A measurement of the mass of the $\tau$ lepton using new methods to study semi-invisible decays

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Measuring the mass of particles whose decay products cannot be detected poses a significant challenge due to the complexity of reconstructing these decays and measuring various parameters. However, studying processes involving undetectable particles is crucial as it enables us to delve deeper into familiar decays involving energy loss, such as Standard Model processes involving neutrinos. Additionally, it provides an opportunity to test models associated with physics beyond the Standard Model that can be generated in leptonic colliders. In this study, the mass of the tau lepton was determined by comparing three different methods for decays with semi-invisible final states. Specifically, the measurement focused on the decay $\tau^- \rightarrow \pi^- \nu_\tau$ (signal). Among the three methods employed, the most accurate result was obtained using the $M_{\text{min}}$ method, yielding a tau lepton mass value of $M_\tau = 1777.06 \pm 0.44$ MeV. The measurement utilized official Monte Carlo data provided by the Belle II collaboration, specifically from the MC13a campaign conducted until 2020, with an integrated luminosity of 100 fb$^{-1}$.

Keywords: $\tau$ lepton mass; methods to study invisible particles; lepton colliders.

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1. Introduction

The $\tau$ lepton is a fundamental particle in the Standard Model of particle physics (SM). There are measurements with extraordinary precision, such as the mass of the electron [1], but the mass of the $\tau$ still has a large uncertainty, and its precise mass determination is crucial for testing the theory and exploring physics beyond it. The mass of the tau lepton is an essential parameter for precision electroweak measurements due to its relationship with the weak mixing angle [2], lepton flavor universality tests, including radiative corrections. [3], and the determination of the strong coupling constant $\alpha_s$ at the $\tau$ mass scale [4]. Important deviations from the SM in one of these could imply a sighting of BSM physics. Considering the reasons stated above, and the fascinating features of the $\tau$ lepton, a better determination of its mass is now essential.

Significant progress has been made in recent years in measuring the mass of the tau lepton. The most accurate measurement recorded in the PDG is $1776.86 \pm 0.12$ MeV, it is dominated for the fit made by BES III [5] which took advantage of $e^-e^+ \rightarrow \tau^-\tau^+$ cross-section near the $\tau$-pairs production threshold, the BaBar [6] and Belle [7] collaborations used the so-called pseudo-mass method (developed by ARGUS [8]) is used to provide the measurement. Currently, the pseudo-mass technique is still employed to determine the mass of the $\tau$ in Belle II from the decay mode $\tau^- \rightarrow \pi^- \pi^+ \pi^- \nu_\tau$, however, the greatest statistic is not concentrated where the pseudo-mass distribution has kinematic fall [9]. The mass of the $\tau$ lepton is connected to such a decline. To enhance the precision of mass measurement, the statistics of the distribution used to obtain this quantity must be redistributed around the kinematic edge.

With the purpose of redistributing better the statistics around the point of interest, in this work, we implemented three different methods ($M_{\text{edge}}, M_{\text{min}}$ and $M_{\text{max}}$) to study decays where there is lost energy in the final states. Such processes are produced via $e^-e^+ \rightarrow X\overline{X}$, where the $X$ could be a heavy particle; consequently, we have the process $X\overline{X} \rightarrow (\sum_{i=1}^{n} Y_{a_i} + N_1)(\sum_{j=1}^{m} Y_{b_j} + N_2)$ where $Y_{a(b)}$ represents any detectable particle and $N_{1(2)}$ a particle that escapes to the detection; processes like these are called semi-invisible decays.

Kinematic edges link the three techniques, and it is possible to create a connection between the kinematic edge and the mass of the mother particle $[10, 11]$. At the same time, a constraint, that connects the mother particle’s masses $X$ and the invisible particles $N_1$ and $N_2$, is created. This relationship is useful to suggest novel searching variables for non-standard unseen particles with well-defined initial state energy and momentum, such as in BSM processes [12]; hence, validation in a more practical situation is required. We used these approaches for $e^-e^+ \rightarrow \tau^-\tau^+$ followed by $\tau^-\tau^+ \rightarrow (\pi^- + \nu_\tau)(\pi^+ + \bar{\nu}_\tau)$, this decay topology is known as $(1 \times 1)$-prong. We have chosen as signal, and tag a process where $\tau$ decays to one pion plus neutrino, just as shown in Fig. 1; there are two challenges here: the first is that neutrinos are impossible to detect entirely, and the second is that $\tau$ has a very short lifetime. The combination of these two technical obstacles make reconstructing this process extremely tough. The measurement of the $\tau$ mass in the $(1 \times 1)$-prong topology in the Belle II collaboration has not yet been investigated for the above reasons. We aimed to apply the three approaches indicated above throughout this study to calculate the mass.
we follow the approaches given is illustrated in Fig. 2, where the dotted lines indicate that N1
hadronic particle in the final state that can be detected, we
particles. Since, in our case we only have for each decay one
count each event with a solution region in the plane compos-
tion level using official Monte Carlo data from the Belle II
of the
The validation made in this work opens the door for the
Belle II collaboration to investigate the mass of the
τ−
experiment for an integrated luminosity of (100 fb −1).

2. Kinematic reconfiguration

In the extended scenario when we have the decay of the X X̄ pair, we choose to start with kinematic
of the τ lepton in the (1 × 1)-prong topology at the simulation
topology by Refs. [11, 12] and N2 escape the detector. We follow the approaches given
of the τ lepton in the (1 × 1)-prong topology at the simulation
level using official Monte Carlo data from the Belle II
for an integrated luminosity of (100 fb −1).

Now, let us now do a kinematic study of the process of
Fig. 2. Let P_a, P_b, P_1 and P_2 the 4-momentum of h_a, h_b, N_1 and N_2 in the center-of-mass system (CMS) respectively, where
P_a = (E_a, p_a), P_b = (E_b, p_b), P_1 = (E_1, p_1) and
P_2 = (E_2, p_2). We have the following kinematic relations
due to energy-momentum conservation and invariant mass
\[ q^\mu = P_a^\mu + P_b^\mu + P_1^\mu + P_2^\mu, \]
\[ P_1^2 = m_1^2, \]
\[ P_2^2 = m_2^2, \]
\[ (P_a + P_1)^2 = (P_b + P_2)^2 = m_X^2, \]
where q^\mu = (\sqrt{s}, 0, 0, 0) is the 4-momentum in CMS framework,
\[ \mu = 0, 1, 2, 3; m_1, m_2 \text{ and } m_X \text{ are the masses of } N_1, \]
\[ N_2 \text{ and } X \text{ respectively. Note that if our process occurs in } \]
a hadronic collider, q^0 and q^3 will be indeterminate, i.e.,
the first equation will be nonviable; however, in colliders like SuperKEKB, the 4-momentum of the system can be determined,
thus P_a^\mu and P_b^\mu can be fully determined when solving the
eight kinematic equations for test values of m_X and m_N.

For simplicity, we redefine the kinematic variables as follows,
the normalized energies \( z_i = P_i^\mu / \sqrt{s} \) (\( i = 1, 2, a, b, X \)),
the normalized 3-momentum \( k_j = p_j / \sqrt{s} \)
(\( j = 1, 2, a \equiv p_a / \sqrt{s}, b \equiv p_b / \sqrt{s} \) and the normalized masses \( \mu_k = m_k / \sqrt{s} \) \( k = 1, 2, X, N \)). Using the above
definitions and the conservation of energy-momentum, the
Eqs. (2)-(4) would remain
\[ |k_1|^2 + \mu_1^2 = z_1^2 = (z_X - z_a)^2, \]
\[ |k_1 + a + b|^2 + \mu_2^2 = (1 - z_a - z_b - z_1)^2, \]
\[ |k_1 + a|^2 + \mu_X^2 = z_X^2, \]
\[ |k_1 + a|^2 + \mu_X^2 = (1 - z_X)^2. \]

As each mother particle takes about half of the energy of
the CMS, we may represent the energy of X as \( z_X = 1/2 \).
We need to eliminate \( k_1 \) from the kinematic equations; for
this, we clear this quantity from the Eq. (5), and we have
\[ K = |k_1|^2 = \left( \frac{1}{2} - z_a \right)^2 - \mu_1^2. \]

To the replacing (9) in (8) we stayed with
\[ a \cdot k_1 = \frac{1}{2} (z_a - z_a^2 - \mu_X^2 + \mu_1^2 - |a|^2), \]
from (6) we get
\[ b \cdot k_1 = \frac{1}{2} (z_b^2 - z_b + \mu_X^2 - \mu_2^2 - |b|^2) - a \cdot b. \]

If we define
\[ A \equiv a \cdot k_1, \]
\[ B \equiv b \cdot k_1, \]
and we develop
\[ K(a_2 b_y - a_y b_2)^2 = k^2_{1x}(a_2 b_y - a_y b_2)^2 \]
\[ + k^2_{1y}(a_2 b_y - a_y b_2)^2 + k^2_{1z}(a_2 b_y - a_y b_2)^2, \]
we come to the quadratic next expression of \( k^2_{1x} \)
\[ (Ab_2 - Ba_2)^2 + (Ab_y - Bay)^2 \]
\[ + 2(( Ab_2 - Ba_2)(a_2 b_y - a_y b_2) \]
\[ + ( Ab_y - Ba_y)(a_2 b_y - a_y b_2) |k_{1x}\]
\[ + |a \times b|^2 k^2_{1x} = 0. \]

In addition, \( k_1 \), \( a \) and \( b \) must comply with
\[ |k_1 \times a \times b|^2 = |(a \cdot k_1)b - (b \cdot k_1)a|^2, \]
\[ = |k_1|^2|a \times b|^2 \sin^2 \theta, \]
\[ \leq |k_1|^2|a \times b|^2. \]  \hspace{1cm} (14)

From the above and using the definitions \((12)\) and \((13)\) we obtain the solution condition for the Eq. \((14)\) in compact form
\[ \sqrt{K} \geq \frac{|Ab - Ba|}{|a \times b|}. \]  \hspace{1cm} (15)

Developing \((15)\) using the equation \((9)\), and the same time \((10)\) and \((11)\), thus we come to
\[ |a \times b| \sqrt{\left( \frac{1}{2} - z_a \right)^2 - \mu_2^2} \geq \]
\[ - \frac{1}{2} \left[ (\mu_2^2 - \mu_1^2)b + (\mu_2^2 - \mu_1^2)a + \mathbf{H} \right], \]  \hspace{1cm} (16)

where
\[ \mathbf{H} = (z_b - z_a - |b|^2 - 2(a \cdot b))a \]
\[ + (z_a^2 - z_a + |a|^2) \mathbf{b}. \]  \hspace{1cm} (17)

Then, by doing some algebra, we get the following inequality
\[ A_1(\mu_2^2 - \mu_1^2)^2 + A_2(\mu_2^2 - \mu_1^2)^2 \]
\[ + A_3(\mu_2^2 - \mu_1^2)(\mu_2^2 - \mu_1^2) \]
\[ + B_1(\mu_2^2 - \mu_1^2) + B_2(\mu_2^2 - \mu_1^2) \]
\[ + C_1 \mu_2^2 + D_1 \leq 0, \]  \hspace{1cm} (18)

here
\[ A_1 \equiv |b|^2, \]
\[ A_2 \equiv |a|^2, \]
\[ A_3 \equiv 2(a \cdot b), \]
\[ B_1 \equiv 2(b \cdot H), \]
\[ B_2 \equiv 2(a \cdot H), \]
\[ C_1 \equiv 4|a \times b|^2, \]
\[ D_1 \equiv |H|^2 - 4|a \times b|^2 \left( \frac{1}{2} - z_a \right)^2. \]  \hspace{1cm} (25)

If we consider the case where \( \mu_1 = \mu_2 \) the Eq. \((18)\) becomes in
\[ A_0(\mu_2^2 - \mu_1^2)^2 + B_0(\mu_2^2 - \mu_1^2) + C_0 \mu_2^2 + D_0 \leq 0, \]  \hspace{1cm} (26)

where
\[ A_0 \equiv |a + b|^2, \]
\[ B_0 \equiv 2|a \cdot H + b \cdot H|, \]
\[ C_0 \equiv 4|a \times b|^2, \]
\[ D_0 \equiv |H|^2 - 4|a \times b|^2 \left( \frac{1}{2} - z_a \right)^2. \]  \hspace{1cm} (30)

We obtained an equation in which the input variables are the kinematic variables of the particles, we can detect \( h_a \) and \( h_b \). It should be noted that the Eq. \((26)\) is an oblique parabola in the plane \((\mu_2^2 - \mu_1^2)\), rather, the solution region is bounded by a parabola (see Fig. 3). All accessible kinematic information for the process \( XX \rightarrow (h_a + N_1)/(h_b + N_2) \) is included in the inequalities \((18)\) or \((25)\), depending on whether \( N_1 \) and \( N_2 \) are distinct or the same kind of particle.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{The representation of the solution zone from the Eq. \((25)\) for a specific event. The true mass of the \( \tau \) is displayed as a red cross, the edge mass as a blue star, and the maximum value as a green point. \( A_0, B_0, C_0 \) and \( D_0 \) were picked from the local simulation.}
\end{figure}
3. **M_{edge}, M_{min} and M_{max} methods**

The equation’s parabola (26) is rotated 45° in the plane \((\mu_X^2 - \mu_Y^2)\). If we apply a rotational transformation where \(\theta = 45°\), and put the expression in terms of the new variables \((\mu_X^2 \text{ and } \mu_Y^2)\), we get the parabola vertex in the new reference system, and if we perform the inverse transformation, we get the original system’s vertex parabola. This gives us our variables \((\mu_X^2)^2 \text{ and } (\mu_Y^2)^2)\, which represent the vertex of the parabola in the plane \((\mu_X^2 - \mu_Y^2)\). The expressions are

\[
(\mu_X^2)^2 = \frac{4B_0^2 + 3C_0^2 - 16A_0D_0 - 8B_0C_0}{16A_0C_0},
\]

\[
(\mu_Y^2)^2 = \frac{4B_0^2 - C_0^2 - 16A_0D_0}{16A_0C_0}. \tag{31}
\]

Up to this point, we can have two alternatives, that the parabola has a vertex in quadrant I or that it is in quadrant IV, but if it is in the latter, we will have a restriction on \((\mu_Y^2)^2\) and it is that this cannot take negative values, in that case, we will assign the minimum value to this variable, that is, if the vertex is in quadrant IV, the furthest physical point will be the intersection of the parabola and the axis \(\mu_Y^2\). Hence, from (26) we will obtain for the situation when \(\mu_Y^2 = 0\)

\[
(\mu_X^2)^2 = \frac{B_0^2 - 4A_0D_0}{2A_0},
\]

\[
(\mu_Y^2)^2 = 0. \tag{32}
\]

Both (31) and (32) will be used to extract the masses \(m_{edge}^X\) and \(m_{edge}^Y\) knowing that \(m_{edge}^X = (\mu_X^2)^\sqrt{\sigma}\, m_{edge}^Y = (\mu_Y^2)^\sqrt{\sigma}\).

Now, to construct the other two methods, let us begin with the Eq. (18) and let us particularize to the decay \(\tau^- \rightarrow \pi^- \nu_\tau\). For practicality, let us consider \(\mu_1 = \mu_\nu\), and \(\mu_X = \mu_\tau\). Assuming \(m_\tau > \min\), the Eq. (18) reduces to

\[
(A_1 + A_2 + A_3)(\mu_1^2)^2 + (B_1 + B_2)(\mu_2^2) + D_1

= A_0(\mu_2^2)^2 + B_0(\mu_2^2)^2 + D_0 \leq 0. \tag{33}
\]

Solving, we obtain

\[
(\mu_2^2)^2 = \frac{\sqrt{B_0^2 - 4A_0D_0}}{2A_0} - \frac{B_0}{2A_0}, \tag{34}
\]

this can be seen as

\[
(\mu_2^2)^2 \leq (\mu_2)\mu_\tau \leq (\mu_2^\max)^2, \tag{35}
\]

given that \(m_\tau = \mu_\tau \sqrt{\sigma}\), thus

\[
M_{\min}^2 \leq m_\tau \leq M_{\max}^2, \tag{36}
\]

where

\[
M_{\min}^2 = (\sqrt{\sigma})^2 \left(\frac{B_0 - \sqrt{B_0^2 - 4A_0D_0}}{2A_0}\right)^2, \tag{37}
\]

\[
M_{\max}^2 = (\sqrt{\sigma})^2 \left(\frac{B_0 + \sqrt{B_0^2 - 4A_0D_0}}{2A_0}\right)^2. \tag{38}
\]

4. **Event selection**

The Belle II detector is composed of several sub-detectors arranged in a cylindrical configuration around \(e^- e^+\) interaction point (IP). We select \(\tau\)-pair candidates by requiring only two final state charged particles; each was less than 3 cm in the z-direction and less than 1 cm in the transverse plane from the mean IP. By merging the information from all sub-detectors into a global discriminator akin to a probability ratio, the particle in the signal hemisphere must be designated as a pion; such hemisphere is created via thrust that is defined by the unit vector \(\vec{u}_{\text{thrust}}\) perpendicular to the line separating the signal and tag (see Fig. 1). The value of thrust is defined as \(V_{\text{thrust}} \equiv \sum_i |\vec{p}_i| |\vec{u}_{\text{thrust}}|/|\vec{p}_i|\), such that said value is the maximum, where \(\vec{p}_i\) is the momentum of each final-state particle in the CMS. The values are split using the vector \(\vec{u}_{\text{thrust}}\) and the thrust. In this manner, we have the production back-to-back of \(\tau\)-pairs in the CMS. The variable \(E_{\text{ECL}}/p\), which is the ratio of the energy deposited in the calorimeter to the momentum of charged particles, requires \(0 \leq E_{\text{ECL}}/p \leq 0.8\) to ensure that there are more pions in the tracks.

The background where \(e^- e^+\) produce in the final-state \(q\bar{q}\) with \(q = u, d, s, c\) (hadronic), \(l^- l^+\gamma\) (dileptonic), and \(e^- e^+ l^- l^+\) (two-photon) needs to be reduced. To identify the criterion for suppressing these backgrounds, we employ simulated events. The KKMC generator is used to create the \(e^- e^+ \rightarrow \tau^- \tau^+\) process \([13, 14]\). The \(\tau\) decays are handled with the software TUAOLA \([15]\), their radiative corrections by PHOTOS \([16]\). We use KKMC to simulate \(\mu^- \mu^+ (\gamma)\) and \(q\bar{q}\) production, PYTHIA \([17]\) for the fragmentation of the \(q\bar{q}\) pair; BabaYaga@NLO \([18-21]\) for \(e^- e^+ \rightarrow e^- e^- (\gamma)\) events, AAFH \([22-24]\) and TREPS \([25]\) for the non-radiative
final states $e^- e^+ l^- l^+$. The Belle II Software Framework (Basf2) [26] uses Geant4 [27] to simulate the response of the detector to the passage of the particles.

To reduce the background noise caused by $e^- e^- q\bar{q}$ we required zero neutrals in the final state. The photons utilized for $\pi^0$ reconstruction are in clusters with an energy of at least 0.1 GeV. Neutral pions are photon pairs with masses between [115,152] MeV/c². Events containing photons that meet the aforementioned criteria but are not included in the $\pi^0$ reconstruction and have an energy greater than 0.2 GeV are likewise excluded. Imposing a cut in the $T_{hrust} \leq 0.99$ we suppressed events from $e^+ e^-(\gamma)$ and $q\bar{q}$.

All other events that are not considered as signal are background, i.e., the remain events of low multiplicity ($ee, ee\mu\mu, \mu\mu, eeee$), $q\bar{q}$ ($q = u, d, s, c$), mixed ($B_0\bar{B}_0$) and charged ($B^+B^-$). To clean our signal $\tau^- \to \pi^- \nu_\tau$ (all other decays of the $\tau$ are considered as noise), we used a Machine Learning (ML) model based on Boosting Decision Trees (BDT) implemented in ROOT [28] through an environment for the processing and evaluation of multivariate classification such as TMVA (Toolkit for Multivariate Data Analysis) [29]. The new variable “BDT” was optimized with the purpose of extracting the best cut for the separation signal/background using the figure of merit (FOM) $2(\sqrt{N_{s信号}} + N_{b背景} - \sqrt{N_{b背景}})$, where $N_{s信号}$ is the number of the signal events and $N_{b背景}$ is the number of background events.

5. Estimation of the $\tau$ mass

5.0.1. $M_{edge}$ method

To determine the mass of the $\tau$, we performed a unbinned likelihood fit [30] using the following parametrization

$$F(M_{edge}) = fSPDF + (1 - f)BPDF,$$  \hspace{1cm} (39)

where $SPDF$ is the probability density function (PDF) for the signal, $BPDF$ is the PDF for the background and $f$ is a coefficient less than 1 that is subject to the normalization conditions. The empirical PDF for the signal has the following form

$$SPDF(M_{edge}) = (P_3 + P_4M_{edge} + P_5M_{edge}^2 + P_7M_{edge}^3) \times erf\left(\frac{M_{edge} - P_1}{P_2}\right) + P_8M_{edge}^2 + P_9M_{edge} + 1,$$  \hspace{1cm} (40)

where $P_1$ is the estimator for the mass of the $\tau$, the estimator is not exactly the mass of the tau, but it is assumed that it will be a range from the real value, so the value of the estimator must be corrected for the selected function in simulated data; for this, we use an official Monte Carlo sample (MC13a) for the production of $\tau$-pairs, for which we know the generation mass (1777 MeV) that was taken as the real value.

The $M_{edge}$ distribution for the background is flat, it can be modeled as a linear function. The PDF for the background is

$$BPDF(M_{edge}) = A_1 + A_2M_{edge}.$$  \hspace{1cm} (41)

For the variable $M_{border}$ in the mass window $1.610 < M_{edge} < 1.824$ and using the addition of PDF’s a value of the estimator $P_1 = 1765.80 \pm 0.60$ MeV was obtained [Fig. 5a]). The difference between the estimator and the true mass is $\Delta_m = 11.2$ MeV, and using this bias value, the final estimated in the remaining sample is adjusted [Fig. 5b)]. After correcting for the bias, we obtain a value for the mass of the $\tau$ of $m_\tau = 1772.40 \pm 7.38$ MeV. The uncertainties were summed in quadrature for this final value.

5.0.2. $M_{min}$ method

For this estimation, we used the same parametrization function presented above [Eq. (39)] and the same PDF for the background $BPDF$. The signal PDF is given by

$$SPDF(M_{min}) = (P_3 + P_4M_{min} + P_5M_{min}^2 + P_7M_{min}^3) \times erf\left(\frac{M_{min} - P_1}{P_2}\right) + P_8M_{min}^2 + P_9M_{min} + 1.$$  \hspace{1cm} (42)
A value for the estimator of $P_1 = 1777.70 \pm 0.16$ MeV is obtained with a difference from the real value of $\Delta m = -0.70$ MeV. Already considering the remaining sample [Fig. 6b)], and with bias, we corrected the estimation of the mass to obtain a value for the mass of the $\tau$ lepton of $m_\tau = 1777.06 \pm 0.47$ MeV.

### 5.0.3. $M_{\text{max}}$ method

For the variable $M_{\text{min}}$ we used the same previous parametrization function, and the same signal PDF (40).

In this case, it was possible to obtain a value for the mass estimator of $P_1 = 1775.16 \pm 0.14$ MeV [Fig. 7a)] for a difference in mass of $\Delta m = 1.84$ MeV. Correcting then with this bias to the value obtained in the adjustment to the remaining data, a value for the mass of $m_\tau = 1781.44 \pm 0.38$ MeV was obtained.

### 6. Conclusions

Three approaches for measuring tau lepton mass based on the solubility of kinematic equations were adopted in this study. The first is for a full solution zone called $M_{\text{edge}}$, where it was discovered that the density of events is higher towards the point of actual mass. The other two techniques, $M_{\text{min}}$ and $M_{\text{max}}$, used zero mass for the tau neutrino as a rough approximation. The key conclusion is that the $M_{\text{min}}$ approach is the best way for measuring the tau lepton mass in the (1 x 1)-prong topology (which has not yet been implemented in the Belle II collaboration), yielding a mass for the $\tau$ lepton of $m_\tau = 1777.06 \pm 0.47$ MeV. This result is obtained with the official Belle II simulated data in the MC13a campaign. It is crucial to note that the methods utilized in this study to do mass measurements in decays with semi-invisible end products are naturally biased due to the inability of reconstructing those decays in their entirety. Also, it is important to highlight that there is a relation between the precision and the uncertainties of the measurements with the redistribution of the events (without increase data) around the real value of the mass. For instance, the measurement made with the $M_{\text{min}}$ method presents major statistics distributed around the point of interest, and we obtained with this the more accurate measurement with the lower uncertainty. Therefore, looking at the $M_{\text{edge}}$ variable we can see that it has the greatest widening in its distribution, thus, it has the largest bias with a value of $\Delta m = 11.2$ MeV and the worst accuracy.

Finally, we have verified three approaches for investigating decay processes originating in leptonic colliders with invisible particles in the final state; such processes can be also BSM decays where it is necessary to deploy sophisticated ways to extract new information with the limited variables that the detectors can give us in order to discover new particles or impose competitive production upper bounds.


