcal{S}}(2y_c,\sigma;\mu,\xi) = \sqrt{\frac{\tilde{S}(2y_c,2y_n;\xi)}{\tilde{S}(\sigma,-2y_c,;\xi)\tilde{S}(\sigma,2y_n;\xi)}}, \quad (7)$$

with

$$\tilde{S}(y_1,y_2;\xi) = \frac{1}{N_c^2-1}\left\langle 0\left|(\mathcal{W}^{n_1(y_1)}_\xi)^\dagger\mathcal{W}^{n_2(y_2)}_\xi(\mathcal{W}^{n_2(y_2)}_\xi)^\dagger\mathcal{W}^{n_1(y_1)}_\xi\right|0\right\rangle, \quad (8)$$

where $n_{1,2}(y_{1,2})$ are tilted Wilson lines such that n_1 is placed at rapidity $y_1/2$ and n_2 at rapidity $-y_2/2$. For a precise definition of the light-cone directions see [12]. Within high energy factorization, we then aim at the determination of the following coefficients C_{gg^*}, implicitly defined through

$$f_g(\eta_a,\eta_b,y_c,\zeta_B,\mathbf{q},\mu) = \int\frac{d^2\mathbf{k}}{\pi}C^f_{gg^*}(\zeta_B,y_c,\eta_a,\mathbf{q},\mathbf{k},\mu)\mathcal{G}(\Delta\eta_{ab},\eta_b;\mathbf{k}),$$
$$h_g(\eta_a,\eta_b,y_c,\zeta_B,\mathbf{q},\mu) = \int\frac{d^2\mathbf{k}}{\pi}C^h_{gg^*}(\zeta_B,y_c,\eta_a,\mathbf{q},\mathbf{k},\mu)\mathcal{G}(\Delta\eta_{ab},\eta_b;\mathbf{k}), \quad (9)$$

where $\mathcal{G}(\Delta\eta_{ab},\eta_b;\mathbf{k})$ denotes the unintegrated gluon distribution of high energy factorization, where Δ_{ab} denotes the evolution parameter and $\mathbf{k}$ the transverse momentum of the high energy factorized gluon, see [12, 19] for details. At leading order in the strong coupling constant we find

$$C^{f,(0)}_{gg^*}(\mathbf{q},\mathbf{k}) = C^{h,(0)}_{gg^*}(\mathbf{q},\mathbf{k}) = \delta^{(2)}(\mathbf{q}-\mathbf{k}), \quad (10)$$

and the TMD gluon distributions agrees up to an overall factor with the unintegrated gluon density. At next-to-leading order, the result is far more complicated: it requires both application of the soft-factor Eq. (7), subtraction and application of a transition function to remove high energy factorized contributions as well as ultraviolet renormalization. The final results reads

$$\hat{C}^{(1)f}_{gg*}(\zeta_B, y_c, \eta_a, \mathbf{q}, \mathbf{k}, \mu, \bar{f}^{(1)}) = \frac{\alpha_s C_A}{2\pi}\Bigg\{\delta^{(2)}(\mathbf{l})\left[\ln\frac{\mathbf{q}^2}{\mu^2}\left(\ln\frac{\zeta_B}{\mu^2} - \frac{\beta_0}{2C_A}\right) - \frac{1}{2}\ln^2\frac{\mathbf{q}^2}{\mu^2} + \frac{67}{18} - \frac{5n_f}{9C_A} - \frac{\pi^2}{3}\right]$$
$$+ 2(y_c - \eta_a)\frac{1}{\pi}\left[\frac{1}{\mathbf{l}^2}\right]_+ + \frac{1}{\pi}\int_0^1 dz\left[\frac{z(1-z)(\mathbf{l}^2 - \mathbf{q}^2)^2}{[z\mathbf{l}^2 + (1-z)\mathbf{q}^2]^2\,\mathbf{k}^2}\right.$$
$$\left.+ \frac{1}{\mathbf{l}^2}\frac{\mathbf{k}^2 - 3(\mathbf{q}-\mathbf{k})^2 - \mathbf{q}^2}{[z(\mathbf{q}-\mathbf{k})^2 + (1-z)\mathbf{q}^2]}\right]\Bigg\} + \bar{f}^{(1)}(\mathbf{q}, \mathbf{k}) + \mathcal{O}(\epsilon), \tag{11}$$

$$\hat{C}^{(1)h}_{gg*}(\zeta_B, y_c, \eta_a, \mathbf{q}, \mathbf{k}, \mu, \bar{f}^{(1)}) = \frac{\alpha_s C_A}{2\pi}\Bigg\{\delta^{(2)}(\mathbf{l})\left[\ln\frac{\mathbf{q}^2}{\mu^2}\left[\ln\frac{\zeta_B}{\mu^2} - \frac{\beta_0}{2C_A}\right] - \frac{1}{2}\ln^2\frac{\mathbf{q}^2}{\mu^2} + \frac{67}{18} - \frac{5n_f}{9C_A} - \frac{\pi^2}{3}\right]$$
$$+ 2(y_c - \eta_a)\frac{1}{\pi}\left[\frac{1}{\mathbf{l}^2}\right]_+ + \left(y_c + \ln\frac{q^-}{|\mathbf{l}|}\right)\frac{4\left((\mathbf{l}\cdot\mathbf{q})^2 - \mathbf{l}^2\mathbf{q}^2\right)}{\pi\mathbf{l}^2\mathbf{q}^2\mathbf{k}^2}$$
$$+ \frac{1}{\pi}\int_0^1 dz\frac{(\mathbf{q}^2 - \mathbf{l}^2)2\mathbf{k}\cdot\mathbf{l}}{[z\mathbf{l}^2 + (1-z)\mathbf{q}^2]\,\mathbf{l}^2\mathbf{k}^2}\Bigg\} + \bar{f}^{(1)}(\mathbf{q}, \mathbf{k}) + \mathcal{O}(\epsilon). \tag{12}$$

Here $\zeta_{,B} = (x_B p_B^-)^2 e^{2y_c}$ denotes a scale related to the evolution of the TMD distribution, with p_B^- the dominant component of the hadron momentum and x_B the momentum fraction carried on by the gluon. y_c is a parameter related to the subtraction of soft gluons, while η_a and $\bar{f}$ is a parameter and function of high energy factorization, see [12]; μ finally denotes the renormalization scale. The corresponding renormalization constant is identical for both unpolarized and linearly polarized gluons and is obtained as

$$\mathcal{Z}_G = 1 - \frac{\alpha_s C_A}{2\pi}\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\ln\frac{(q^-)^2 e^{2y_c}}{\mu^2} - \frac{\beta_0}{2C_A}\right)\right], \tag{13}$$

which gives rise to the following anomalous dimension

$$\gamma_G = \frac{d\ln\mathcal{Z}_G}{d\ln\mu} = \frac{\alpha_s}{2\pi}\left[\beta_0 + 2C_A\ln\frac{\mu^2}{(q^-)^2 e^{2y_c}}\right], \tag{14}$$

where we used $d\alpha_s/d\ln\mu = 2\epsilon\alpha_s$. Note that the above anomalous dimension agrees with the corresponding result obtained within a treatment based on collinear factorization [11]. This is indeed to be expected since it arises due to the renormalization of ultraviolet divergences, which are naturally independent of the non-zero transverse momentum of the initial state gluon. We however stress again that linearly polarized TMD gluon distribution does not give rise to the above anomalous dimension within collinear factorization. Again this is natural, since the corresponding distribution vanishes within collinear factorization at tree-level, and the 1-loop result is therefore not renormalized.

3. Conclusions

In this short contributions, we gave a few details on the 1-loop calculation of the Transverse Momentum Dependent Gluon distribution in the high energy or low x limit. The result is of interest, since it demonstrates that the anomalous dimension - which governs the renormalization group evolution of this distribution – is indeed universal, as one would expect on general grounds. Future work must focus on both the numerical implementation of this results, its extension to high gluon densities (as needed for EIC phenomenology) as well as to the combination with actual observables. While the final goal would be the description of EIC processes such as di-hadron production, an easier intermediate step is given by the study of a colorless final state, *e.g.* a Higgs boson, as done in Ref. [19].

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1. F. Gelis, Initial state and thermalization in the Color Glass Condensate framework, *Int. J. Mod. Phys. E* **24** (2015) 1530008, `https://doi.org/10.1142/S0218301315300088`.
2. F. Gelis, E. Iancu, J. Jalilian-Marian and R. Venugopalan, The Color Glass Condensate, *Ann. Rev. Nucl. Part. Sci.* **60** (2010) 463-489, `https://doi.org/10.1146/`

annurev.nucl.010909.083629.

3. L. V. Gribov, E. M. Levin and M. G. Ryskin, Semihard Processes in QCD, *Phys. Rept.* **100** (1983) 1-150, https://doi.org/10.1016/0370-1573(83)90022-4.
4. L. D. McLerran and R. Venugopalan, Computing quark and gluon distribution functions for very large nuclei, *Phys. Rev. D* **49** (1994) 2233-2241, https://doi.org/10.1103/PhysRevD.49.2233.
5. L. D. McLerran and R. Venugopalan, Gluon distribution functions for very large nuclei at small transverse momentum, *Phys. Rev. D* **49** (1994) 3352-3355, https://doi.org/10.1103/PhysRevD.49.3352.
6. A. Accardi *et al.*, Electron Ion Collider: The Next QCD Frontier: Understanding the glue that binds us all, *Eur. Phys. J. A* **52** (2016) 268, https://doi.org/10.1140/epja/i2016-16268-9.
7. L. Zheng, E. C. Aschenauer, J. H. Lee and B. W. Xiao, Probing Gluon Saturation through Dihadron Correlations at an Electron-Ion Collider, *Phys. Rev. D* **89** (2014) 074037, https://doi.org/10.1103/PhysRevD.89.074037.
8. F. Dominguez, C. Marquet, B. W. Xiao and F. Yuan, Universality of Unintegrated Gluon Distributions at small x, *Phys. Rev. D* **83** (2011) 105005, https://doi.org/10.1103/PhysRevD.83.105005.
9. J. Zhou, "The evolution of the small x gluon TMD, *JHEP* **06** (2016) 151, https://doi.org/10.1007/JHEP06(2016)151.
10. B. W. Xiao, F. Yuan and J. Zhou, "Transverse Momentum Dependent Parton Distributions at Small-x, *Nucl. Phys. B* **921** (2017) 104, https://doi.org/10.1016/j.nuclphysb.2017.05.012.
11. M. G. Echevarria, T. Kasemets, P. J. Mulders and C. Pisano, "QCD evolution of (un)polarized gluon TMDPDFs and the Higgs q_T-distribution, *JHEP* **07** (2015) 158, [erratum: *JHEP* **05** (2017) 073] https://doi.org/10.1007/JHEP07(2015)158.
12. M. Hentschinski, Transverse momentum dependent gluon distribution within high energy factorization at next-to-leading order, *Phys. Rev. D* **104** (2021) 054014, https://doi.org/10.1103/PhysRevD.104.054014.
13. L. N. Lipatov, "Gauge invariant effective action for high-energy processes in QCD, *Nucl. Phys.* **B452** (1995) 369-400, [arXiv:hep-ph/9502308 [hep-ph]].
14. L. N. Lipatov, "Small x physics in perturbative QCD, *Phys. Rept.* **286** (1997) 131-198, [hep-ph/9610276].
15. M. Hentschinski, Lipatov's QCD high energy effective action: past and future, in "From the Past to the Future The Legacy of Lev Lipatov, edited by J. Bartels, V. Fadin, E. Levin, A. Levin, V. Kim, A. Sabio Vera, World Scientific, https://doi.org/10.1142/12127.
16. P. J. Mulders and J. Rodrigues, Transverse momentum dependence in gluon distribution and fragmentation functions, *Phys. Rev. D* **63** (2001) 094021, https://doi.org/10.1103/PhysRevD.63.094021.
17. X. d. Ji, J. P. Ma and F. Yuan, Transverse-momentum-dependent gluon distributions and semi-inclusive processes at hadron colliders, *JHEP* **07** (2005) 020, https://doi.org/10.1088/1126-6708/2005/07/020.
18. J. Collins, Foundations of perturbative QCD, *Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.* **32** (2011) 1-624.
19. M. Hentschinski, K. Kutak and A. van Hameren, Forward Higgs production within high energy factorization in the heavy quark limit at next-to-leading order accuracy, *Eur. Phys. J. C* **81** (2021) 112, [erratum: *Eur. Phys. J. C* **81** (2021) 262] https://doi.org/10.1140/epjc/s10052-021-08902-6.