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Particle radiation produced by accelerated systems and their analogy with damped oscillators

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The Unruh effect predicts how uniformly accelerated observers will perceive a change in the vacuum state. This shows that the concept of particle number depends on the acceleration of the reference frame. Although this is a result of quantum field theory, its experimental verification is still questioned, mainly due to the high accelerations required. In this work we study a quantum oscillator with only one complex coordinate and a damping term acting as perturbation, which has all the characteristics of the Unruh effect in second quantization for an accelerated observer. The Bogoliubov transformation connecting the two different vacuum states is obtained. This leads to an explicit formula for the particle occupation number as a function of energy and acceleration. Furthermore, it is shown that our analogue system contains an effective temperature that depends on the observer's sudden acceleration, seen as a friction force. The purpose of this work is to demonstrate that quanta production (particles or energy packets) is inevitable under the premises of quantum field theory.

Keywords: Unruh effect; Rindler coordinates; Klein-Gordon equation in curved space; micro electro-mechanical oscillators.

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1. Introduction

Relativistic quantum field theory complies with the principle of invariance under changes of inertial frames, which is to say, that Poincaré invariance is a cardinal principle of all physical theories. However, Unruh concluded that the concept of vacuum depends on whether frames of reference are inertial or non inertial [1, 2]. The lack of invariance of vacuum under transformations for accelerated observers entails particle production, but this effect has not been experimentally verified due to the need of large accelerations and expensive setups. Nowadays there are interesting proposals to demonstrate it empirically; nonetheless, simplified analogue setups in top table experiments should enable us to demonstrate its inevitability. The present contribution is devoted to the study of a single mode quantum oscillator that receives corrections due to a non inertial coordinate frame. Quanta production, thermal distributions and transition amplitudes can be computed analytically without divergences, with the aim of implementing the effect in micro electro-mechanical oscillators (MEMs) [3].

Despite the early objections on the plausibility of observing Unruh radiation –*e.g.* the nature of particle detectors in accelerated frames– here we argue that the modification of the average particle number depends crucially on the initial state selected by the observer, as well as a sudden acceleration process that modifies the observables, but not the preselected state, which is the inertial vacuum. Evidently, due to the presence of vacuum expectation values (VEVs) this would be equivalent to the transformation of the state without change in the relevant observables, *e.g.* particle number or excitation number.

After a short review of the Unruh effect for a scalar field in Sec. 2, we shall establish in Sec. 3 an analogy with a single harmonic oscillator subjected to a damping force; this shall be done by means of a dimensional reduction of the Klein-Gordon equation in curved space. The results of this paper increase the collection of successful quantum-dynamical emulations in simplified systems produced in [4, 5] (Dirac oscillator) and [6, 7] (Peierls model of interacting channels). Some concluding remarks are given in Sec. 4.

2. Klein-Gordon in curved space

We know that the Lorentz transformation connecting a frame S' moving with a velocity v in the axis x with respect to a second frame S, is

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh(\phi) & \sinh(\phi) \\ \sinh(\phi) & \cosh(\phi) \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}, \qquad (1)$$

where $\cosh(\phi) = \gamma$ y $\sinh(\phi) = \gamma\beta$. A Rindler observer [8] is one that moves with a constant acceleration, see Fig. 1. Therefore, using the equivalence principle, its motion is given by (1) with $\alpha = \dot{\phi}$ and $-\alpha^2 = -\left(a^0\right)^2 + \left(a^1\right)^2$ for the acceleration vector α^μ. The Rindler metric is $g_{\mu\nu} = \mathrm{diag}\left(\alpha^2 x^2, -1, -1, -1\right)$. As we know, the scalar curvature in this case is $R = 0$ and the Christoffel symbols corresponding to this metric are $\Gamma^x_{\tau\tau} = \alpha x$, $\Gamma^\tau_{\tau x} = \Gamma^\tau_{x\tau} = 1/\alpha x$ and the rest vanish. In flat Minkowski spacetime with the metric $\eta = (1, -1, -1, -1)$ we have the well-known Klein-Gordon equation $\left(\Box + m^2\right)\phi\left(x_\mu\right) = 0$, in natural units. Furthermore, we know that this scalar field can be expanded in terms of the momentum eigenfunctions $\phi_k = \mathcal{N} e^{-i\omega_k t} e^{i\mathbf{k}\cdot\mathbf{x}}$, with their corresponding coefficients C_k y C_k^*. However when we work in a curved space we need to replace the operator ∂_μ by the covariant derivative operator ∇_μ, which satisfies $\nabla_\mu g_{\nu\sigma} = 0$. We have

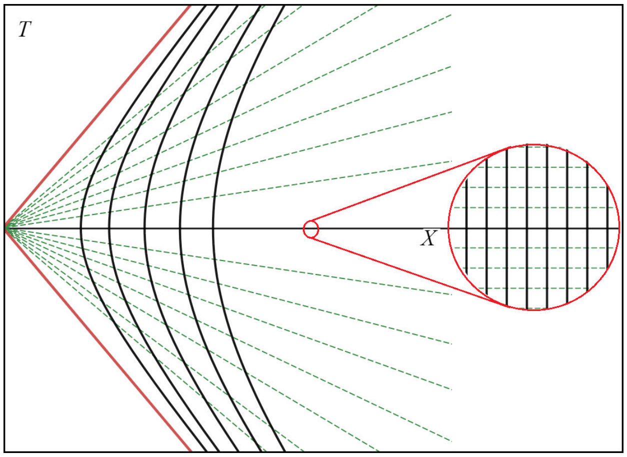

FIGURE 1. A diagram of Rindler space. The red line limits the region known as Rindler wedge. The black hyperbolas correspond to the trajectories for observers with constant acceleration. A magnification in the asymptotic region shows the limit of flat space-time.

$$\left(\nabla_\mu \nabla^\mu + m^2\right)\phi = \nabla_\mu \left[\nabla^\mu \phi\right] + m^2\phi$$
$$= \partial_\mu \partial^\mu \phi + \Gamma^\lambda_{\mu\lambda}\partial^\mu \phi + m^2 \phi = 0. \quad (2)$$

By the equivalence principle, the equation for a curved space is the same as having an accelerated observer, *i.e.* it is equivalent to a change of coordinates. Similar expansions can be made in the curved case; let $\psi_n(x)$ be the stationary solutions of the problem, such that

$$\phi(x) = \sum_n \left[D_n \psi_n(x) + D_n^* \psi_n^*(x)\right]. \quad (3)$$

Using the corresponding inner product, we will have the following projections:

$$(\phi_k, \phi) = \int d^3k' C_{k'} \left(\phi_k, \phi_{k'}\right) = 2\omega_k C_k,$$
$$(\psi_n, \phi) = \sum_{n'} D_{n'} \left(\psi_n, \psi_{n'}\right) = 2\omega_n D_n. \quad (4)$$

With this last result (4) we arrive at a transformation between the coefficients of the different modes:

$$C_k = \frac{1}{2\omega_k}\sum_n \left[D_n \langle \phi_k, \psi_n\rangle + D_n^* \langle \phi_k, \psi_n^*\rangle\right]. \quad (5)$$

The next step is to use the second quantization postulates on the fields, which implies a substitution of the coefficients C_k, D_n by operators $\hat{C}_k, \hat{D}_n$. Making a direct substitution of the Fourier expansion (3) in commutation relations $\left[\hat{\phi}(t,\mathbf{x}), (g_{00})^{-1/2}\, \partial_t \hat{\phi}(t,\mathbf{y})\right] = i\hbar\delta^3(\mathbf{x}-\mathbf{y})$, $[\hat{\phi}(t,\mathbf{x}), \hat{\phi}(t,\mathbf{y})] = g^{00}\left[\partial_t\hat{\phi}(t,\mathbf{x}), \partial_t\hat{\phi}(t,\mathbf{y})\right] = 0$, and making the substitution $\hat{C}_k = \hat{A}_k^\dagger/\sqrt{2\omega_k}$, we find the following Heisenberg algebra

$$\left[\hat{A}_k, \hat{A}_{k'}^\dagger\right] = \delta^3\left(\mathbf{k}-\mathbf{k}'\right), \quad \left[\hat{A}_k, \hat{A}_{k'}\right] = 0. \quad (6)$$

These commutation relations reveal the usual infinite set of harmonic oscillators that conforms the field. We finally obtain the celebrated Bogoliubov transformation:

$$\hat{A}_k = \sum_n \left[\left(\frac{\langle \phi_k, \psi_n\rangle^*}{2\sqrt{\omega_k \omega_n}}\right)\hat{B}_n + \left(\frac{\langle \phi_k^*, \psi_n\rangle}{2\sqrt{\omega_k\omega_n}}\right)\hat{B}_n^\dagger\right]. \quad (7)$$

We can see that the old inertial vacuum $|0\rangle_A$ is not annihilated by the accelerated operators, *i.e.* $\hat{B}_n|0\rangle_A \neq 0$. We come to the conclusion that the vacuum state and the concept of number depend on whether our framework is accelerated or not. Crearly, if the inertial vacuum $|0\rangle_A$ is pre-selected by an inertial observer, no particle radiation should be visible. Upon a sudden acceleration of such an observer, the system remains in state $|0\rangle_A$ but the observables evolve into accelerated operators $\hat{B}_n$. Thus $\langle N_B\rangle_A = \langle N_A\rangle_B \neq 0$ which shows particle production.

3. Reduction to a single harmonic oscillator: an analogy

We study the analogy between the Klein-Gordon equation in Rindler space and the equation for a classical harmonic oscillator with a viscous force. Our goal is to obtain the Bogoliubov transformation for a single oscillation mode. First we observe that the term affected by Γ in Eq. (2) is linear in the derivatives, which is similar to a velocity-dependent force that perturbs the acceleration $\partial^2\phi/\partial t^2$. If we apply a dimensional reduction such that $\mu = 0$ is the only allowed value, we will have $g_{00} \equiv 1$, $\Gamma^0_{00} \equiv a$, $m^2 \equiv \omega^2$, $\phi(x,t) \rightarrow \hat{X}(t)$, which leads to the equivalence between the two following equations

$$\left[\frac{d^2}{dt^2} + a\frac{d}{dt} + \omega^2\right]\hat{X}(t) = 0 \leftrightarrow \left[\Box + \Gamma^\lambda_{\mu\lambda}\partial^\lambda + m^2\right]\phi = 0. \quad (8)$$

The factor a is associated with the friction of the system. In general it can be complex but for systems with dissipation it is taken as real and positive. Equation (8) for $\hat{X}(t)$, can also be obtained using the Caldirola-Kanai Hamiltonian [9]

$$\hat{H} = e^{-at}\frac{\hat{P}^2}{2} + e^{+at}\omega^2\frac{\hat{X}^2}{2}. \quad (9)$$

With the use of a canonical transformation $\hat{Y} = e^{+at/2}\hat{X}$, $\hat{\Pi} = e^{-at/2}\hat{P}$, we obtain a new equation corresponding to an oscillator with a new frequency $\Omega^2 \equiv \omega^2 - (a/2)^2$. We have

$$\frac{d^2\hat{Y}}{dt^2} = -\left[\omega^2 - (a/2)^2\right]\hat{Y} \equiv -\Omega^2\hat{Y}. \quad (10)$$

The new solution of the damped oscillator $\hat{Y}$ can be expressed in terms of the orthonormal basis $\left\{e^{-i\omega t}, e^{+i\omega t}\right\}$ and the old coefficients C and $C^\dagger$, corresponding to the case $a = 0$, which gives us

$$\hat{Y}(t) = e^{-i\omega t}\hat{C}^\dagger(t) + e^{+i\omega t}\hat{C}(t). \quad (11)$$

By means of the inner product obtained from the equation of motion (8), we calculate the projection between $\hat{Y}$ and the eigenmodes of the unperturbed problem. This operation will result in a connection between old coefficients $\hat{C}$ and $\hat{C}^\dagger$, with new coefficients $\hat{D}$ and $\hat{D}^\dagger$, corresponding to the damped problem. The simplified Bogoliubov transformation is

$$\hat{C} = \frac{\omega - \Omega}{2\omega} e^{-i(\omega+\Omega)t} \hat{D}^\dagger + \frac{\omega + \Omega}{2\omega} e^{-i(\omega-\Omega)t} \hat{D}. \qquad (12)$$

Furthermore, the creation and annihilation operators will be given by

$$\hat{D} = \frac{\hat{B}^\dagger}{\sqrt{2\Omega}}, \quad \left[\hat{B}, \hat{B}^\dagger\right] = 1,$$
$$\hat{C} = \frac{\hat{A}^\dagger}{\sqrt{2\omega}}, \quad \left[\hat{A}, \hat{A}^\dagger\right] = 1. \qquad (13)$$

Finally using the notation $\zeta = \sqrt{\Omega/\omega}$, obtains

$$\hat{A} = \frac{\zeta - \zeta^{-1}}{2} e^{i(\omega+\Omega)t} \hat{B}^\dagger + \frac{\zeta + \zeta^{-1}}{2} e^{i(\omega-\Omega)t} \hat{B},$$
$$\hat{B} = \frac{\zeta^{-1} - \zeta}{2} e^{i(\omega+\Omega)t} \hat{A}^\dagger + \frac{\zeta^{-1} + \zeta}{2} e^{i(\Omega-\omega)t} \hat{A}. \qquad (14)$$

We see that the old vacuum in the case $a = 0$, *i.e.* $|0\rangle_x$, is not annihilated by the new operators $\hat{B}|0\rangle_x \neq 0$. We conclude that the vacuum state will depend on whether or not our system has *friction*. To support this statement, we take this transformation to phase space as the application of a rotation, a rescaling of variables and a reverse rotation.

$$\begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} \times \begin{pmatrix} e^{\theta} & 0 \\ 0 & e^{-\theta} \end{pmatrix} \begin{pmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t & \cos\Omega t \end{pmatrix} \times \begin{pmatrix} \hat{X} \\ \hat{P} \end{pmatrix} = R_\omega^\top S_\theta R_\Omega \begin{pmatrix} \hat{X} \\ \hat{P} \end{pmatrix}, \qquad (15)$$

with $e^\theta = \zeta$. In Hilbert space, the rotations R_ω and R_Ω are generated by the number operator, while S_θ is the squeeze operator, a rescaling of variables, *i.e.* $R_\omega \rightarrow e^{i\hat{B}^\dagger \hat{B}\omega t}$, $R_\Omega \rightarrow e^{-i\hat{B}^\dagger \hat{B}\Omega t}$, $S_0 \rightarrow e^{\frac{\theta}{2}\left(\hat{B}^{\dagger 2} - \hat{B}^2\right)}$.

Then U, the transformation that consists of the successive application of these three operators, will connect both vacuum states. Therefore, the old vacuum $|0\rangle_x$ in terms of the new modes with friction $|n\rangle \equiv |n\rangle_y$, will be:

$$|0\rangle_x = U^\dagger|0\rangle = \sum_{n=0}^{\infty} \langle n| e^{iB^\dagger B\Omega t} e^{\frac{\theta}{2}\left\{\hat{B}^{\dagger 2} - \hat{B}^2\right\}} e^{-iB^\dagger B\omega t} |0\rangle |n\rangle = \sum_{n=0}^{\infty} e^{in\Omega t} \langle n| e^{-\theta\left(\hat{B}^{\dagger 2} - \hat{B}^2\right)/2} |0\rangle |n\rangle. \qquad (16)$$

There are several methods to determine the matrix elements of the operator that appears in (16). Here we use the Baker-Campbell-Hausdorff formula for bilinear operators of the harmonic oscillator, found in [10]

$$\begin{aligned} S(z) &= \exp\left[\frac{1}{2}\left(zB^\dagger B^\dagger - z^* BB\right)\right] \\ &= \exp\left[\frac{1}{2}\left(e^{i\phi}\tanh r\right) B^\dagger B^\dagger\right] \\ &\times \exp\left[-2(\ln\cosh r)\left(\frac{1}{2}B^\dagger B + \frac{1}{4}\right)\right] \\ &\times \exp\left[-\frac{1}{2}\left(e^{i\phi}\tanh r\right) BB\right]. \end{aligned} \qquad (17)$$

In our case $z = re^{i\phi}$, $z = z^* = -\theta = |1|e^{i\phi} \Rightarrow e^{i\phi} = -1$ and $r = \theta$. Applying the formula to (16), obtains:

$$|0\rangle_x = \sum_n C_n |n\rangle, \qquad (18)$$

$$C_n = \frac{e^{in\Omega t}}{\sqrt{\cosh\theta}} \begin{cases} \left(\frac{-\tanh\theta}{2}\right)^{n/2} \frac{\sqrt{n!}}{(n/2)!}, & \text{if } n \text{ even} \\ 0, & \text{if } n \text{ odd} \end{cases}. \qquad (19)$$

The coefficient $|C_n|^2$ is interpreted as the transition probability between the old ground state and the new quantum occupation numbers. Since $e^\theta = (\omega/(\omega^2 - a^2/4)^{1/2})^{1/2}$, we have now a relationship between the number of excitations produced in the oscillator and the viscous force factor a in Eq. (19). We know that $|C_n|^2$ is a discrete probability distribution but we may consider continuous variables by writing $n = E/\Omega - 1/2$. The probability of having n particles with energy E as a function of viscosity parameter a is:

$$\begin{aligned} P_a(E) &= \frac{2\sqrt{\omega}\left(\omega^2 - (a/2)^2\right)^{1/4}}{\omega + \left(\omega^2 - (a/2)^2\right)^{1/2}} \\ &\times \frac{\Gamma\left(\frac{E}{\sqrt{\omega^2 - (a/2)^2}} + \frac{1}{2}\right)}{\left[\Gamma\left(\frac{E}{2\sqrt{\omega^2-(a/2)^2}} + \frac{3}{4}\right)\right]^2} \\ &\times \exp\left\{-\left(\frac{E}{\sqrt{\omega^2 - (a/2)^2}} - \frac{1}{2}\right) \right. \\ &\left. \times \ln\left(2\frac{\omega + \left(\omega^2 - a^2/4\right)^{1/2}}{\omega - \left(\omega^2 - a^2/4\right)^{1/2}}\right)\right\}. \end{aligned} \qquad (20)$$

Equation (20) can be treated using a Stirling approximation, leading to a Boltzmann-type law, see Fig. 2. For large values of $n (n \geq 10)$, we have:

$$P_a(E) = 2\sqrt{\frac{\Omega\omega}{\omega^2 - \Omega^2}} \exp\left[-\frac{1}{\Omega}\ln\left(\frac{\omega + \Omega}{\omega - \Omega}\right) E\right] \qquad (21)$$

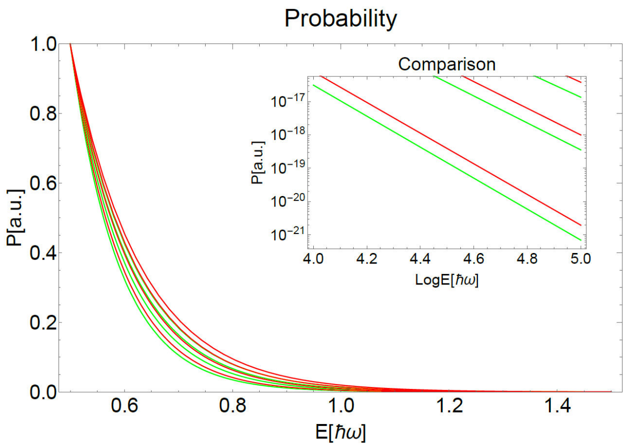

FIGURE 2. Comparison between approximation (21) (red curves) and exact result (20) (green curves). Different curves are generated for values $a = 0.02s^{-1}$, $a = 0.04s^{-1}$, $a = 0.06s^{-1}$, $a = 0.08s^{-1}$. The Stirling approximation works better for large n.

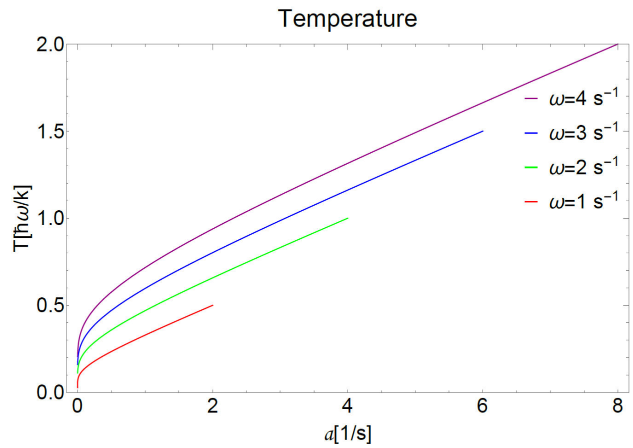

FIGURE 3. Effective temperature as a function of a. We see an asymptotic linear relation bounded by the condition $a = 2\omega$, complying with $0 < \Omega^2$.

We can deduce that our probability density as a function of energy E has the form of a Boltzmann factor

$$P_a(E) = \beta e^{-\beta E}, \quad \beta \equiv \frac{1}{\Omega} \ln\left(\frac{\omega + \Omega}{\omega - \Omega}\right). \tag{22}$$

And finally, we have now an expression of effective temperature as a function of frequency and viscosity, as plotted in Fig. 3:

$$T = \frac{\hbar\sqrt{(\omega^2 - (a/2)^2)}}{k_B \ln\left(\frac{\omega + (\omega^2 - a^2/4)^{1/2}}{\omega - (\omega^2 - a^2/4)^{1/2}}\right)}. \tag{23}$$

4. Concluding remarks

We have found an analogy between the Klein-Gordon equation in curved space for accelerated observers and the equation of a harmonic oscillator with rheological force, which can be seen in the Bogoliubov transformation (14). We have calculated the coefficients for this transformation in (19). Finally, we have arrived at a particle occupation number distribution as a function of an effective temperature in terms of the observer's acceleration, in analogy with a rheological force.

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