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Entanglement entropy in high energy collisions of electrons and protons

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We investigate the proposal by Kharzeev and Levin of a maximally entangled proton wave function in Deep Inelastic Scattering of electrons and proton in the region of low Bjorken x. Using their proposed relation between parton number and entanglement entropy, we determine the latter using both conventional parton distribution functions and parton distribution functions obtained from an unintegrated gluon distribution subject to next-to-leading order Balitsky-Fadin-Kuraev-Lipatov (BFKL) evolution as well as from a dipole cross-section, subject to running coupling Balitsky-Kovchegov (rcBK) evolution. We compare our results to hadronic entropy obtained from final state hadron multiplicity.

Keywords: Quantum chromodynamics; deep inelastic scattering; quantum information; entanglement.

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1. Introduction

Deep Inelastic Scattering (DIS) of electrons on protons is for more than 60 years a key process for the exploration of nuclear structure, in particular of the proton. It was in the analysis of this process where for the first time the presence of point-like particles inside the proton manifested itself. When a highly virtual photon with virtuality $q^2 = -Q^2$, exchanged between scattering electron and proton, hits the proton, the photon seems to scatter on quasi-free particles, instead of a strongly bound state, usually associated with the proton.

In the DIS reaction, the proton can therefore be described as a collection of partons, where each parton is subject to a certain distribution function, called the parton distribution function or short "PDF". From a modern point of view, this simple parton model can be understood as a consequence of asymptotic freedom and collinear factorization. At high renormalization scales, the running coupling constant of Quantum Chromodynamics (the fundamental gauge theory of strong interactions) turns weak (asymptotic freedom) which then explains the observation of quasi-free partons, when the the proton is hit by a highly virtual photon; nowadays these partons are identified as quarks and gluons. Collinear factorization provides on the other hand a factorization framework, which factorizes the cross-section of the DIS reaction into a partonic cross-section (interaction of the electron with quarks and gluons through the exchange of a virtual photon) and corresponding PDFs, defined as certain matrix elements of quark and gluon field operators.

While this framework turns out to be very robust (resulting from a continuous development during various decades) and is now also applied to the description of more complicated reactions, such as proton-proton collisions, there remains at least one last concern: the proton is known to be a pure quantum state, albeit with its wave function in general unknown. Pure states are known to have vanishing von Neumann entropy and one therefore expects this to the case also for the proton. On the other hand, the parton model describes the proton as a lost collection of several quasi-free particles, which are only constrained by their corresponding parton distribution functions. Such a system of quasi-free particles is clearly no longer a pure quantum state, but it corresponds to a mixed state which non-zero von Neumann entropy. Now it is of course clear that the parton model does not provide a complete description of the proton; it only applies to the description of the proton structure within the DIS process, whenever the proton is probed by a highly virtual photon. Nevertheless there is a need to systematically coincide both pictures.

In Ref. [1] Kharzeev and Levin made an interesting proposal on how to reconcile both pictures, which we will study in some detail in the following. They proposed that the entropy observed in the parton model arises due to entanglement of colored degrees of freedom in the proton and should be therefore identified with entanglement entropy. Referring to the classical analysis by Gribov, Ioffe and Pomeranchuck [2], they recall that the virtual photon resolves in the DIS reaction only an area of a size inversely proportional to the photon virtuality $A_{\text{resolved}} \sim 1/Q^2$. Depending on the value of Q^2, there remains therefore a significant region of the proton which is unobserved. Such a partial observation of a pure state is known to lead to entanglement entropy [3–5]. In simple terms, summing for the density matrix of a pure but entangled state over unobserved components, one obtains the density matrix of a mixed state. The latter is characterized by non-vanishing von Neumann entropy. Technically the description of the transition of the pure quantum state towards a mixed state is very challenging: perturbative QCD tools work only well in the presence of a $Q^2 \gg \Lambda_{\text{QCD}^2}$, where Λ_{QCD^2} denotes the QCD characteristic scale of the order of a few hundred MeV. A description of the pure state, expected for $Q^2 \simeq 0$ is therefore not possible using these kinds of methods. Kharzeev and Levin proposed therefore the use of a certain 1+1 dimensional dipole model, to get a first glimpse on the transition from the pure towards the mixed state. This model turns out to catch surprisingly well the essential features of this transition, in particular if confronted with data. In the following we will present a few details on this model as well its applications to phenomenology. For more details we refer the interested reader to the publications [6,7] as well as our recent preprint [8], which deals with the extension of the formalism to diffractive DIS.

2. Review of the theoretical framework

In Ref. [1] the following model, original proposed for a different purpose in Refs. [9, 10] has been used. The model is based on the low x description of the DIS reaction, see also Fig. 1. In this kinematic limit, the center of mass energy W with $W^2 \simeq (1 - x/x)Q^2$ is large and the scattering virtual photon and proton are separated by a large relative boost factor. If one then turns to the proton rest frame, the proton degrees of freedom do not evolve if x is decreased: the entire evolution takes place within the virtual photon. In this picture, the virtual photon forms first a color dipole (a quark-antiquark pair) before finally interacting with the proton, *i.e.*, the formation of the dipole is time dilated in the proton rest frame. Decreasing x further, one increases the center of mass energy which in turn leads to the subsequent emission of gluons; seen from the proton rest frame, the original color dipole, develops a cascade of quarks and gluons. In the large color approximation, this partonic cascade can be cast into a cascade of color dipoles. This cascade can then be described within a simple 1+1 dimensional model, which is the basis for the following analysis. Within this model, such a cascade can be described through the following simple equation,

$$\frac{d}{dy}p_n(y) = -\Delta n p_n(y) - \Delta(n-1)p_{n-1}(y), \qquad (1)$$

where $p_n(y)$ denotes the probability to encounter n dipoles at a certain value of y and Δ yields the BFKL intercept in the (1+1) dimensional model while $y = \ln 1/x$. Evolving towards smaller values of x requires therefore the emission of another dipole with probability Δ multiplied by the number of dipoles, *i.e.* , each dipole is a potential source of a new

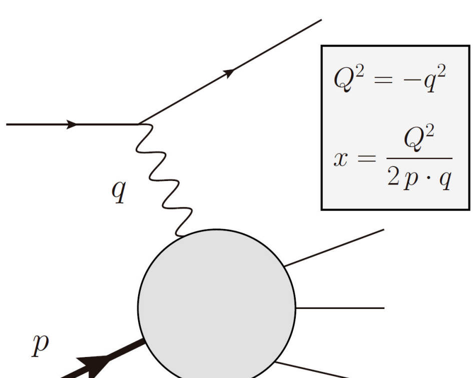

FIGURE 1. Schematic DIS reaction with the kinematic variables Q^2 and Bjorken x.

dipole. Imposing finally the requirement that $p_1(0) = 1$ and $p_{n>1}(0) = 0$, one finds

$$p_n(y) = e^{-\Delta y}\left(1 - e^{-\Delta y}\right)^{n-1}, \qquad (2)$$

as a solution. The above initial condition imply that at $y = 0$ corresponding to $x = 1$ the system consists of a single dipole, which is the analog of a pure quantum state. Decreasing x, one encounters on the other hand the above mentioned mixed state. Von Neumann entropy is then obtained from there, using

$$S_{\text{part.}} = -\sum p_n \ln p_n. \qquad (3)$$

The mean number of dipoles is on the other hand found as

$$\langle n \rangle = \sum_n n p_n = e^{\Delta y}. \qquad (4)$$

In Ref. [1] identify this mean number of dipoles as the gluon PDF, while in Ref. [6] this mean number of dipoles has been taken as the sum of quark and gluon PDFs, corresponding to the mean number of partons in the proton. Note that taking into account that PDFs have a certain interpretation as number densities, *i.e.*, their integral over momentum fraction x yields the expectation value of the parton number operator, it is somehow natural to interpret Eq. (4) as the gluon distribution. Even though $xg(x)$ usually denotes the momentum fraction carried by gluons, while the number density is associated with the integral of $g(x)$, the above identification is correct, since one really determines the mean value of the number of partons per $\ln(1/x)$, *i.e.*, $\langle dn/dy \rangle$, see also the discussion in Ref. [7].

The above expression allows us to determine partonic entropy as a function of the average number of partons in the system. In the low x-region $x \to 0$, corresponding to $y \to \infty$, one finds finally

$$S_{\text{part.}} \simeq \ln\langle n \rangle, \qquad (5)$$

i.e. partonic entropy coincides with the logarithm of the average parton number. Note that if one assumes that $n = 1, \ldots, n_{\max}$ and $n_{\max} \simeq \langle n \rangle$, it is possible to identify this entropy as the entropy of a homogeneous distribution, which yields the maximal value of entropy. This observation is then the origin of the statement that the proton probed in DIS is “maximally entangled”. Combining both results Eq. (4) and Eq. (5) it is then possible to arrive at phenomenological predictions and to compare to hadronic entropy, determined from final state multiplicities in DIS experiments, such as H1 at HERA.

Since the detection of neutral hadrons is experimentally challenging, this hadronic multiplicity distribution is usually only extracted for charged hadrons. Since pions are the predominantly produced hadron species, one can as a first estimate assume that the total number of produced hadrons is roughly $3/2$ times the number of charged hadrons observed in experiment, if one assumes that final state gluons turn with equal probabilities into positively, negatively, and neutral pion states. The number of gluons and possibly sea quarks which yield charged hadrons is therefore approximate the fraction $2/3$ of the total parton number. This suggests correcting the partonic entropy by a corresponding factor,

$$S_{\text{part.}} \to S_{\text{charged}} = S_{\text{part.}} + \ln(2/3). \tag{6}$$

Clearly this factor is not exact, but merely an estimate of order of magnitude.

3. Numerical results

To compare to hadronic entropy extracted from H1 data in Ref. [11], we require suitable sets of parton distribution functions. In this work, our comparison uses two low x frameworks with and without high density or saturation effects. Investigating the relevance of such effects is of particular interest, if one probes the region $Q^2 \to 0$ for small values of x. While this is not precisely the region where the proton turns again into a pure state, it is nevertheless a region of phase space where the area probed by the photon is large; seen from the low x perspective, one also expects that the dominant contributions to the DIS process arise from a) the splitting of the virtual photon into a color dipole (consisting of a quark anti-quark pair) which subsequently scatters b) on a coherent gluonic field, which characterizes the proton target. One would then expect for this case a reduction in the observed entropy. Our comparison is then based on the unintegrated gluon distribution obtained from the HSS fit [12,13], which provides an unintegrated gluon distribution subject to NLO BFKL evolution (which does not contain high density effects) as well as the AAMQS fit [14], which contains high density effects and which solves the leading order running coupling improved BK equation (rcBK). Last but not least, as an additional cross-check of our results, we will also employ leading order HERA PDFs [15] for a comparison to data. To determine PDFs *i.e.* integrated parton densities from their unintegrated counterparts, we employ the Catani-Hautmann procedure [16]: The gluon density is obtained from

$$xg(x,\mu_F) = \int_0^{\mu_F^2} d\boldsymbol{k}^2 \mathcal{F}(x,\boldsymbol{k}^2), \tag{7}$$

where $\mathcal{F}(x,\boldsymbol{k}^2)$ is the unintegrated gluon distribution, while for the quark PDF one uses

$$x\Sigma(x,Q) = \int_0^\infty \frac{d\boldsymbol{\Delta}^2}{\boldsymbol{\Delta}^2} \int_0^\infty d\boldsymbol{k}^2 \int_0^1 dz \times \Theta\left(Q^2 - \frac{\boldsymbol{\Delta}^2}{1-z} - z\boldsymbol{k}^2\right) \tilde{P}_{qg}\left(z, \frac{\boldsymbol{k}^2}{\boldsymbol{\Delta}^2}\right) \times \mathcal{F}(x/z, \boldsymbol{k}^2), \tag{8}$$

where the splitting function reads [16]

$$\tilde{P}_{qg}\left(z, \frac{\boldsymbol{k}^2}{\boldsymbol{\Delta}^2}\right) = \frac{\alpha_s 2n_f}{2\pi} T_F \frac{\boldsymbol{\Delta}^2}{[\boldsymbol{\Delta}^2 + z(1-z)\boldsymbol{k}^2]^2} \times \left[z^2 + (1-z)^2 + 4z^2(1-z)^2 \frac{\boldsymbol{k}^2}{\boldsymbol{\Delta}^2}\right], \tag{9}$$

and μ_F denotes the factorization scale which we identify for the current study with the photon virtuality Q. $\boldsymbol{k}$ denotes the gluon transverse momentum and $\boldsymbol{\Delta} = \boldsymbol{q} - z\boldsymbol{k}$ with $\boldsymbol{q}$ the t-channel quark transverse momentum; $T_F = 1/2$.

Our results, including a comparison to data are then shown in Figs. 2 and 3. For this comparison we obtain the relevant mean number of partons through the following average over the bin size in Q^2,

$$\langle \bar{n}(x,Q^2)\rangle_{Q^2} = \frac{1}{Q^2_{\text{max}} - Q^2_{\text{min}}} \times \int_{Q^2_{\text{min}}}^{Q^2_{\text{max}}} dQ^2 \left[xg(x,Q^2) + x\Sigma(x,Q^2)\right], \tag{10}$$

which finally yields

$$\langle S(x,Q^2)\rangle_{Q^2} = \ln\langle \bar{n}(x,Q^2)\rangle_{Q^2}. \tag{11}$$

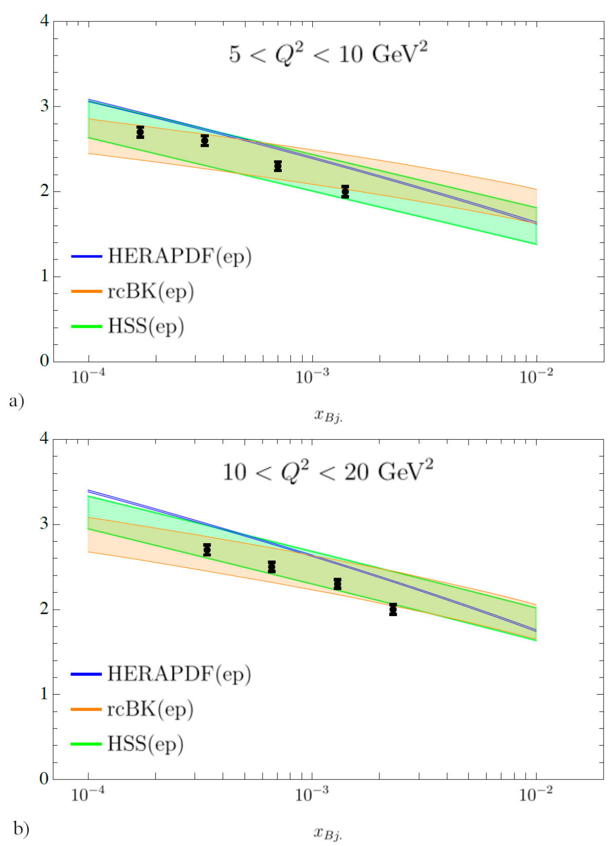

FIGURE 2. Partonic entropy corrected for charged hadrons only $\ln(xg + x\Sigma) + \ln(2/3)$ versus Bjorken x. Results are compared to the final state hadron entropy derived from the charged multiplicity distributions measured by the H1 collaboration [11] for track pseudo-rapidities η^* in the hadronic centre-of-mass frame restricted to the range $0 < \eta^* < 4$.

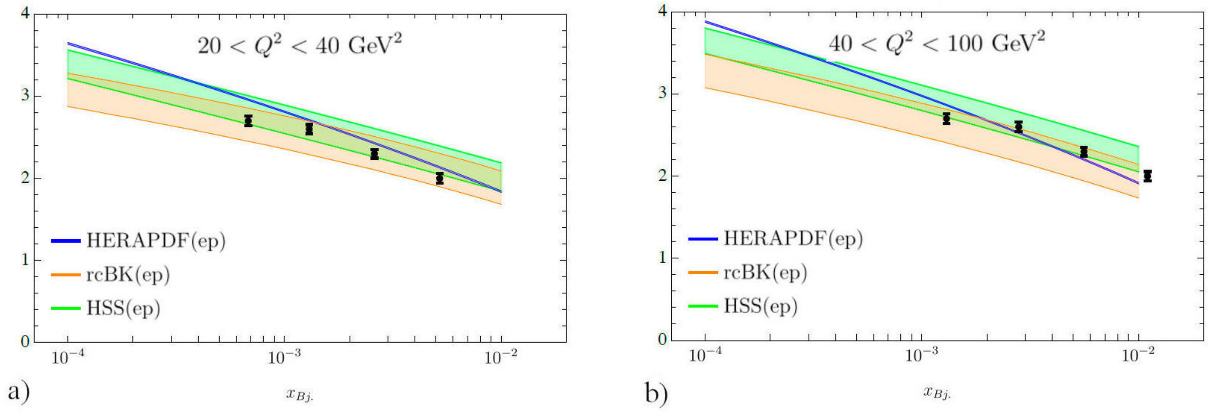

FIGURE 3. Same as Fig. 2, but for the bins $20 < Q^2/\mathrm{GeV}^2 < 40$ and $40 < Q^2/\mathrm{GeV}^2 100$.

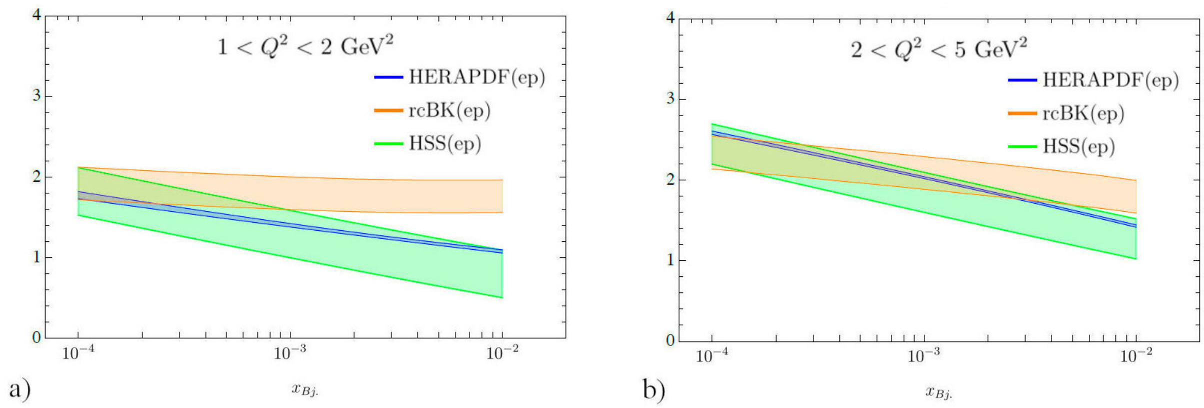

FIGURE 4. Partonic entropy corrected for charged hadrons only, $\ln(xg + x\Sigma) + \ln(2/3)$ versus Bjorken x, calculated for low Q^2 bins. The result demonstrates saturation of entropy at low Q^2 and low x.

In this expression, $g(x, \mu_f^2)$ denotes the gluon distribution function at the factorization scale μ_f and $\Sigma(x, \mu_f^2) = \sum_{a=1}^{n_f} \left(q_a(x, \mu^2) + \bar{q}_a(x, \mu^2)\right)$ the quark flavor singlet distribution, with $q(x, \mu^2)$ and $\bar{q}(x, \mu^2)$ quark and antiquark distribution functions for flavor a. We find in general a very good agreement of both the NLO BFKL (HSS) and rcBK framework. While the growth obtained from leading order HERA PDF is slightly too steep, the general features of the data are also well described.

In Fig. 4 we finally show our predictions for low Q^2 values. As somehow expected we find that the x dependence found for leading order HERA PDFs agrees qualitatively with the one found for NLO BFKL (since both assume a linear, perturbative x-dependence) while the rcBK framework leads to a flatter x-dependence.

4. Conclusions

The currently presented results are certainly subject to assumptions and certain approximations. In particular a direct identification of entropy (which is a physical observable) and the logarithm of the parton distribution functions (which is a matrix element of certain QCD field operators) is somehow not possible as an all order result. PDFs are subject to renormalization and are therefore at first unphysical. The proposed relation can be therefore at best serve as a leading order relation, similar to the relation between the structure function F_2 and the sum over quark PDFs times the charge squared of the corresponding quark. In general it is therefore needed to make this framework more robust and to derive at an ultimate all order definition of this relation. Nevertheless, the results obtained from the 1+1 dimensional model are at the very least impressive. Relating PDFs to entropy in the above simple way, gives us indeed a possibility to describe correctly H1 data. We find this observation very encouraging and believe that it justifies further investigations into this direction. For further discussion of related aspects and a new proposal related to diffractive DIS, we refer the interested reader to the publications [6, 7] as well as our recent preprint [8].

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