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Quantum entropy production rate of quantum markov semigroups

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This paper explores various perspectives on Quantum Detailed Balance and the Entropy Production Rate within the framework of Quantum Markov Semigroups. Using the generators of these semigroups, formulated according to the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) theorem, and their respective adjoints, we identify two contrasting families of Quantum Markov Semigroups. The first family demonstrates a situation where the condition for Quantum Detailed Balance is violated, yet the entropy production rate is zero. In contrast, the second family reveals cases where the quantum entropy production rate aligns with an interpretation of Quantum Detailed Balance. These findings provide insights into the relationship between quantum detailed balance and entropy production rate in open systems.

Keywords: Quantum markov semigroups; entropy production; equilibrium; detailed balance; reversibility; circulant; G-circulant.

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1. Introduction

The concept of equilibrium states in physical systems is well established, with several conditions characterizing such states, including detailed balance and zero entropy production. In the case of classical Markov chains, Qian *et al*. [18] demonstrated the equivalence of these two equilibrium criteria using Kalpazidou's cycle representation for Markov chains [14]. In contrast, in the quantum case the notion of non-equilibrium states is far more intricate, as it encompasses a wide range of complex behaviors.

In the theory of open quantum systems, the Lindblad master equation, also known as the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation, is integral to this theory, as it represents the most general framework for describing Markovian quantum dynamics. A wonderful reference where the derivation of this important equation, both from the completely positive, trace preserving framework and from the microscopic dynamics one, can be found in [16]. The semigroups arising from this equation, namely, Quantum Markov Semigroups (QMS), are the key object describing the evolution of an open quantum system. This note aims to present the main results of the Quantum Entropy Production Rate (QEPR) as a means to characterize the equilibrium or Quantum Detailed Balance in the framework of QMS.

The full details of the exposition encompass the span of [4, 5] and [9–12] which form part of the program outlined in Reference [1], namely, *to look for some interesting GKSL generators with properties that are rich enough to go beyond the equilibrium situation, but concrete enough to allow explicit study and, in some cases, explicit solutions*. In the last section we present recent developments for G-circulant QMS that generalize known results of circulant QMS, where the underlying group structure is $\mathbb{Z}_n$, to a general, possibly non-commutative, finite group G. This family, recently introduced in Ref. [6], admits non-equilibrium steady states but exhibits nice symmetries that allow explicit computation of the QEPR. The symmetry properties of our semigroups arise from the group structure on the state space of the associated classical Markov chain.

2. Entropy production rate for Markov chains

According to Qian *et al*. [18], the Classical Entropy Production Rate of an irreducible Markov chain $\xi = (\xi_t)_{t\in\mathbb{R}}$ with intensity transition matrix $Q = (q_{ij})_{i,j\in S}$ and stationary measure $\pi = (\pi_i)_{i\in S}$, over a finite state space S is given by

$$e_p = \frac{d}{dt} H\left(\mathbb{P}_t, \mathbb{P}_t^-\right)\Big|_{t=0}, \tag{1}$$

where H is the Kullback-Leibler divergence or relative entropy for probability distributions and $\mathbb{P}_t, \mathbb{P}_t^-$ are the restriction to $[0, t]$ of the distributions of ξ and the reverse chain ξ^-, respectively. The Q-matrix of the chain $\xi^-$is given by $Q^- = \left(\pi_i q_{ij}/\pi_j\right)_{i,j}$. This chain is also known as the adjoint chain with respect to π. A closed explicit expression of the entropy production rate may be derived

$$e_p = \frac{1}{2}\sum_{i,j\in S}(\pi_i q_{ij} - \pi_j q_{ji}) \log \frac{\pi_i q_{ij}}{\pi_j q_{ji}}.$$

It is immediate to see that the reversibility of the process, which is equivalent to the well-known detailed balance condition

$$\pi_i q_{ij} = \pi_j q_{ji}, \ \ i, j \in S, \tag{2}$$

may be characterized by the zero entropy production rate $e_p = 0$.

3. Quantum detailed balance

A Quantum Markov Semigroup (QMS) is a family of trace preserving, completely positive maps $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ acting on the bounded operators $\mathcal{B}(\mathsf{h})$ of a Hilbert space h satisfying the semigroup properties and a continuity condition. The celebrated Gorini-Kossakowski-Sudarshan and Lindblad theorem establishes the structure of the infinitesimal generator $\mathcal{L}$ of a uniformly continuos QMS $\mathcal{T}_t = e^{t\mathcal{L}}$.

$$\mathcal{L}(x) = i[H,x] - \frac{1}{2}\sum_{k\geq 1}(L_k^* L_k x - 2L_k^* x L_k + x L_k^* L_k), \tag{3}$$

where $H = H^*$, $L_k \in \mathcal{B}(\mathsf{h})$ and the series $\sum_{k\geq 1} L_k^* L_k$ is convergent in norm. The QMS in the Schrödinger picture is obtained via the duality relation

$$\mathrm{Tr}\left(\rho\mathcal{L}(x)\right) = \mathrm{Tr}\left(\mathcal{L}_*(\rho)x\right), \;\; x \in \mathcal{B}(\mathsf{h}), \;\; \rho \in L_1(\mathsf{h}).$$

The operator $\mathcal{L}_*$ defined by the above is referred to as the generator of the predual semigroup $\mathcal{T}_* = (\mathcal{T}_{*t})_{t\geq 0}$. A state ρ is said to be a stationary state for $\mathcal{T}$ if $\mathcal{L}_*(\rho) = 0$, equivalently $\mathrm{Tr}\left(\rho\mathcal{L}(x)\right) = 0$ for all $x \in \mathcal{B}(\mathsf{h})$.

Several notions of quantum detailed balance for QMS have been proposed. Roughly speaking, all of these conditions are based on a notion of dual or adjoint. Indeed, for uniformly continuous QMS on $\mathcal{B}(\mathsf{h})$ with h a separable Hilbert space, a notion of detailed balance was introduced first by Agarwal [2], see also the works of Alicki [3] and Frigerio-Gorini-Kossakowski-Verri [15]. A QMS satisfies the quantum detailed balance condition in the sense of [3, 15] with respect to a stationary state ρ, if there exists an operator $\tilde{\mathcal{L}}$ on $\mathcal{B}(\mathsf{h})$ and a self-adjoint operator K on h such that for all $x, y \in \mathcal{B}(\mathsf{h})$ the following relations hold:

$$\begin{aligned} \mathrm{Tr}(\rho\tilde{\mathcal{L}}(x)y) &= \mathrm{Tr}(\rho x\mathcal{L}(y)), \\ \tilde{\mathcal{L}}(x) - \mathcal{L}(x) &= 2i[K,x]. \end{aligned} \tag{4}$$

The operator $\tilde{\mathcal{L}}$ is called the ρ-adjoint of $\mathcal{L}$. For a wide class of GKSL generators, including those deduced from the stochastic limit of quantum theory, the ρ-adjoint coincides with the time-reversed generator if quantum detailed balance holds. Therefore, $\tilde{\mathcal{L}}$ can be considered as an extension of the time-reversed GKSL generator to the non-equilibrium situation, see Accardi-Fagnola-Quezada [1] and the references therein.

Other notions of quantum detailed balance have been introduced by Fagnola and Umanità [11, 12]. The main idea is to decompose the invariant state ρ into two pieces or, equivalently, define the ρ-adjoint using the inner product $\langle a,b\rangle_s = \mathrm{Tr}(\rho^{1-s}a^*\rho^s b)$ for $0 \leq s \leq 1/2$, and replace relations (4) by

$$\begin{aligned} \mathrm{Tr}(\rho^{1-s}\mathcal{L}'(x)\rho^s y) &= \mathrm{Tr}(\rho^{1-s}x\rho^s\mathcal{L}(y)), \\ \mathcal{L}'(x) - \mathcal{L}(x) &= 2i[K,x]. \end{aligned} \tag{5}$$

Due to the non-commutativity, these two definitions are not equivalent in general. Clearly, detailed balance in the sense of (4) corresponds with the case $s = 0$ in (5).

It has been proved [12] that among all $s \in [0, 1/2]$, the two cases above are the two prototypical values: $s = 0$ obtaining $\tilde{\mathcal{L}}$ and $s = 1/2$ obtaining $\mathcal{L}'$. The latter corresponds with the KMS symmetry discussed by Petz, [17] Goldstein and Lindsay [13] and Cipriani [7,8] *i.e.*, the QMS $\mathcal{T}$ is KMS symmetric if and only if $\mathcal{T} = \mathcal{T}'$.

Definition 1. *We say $\mathcal{L}$ and a faithful invariant state ρ satisfy*

1. *the **SQDB** condition if*

$$\mathcal{L}(x) - \mathcal{L}'(x) = i[K,x], \quad K = K^*,$$

where $\mathcal{L}'$ is the generator of the KMS-adjoint semigroup $\mathcal{T}'$ defined by

$$\mathrm{Tr}\left(\rho^{\frac{1}{2}}\mathcal{T}_t'(x)\rho^{\frac{1}{2}}y\right) = \mathrm{Tr}\left(\rho^{\frac{1}{2}}x\rho^{\frac{1}{2}}\mathcal{T}_t(y)\right),$$

for all $x, y \in \mathcal{B}(\mathsf{h})$.

3.1. The Θ-KMS adjoint QMS

Associated with an anti-unitary operator θ, a reversing operation on the observables is defined as $\Theta(x) = \theta x^*\theta$. This reversing operation allows us to incorporate in the KMS symmetry, typical quantum notions such as that of parity of observables. We include some useful properties of anti-unitary operators in the Appendix A.

Definition 2. *A uniformly continuous QMS $(\mathcal{T})_{t\geq 0}$ with a faithful invariant state ρ and a KMS adjoint semigroup $(\mathcal{T}')_{t\geq 0}$, generated by $\mathcal{L}'$, satisfies a Standard Quantum Detailed balance condition with respect to the reversing operation Θ (namely Θ-SQDB) if*

$$\mathcal{T}_t = \Theta\circ\mathcal{T}_t'\circ\Theta.$$

where Θ-SQDB seems to be the most appropriate extension of detailed balance to the non-commutative case; we present a notion of adjoint associated with Θ-SQDB condition as follows.

Definition 3. *Given a reversing operation Θ and a uniformly continuous QMS $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$* on $\mathcal{B}(\mathsf{h})$ *with a faithful invariant state ρ, we say that $\mathcal{T}$ admits a Θ-KMS adjoint (or dual) QMS with respect to the state ρ if there exists a QMS $\mathcal{T}^\Theta = (\mathcal{T}_t^\Theta)_{t\geq 0}$ satisfying the duality relation for all $x, y \in \mathcal{B}(\mathsf{h})$*

$$\mathrm{Tr}\left(\rho^{\frac{1}{2}}\Theta(x^*)\rho^{\frac{1}{2}}\mathcal{T}_t(y)\right) = \mathrm{Tr}\left(\rho^{\frac{1}{2}}\Theta(\mathcal{T}_t^\Theta(x^*))\rho^{\frac{1}{2}}y\right). \tag{6}$$

In the above case, it can be seen that $\mathcal{T}^\Theta = \Theta\circ\mathcal{T}_t'\circ\Theta$. Conditions for the existence of the uniformly continuos QMS $\mathcal{T}^\Theta$ have been studied, for instance the weighted detailed balance condition introduced in Ref. [1].

From now on, we consider the class of uniformly continuous QMS $\mathcal{T}$ that admits a uniformly continuous Θ-KMS adjoint QMS $\mathcal{T}_t^\Theta$. We denote by $\mathcal{T}_{*t}$ and $\mathcal{T}_{*t}^\Theta$ the corresponding predual semigroups.

3.2. States associated with to trace-preserving CP maps

In this section we present a variation of the Choi-Jamiołkowski where (in the infinite dimensional case) a state ρ is used to control any possible divergences. It was first introduced in see [4].

Definition 4. *The Fagnola-Rebolledo entangled state ω_ρ on $\mathcal{B}(\mathsf{h}\otimes\mathsf{h})$ is defined as*

$$\omega_\rho = \left| \sum_i \rho^{\frac{1}{2}} e_i \otimes e_i \right\rangle \left\langle \sum_j \rho^{\frac{1}{2}} e_j \otimes e_j \right|.$$

Clearly ω_ρ depends both on ρ and the chosen orthonormal basis. A simple computation shows that it is a state, introduced for the first time by Fagnola and Rebolledo [9] in 2009.

Definition 5. *Let ρ be a state in $\mathcal{B}(\mathsf{h})$, $\{e_i\}_i$ an orthonormal basis of h and take $\Phi_* \in \mathcal{CP}\big(L_1(\mathsf{h})\big)$, the space of all bounded CP maps on $L_1(\mathsf{h})$. We define the state $\mathcal{E}_\rho(\Phi_*)$ in $\mathcal{B}(\mathsf{h}\otimes\mathsf{h})$ as*

$$\mathcal{E}_\rho(\Phi_*) = (\mathbb{1}\otimes\Phi_*)(\omega_\rho).$$

If one considers the trace state $\rho = (1/n)\mathbb{1}$, the above reduces to the usual Choi-Jamiołkowski isomorphism.

Thus, we can associate to each map $\mathcal{T}_{*t}$ and $\mathcal{T}_{*t}^{\Theta}$ a state $\mathcal{E}_\rho(\mathcal{T}_{*t})$ and $\mathcal{E}_\rho(\mathcal{T}_{*t}^{\Theta})$ respectively, in $\mathsf{h}\otimes\mathsf{h}$.

The von-Neumann relative entropy or Quantum Relative Entropy, is well known in the literature as a generalization of the classical Relative Entropy which is also known as the Kullback-Leibler divergence.

Definition 6. *The Quantum Relative Entropy (QRE) of two states η and σ is defined as*

$$S(\eta,\sigma) = tr\Big(\eta\log\eta - \eta\log\sigma\Big),$$

if $\ker(\sigma)\subset\ker(\eta)$ and ∞ otherwise.

It is well known that $S(\eta,\sigma)\geq 0$ for all η, σ. Moreover, $S(\eta,\sigma)=0$ if and only if $\eta=\sigma$. In the following section, we define the Quantum Entropy Production Rate in analogy to (1).

3.3. Quantum entropy production rate

The relative entropy $S(\mathcal{E}_\rho(\mathcal{T}_{*t}),\mathcal{E}_\rho(\mathcal{T}_{*t}^{\Theta}))$ is a measure of the deviation from Θ-SQDB of the semigroup $\mathcal{T}$. Moreover, one can define the rate of change of relative entropy as follows.

Definition 7. *The Quantum Entropy Production Rate of the uniformly continuos QMS $\mathcal{T}_*$, with respect to the invariant state ρ, is defined as*

$$e_p(\mathcal{T}_*,\rho) = \left.\frac{d}{dt}S(\mathcal{E}_\rho(\mathcal{T}_{*t}),\mathcal{E}_\rho(\mathcal{T}_{*t}^{\Theta}))\right|_{t=0}. \tag{7}$$

Notice that in the last definition there is no reference to the orthonormal basis used to compute the $\mathcal{E}_\rho$ states of $\mathcal{T}_*$ and $\mathcal{T}_*^{\Theta}$, this is justified by the following theorem.

Theorem 8. *Let $\{e_i\}_i$ be an orthonormal basis of h, Φ_*, Ψ_* two CP trace preserving maps acting on $L_1(\mathsf{h})$, and $\mathcal{E}_\rho(\Phi_*)$, $\mathcal{E}_\rho(\Psi_*)$, the $\mathcal{E}_\rho$ states on $\mathcal{B}(\mathsf{h}\otimes\mathsf{h})$, associated with Φ_* and Ψ_*, respectively. The relative entropy $S\big(\mathcal{E}_\rho(\Phi_*),\mathcal{E}_\rho(\Psi_*)\big)$ does not depend on the orthonormal basis $\{e_i\}_i$.*

Proof. It suffices to prove that if $\{e_i'\}_i$ is another orthonormal basis of h and $\mathcal{E}_\rho'(\Phi_*)$, $\mathcal{E}_\rho'(\Psi_*)$ are the corresponding states associated with Φ_* and Ψ_*, then

$$S\big(\mathcal{E}_\rho'(\Phi_*),\mathcal{E}_\rho'(\Psi_*)\big) = S\big(\mathcal{E}_\rho(\Phi_*),\mathcal{E}_\rho(\Psi_*)\big). \tag{8}$$

Using the properties of the anti-unitary operator θ, it follows that $U\theta U^*\theta\otimes\mathbb{1}$ is a unitary operator, this identity follows from an application of the well known invariance of relative entropy with respect to unitary conjugations, which is a consequence of its monotonicity with respect to CP maps (Petz-Uhlmann Theorem). □

Thus, in practice one uses the basis of ρ and takes θ to be the conjugation with respect to this basis, which results in the following commutation, if $u\in\mathsf{h}$ then

$$\theta\rho u = \sum_i \rho_i\langle u,e_i\rangle e_i = \sum_i \rho_i\langle e_i,\theta u\rangle e_i = \rho\theta u.$$

Theorem 9. *Let $(\mathcal{T}_t)_{t\geq 0}$ be a QMS with a faithful invariant state ρ such that $\overline{Im\rho^{1/2}}=\mathsf{h}$ and Θ-KMS adjoint $\mathcal{T}_t^{\Theta}$, then the following are equivalent:*

i) *$(\mathcal{T}_t)_{t\geq 0}$ satisfies a Θ-SQDB condition.*

ii) *The Quantum Relative Entropy vanishes*

$$S\Big(\mathcal{E}_\rho(\mathcal{T}_{*t}),\mathcal{E}_\rho(\mathcal{T}_{*t}^{\Theta})\Big) = 0 \quad \text{for all } t\geq 0.$$

Consequently, the Θ-SQDB condition implies that $e_p(\mathcal{T}_*,\rho)=0$.

As a consequence of the above theorem, we call a non-equilibrium steady state any invariant state ρ of $\mathcal{T}$ for which $e_p(\mathcal{T}_*,\rho)\neq 0$.

It turns out that the converse of Theorem 9 does not hold. Indeed, consider the semigroup on $\mathsf{h}=\mathbb{C}^2$ with GKSL generator given by

$$\mathcal{L}(x) = L_1^*xL_1 + L_2^*xL_2 + G^*x + xG, \tag{9}$$

where $L_1=|e_2\rangle\langle e_1|$, $L_2=|e_1\rangle\langle e_2|$, $G=-(1/2)\mathbb{1}-iH$ and and $H=ik(L_1-L_2)$, $k\in\mathbb{R}-\{0\}$. Let θ be the conjugation w.r.t. the canonical basis and $\rho=(1/2)\mathbb{1}$, which is a faithful invariant state of $\mathcal{L}$. On the one hand, it can be checked that the Θ-SQDB condition is satisfied by $\mathcal{L}$ if and only if $G=G^*$, which directly fails. On the other hand, the explicit computation of the QEPR yields that $e_p(\mathcal{T}_*,\rho)=0$.

We finish this section by a condition under which the Θ-KMS adjoint and the KMS adjoint coincide, rendering the computation of the QEPR simpler.

Definition 10. *A CP operator Φ is called parity preserving, with respect to the anti-unitary operator θ, if it commutes with the time reversing operation $\Theta(x)=\theta x^*\theta$, i.e., $\Theta(\Phi(x))=\Phi(\Theta(x))$ for all $x\in\mathcal{B}(\mathsf{h})$.*

A QMS $(\mathcal{T}_t)_{t\geq 0}$ is parity-preserving w.r.t. θ if and only if $\mathcal{T}_t$ is parity preserving w.r.t. θ for every $t \geq 0$.

Corollary 11. *The QMS is parity-preserving, the Quantum Relative Entropy satisfies*

$$S(\mathcal{T}_{*t}, \mathcal{T}_{*t}^{\Theta}) = S(\mathcal{T}_{*t}, \mathcal{T}_{*t}').$$

In other words, the QEPR can be computed using either the usual KMS adjoint or the Θ-KMS adjoint.

Proof. Recall that $\theta^2 = \mathbb{1}$, it is then immediate that $\mathcal{T}_t' = \mathcal{T}_t^{\Theta}$ when the QMS is parity-preserving. □

4. G-circulant QMS

Let G be a finite group and consider $\ell_2(G)$, the Hilbert space of all functions $\alpha : G \to \mathbb{C}$ with point-wise addition and inner product $\langle \alpha, \beta\rangle = \sum_{g\in G} \overline{\alpha_g}\beta_g$. Denote by $(e_g)_{g\in G}$ the orthonormal basis of $\ell_2(G)$ where $e_g(h) = 1$, if $g = h$, and $e_g(h) = 0$, otherwise. Each element $\alpha \in \ell_2(G)$ can be then written as $\alpha = \sum_{g\in G} \alpha_g e_g$.

The *left regular representation* of G is the unitary representation that maps each group element $g \in G$ to the operator $U_g : \ell_2(G) \to \ell_2(G)$ determined by $U_g(e_h) = e_{gh}$ for every $h \in G$. Note that U_g and U_g^* can be written in terms of the matrix units $|e_k\rangle\langle e_l|$ as

$$U_g = \sum_{k\in G} |e_k\rangle\langle e_{g^{-1}k}| \quad \text{and} \quad U_g^* = \sum_{k\in G} |e_k\rangle\langle e_{gk}|. \quad (10)$$

The *right regular representation* of G is the unitary representation that maps every $g \in G$ to the operator $V_g : \ell_2(G) \to \ell_2(G)$ determined by $V_g(e_h) = e_{hg}$ for every $h \in G$. Note that V_g and V_g^* can be written in terms of the matrix units $|e_k\rangle\langle e_l|$ as

$$V_g = \sum_{k\in G} |e_k\rangle\langle e_{kg^{-1}}| \quad \text{and} \quad V_g^* = \sum_{k\in G} |e_k\rangle\langle e_{kg}|. \quad (11)$$

The families of left G-circulant (resp. right G-circulant) matrices are defined as the matrix algebra generated by $\{U_g : g \in G\}$ (resp. $\{V_g : g \in G\}$). These two families commute with each other and coincide whenever the G is abelian. In particular, when $G = \mathbb{Z}_n$ the algebra of circulant matrices is recovered.

Definition 12. *A linear map $\mathcal{L} : \mathcal{B}(\ell_2(G)) \to \mathcal{B}(\ell_2(G))$ is said to be a left G-circulant GKSL generator if there is a vector $(\alpha_g)_{g\in G} \in \ell_2(G)$ with $\alpha_g \geq 0$ for $g \neq e$ and $\alpha_e = -1 = -\sum_{g\neq e} \alpha_g$ and $H = H^*$ such that*

$$\mathcal{L}(x) = i[H, x] + \sum_{g\in G} \alpha_g U_g^* x U_g \quad \text{for all } x \in \mathcal{B}(\ell_2(G)).$$

Taking $L_k = \sqrt{\alpha_k} U_k$ for $k \in G \setminus \{e\}$, we can express the above in the GKSL standard form (3). Moreover, since they act on finite dimensional algebras $\mathcal{B}(\ell_2(G))$, left G-circulant GKSL generators are bounded operators, and, consequently, give rise to uniformly continuous quantum Markov semigroups.

Definition 13. *A uniformly continuous QMS $(\mathcal{T}_t)_{t\geq 0}$ acting on a matrix algebra $\mathcal{B}(\ell_2(G))$ is said to be left G-circulant if $\mathcal{T}_t = \exp(t\mathcal{L})$ for all $t \geq 0$ for some left G-circulant GKSL generator $\mathcal{L}$.*

The circulant generators $\mathcal{L}(x) = \sum_{k=0}^{n-1} \alpha_{n-k} J^{*k} x J^k$ from [5] correspond to left $\mathbb{Z}_n$-circulant GKSL generators in the above definition; in particular, each shift matrix $J^k = \sum_{i=0}^{n-1} |e_i\rangle\langle e_{i+k}| \in \mathrm{Mat}_n(\mathbb{C})$ corresponds to the unitary operator U_k^* for each k in the cyclic group $\mathbb{Z}_n$.

A key feature of left G-circulant generators is that their preduals are of the same type. Namely, if $\mathcal{L}(x)$ is a left G-circulant GKSL with vector $(\alpha_g)_{g\in G}$, then its predual $\mathcal{L}_*$ is also a left G-circulant GKSL generator with vector $(\alpha_{g^{-1}})_{g\in G}$ and $-H$.

A wide study of the structure of the invariant states, invariant subspaces and some spectral properties of G-circulant semigroups can be found in [6]. We recall that if the left G-circulant matrix associated to $\mathcal{L}$ is irreducible, then the set of invariant states is precisely the set of right G-circulant states, *i.e.*,

$$\mathcal{L}_*(\rho) = 0 \Leftrightarrow \rho = \sum_{g\in G} \beta_g V_g^* \text{ and is a state.}$$

Theorem 14. *A faithful invariant state ρ and an irreducible G-circulant GKSL generator $\mathcal{L}$ satisfy the $SQDB$ condition if and only if $\alpha_g = \alpha_{g^{-1}}$.*

Proof. If ρ is an invariant state, then it commutes with U_g and thus $\rho^{1/2}$ commutes with U_g as well. This implies that $\mathcal{T}_t' = \mathcal{T}_{*t}$ and $\mathcal{L}' = \mathcal{L}_*$, thus

$$\mathcal{L} - \mathcal{L}' = 2i[H, x] + \sum_{g\in G} (\alpha_g - \alpha_{g^{-1}}) U_g^* x U_g.$$

Recall that the GKSL generator $\mathcal{L}(x) = i[H, x] - (1/2)\sum_j (L_j^* L_j x + x L_j^* L_j - 2L_j^* x L_j)$ can be written as

$$\mathcal{L}(x) = G^* x + \sum_j L_j^* x L_j + xG,$$

where $G = -(1/2)(\sum_j L_j^* L_j - iH)$, and its predual is

$$\mathcal{L}_*(x) = xG^* + \sum_j L_j x L_j^* + Gx.$$

Let $\theta : \ell_2(G) \to \ell_2(G)$ be an anti-unitary operator satisfying $\theta^2 = \mathbb{1}$, say the conjugation w.r.t. the basis $(e_g)_{g\in G}$ and the time reversal operation $\Theta(x) = \theta x^* \theta$, we will assume that $\theta G^* \theta = G$ and $\theta\rho = \rho\theta$.

The assumption $\theta\rho = \rho\theta$ is known to be equivalent to ρ being symmetric $\rho = \rho^T$. In our case, since $\rho = \sum_{g\in G} \beta_g V_g^*$ must be right G-circulant we have the following equivalent condition:

$$\beta_g = \langle e_k, \rho e_{kg}\rangle = \langle e_{kg}, \rho e_k\rangle = \beta_{g^{-1}}.$$

Theorem 15. *A faithful invariant state ρ and an irreducible G-circulant GKSL generator $\mathcal{L}$ satisfy the $\Theta - SQDB$ condition if and only if $\alpha_g = \alpha_{g^{-1}}$.*

Using that $(\theta x\theta)^* = \theta x^*\theta$ direct computations show that

$$\begin{aligned}\Theta\circ\mathcal{L}\circ\Theta(x)&=\theta\Big(G^*\theta x^*\theta+\sum_{g\in G}\alpha_g U_g^*\theta x^*\theta U_g+\theta x^*\theta G\Big)^*\theta\\ &=\theta\Big(\theta x\theta G+\sum_{g\in G}\alpha_g U_g^*\theta x\theta U_g+G^*\theta x\theta\Big)\theta\\ &=\theta^2 x\theta G\theta+\sum_{g\in G}\alpha_g\theta U_g^*\theta x\theta U_g\theta+\theta G^*\theta x\theta^2\\ &=xG^*+\sum_{g\in G}\alpha_g U_g^* x U_g+Gx,\end{aligned}$$

thus, since $\mathcal{L}' = \mathcal{L}_*$ then

$$\Theta\circ\mathcal{L}\circ\Theta(x)-\mathcal{L}'(x)=\sum_{g\in G}(\alpha_g-\alpha_{g^{-1}})U_g^* x U_g.$$

This condition for reversibility coincides with the one obtained from the classical detailed balance condition (2) of a left G-circulant Markov chain ξ.

5. Conclusion

The concept of equilibrium has long served as a cornerstone in both classical and quantum statistical mechanics. In classical settings, conditions such as detailed balance and vanishing entropy production are known to be equivalent characterizations of equilibrium behavior. However, in the quantum setting, this equivalence breaks down in subtle but fundamental ways.

In this work, we have studied the Quantum Entropy Production Rate (QEPR) as a tool to characterize and quantify the departure from equilibrium in quantum systems governed by Quantum Markov Semigroups (QMS). These semigroups arise from the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation, which provides a general framework for modeling the evolution of open quantum systems under Markovian dynamics.

While it is true that a QMS satisfying the Θ-Standard Quantum Detailed Balance (Θ-SQDB) condition has vanishing QEPR, the converse fails. We exhibited (see (9)) an explicit example of a QMS acting on a two-level system where the entropy production vanishes, yet the generator does not satisfy the Θ-SQDB condition. This highlights the richer and more nuanced structure of non-equilibrium phenomena in quantum systems, where symmetry or reversibility at the level of entropy production does not necessarily reflect a full equilibrium structure.

Our study contributes to the broader program of going beyond equilibrium, a research direction that seeks to understand and classify open quantum systems that, while not in thermal equilibrium, still exhibit enough structure to allow explicit mathematical and physical analysis. Within this framework, we explored a recently introduced family of G-circulant QMS, which generalizes circulant QMS to a finite (possibly non-abelian) group. These systems support non-equilibrium steady states and display sufficient symmetry to permit explicit calculation of entropy production, a rare feature in quantum dynamics.

Future lines of research include extending these results to broader classes of group-indexed semigroups, examining whether other notions of quantum reversibility correspond more closely with QEPR. An ambitious direction would be to study experimental realizations of such structured non-equilibrium quantum processes, potentially providing a testbed for the concepts explored here.

Appendix

A. Anti-unitary Operators

We recover some useful properties regarding anti-unitary operators.

Definition 16. *A bijective, anti-linear operator $\theta : \mathsf{h} \longrightarrow \mathsf{h}$ is called anti-unitary if*

$$\langle\theta u,\theta v\rangle=\langle v,u\rangle,\quad \text{for all } x,y\in\mathsf{h}.$$

It is immediate from the definition that anti-unitary operators are bounded operators. Even more, they are antilinear isometries, and so they send orthonormal bases on orthonormal bases. The most used anti-unitary operators in physics are those satisfying $\theta^2 = \mathbb{1}$, a property that we assume from now on.

The following properties are straightforward.

Proposition 17. *An anti-unitary operator θ has the following properties:*

(i) *Its adjoint θ^* is also antilinear and it is defined by $\langle u,\theta v\rangle = \langle v,\theta^* u\rangle$. If $\theta^2 = \mathbb{1}$, then $\theta = \theta^*$.*

(ii) $\theta\theta^* = \theta^*\theta = \mathbb{1}$.

(iii) *$\theta x\theta$ is a linear operator satisfying $(\theta x\theta)^* = \theta^* x^*\theta^*$. If $\theta^2 = \mathbb{1}$, then $(\theta x\theta)^* = \theta x^*\theta$.*

(iv) *The composition of two anti-unitary operators is unitary.*

(v) *The composition of a unitary and an anti-unitary operator is an anti-unitary operator.*

(vi) *Each anti-unitary operator θ is the composition of an unitary and a conjugation w.r.t. an orthonormal basis.*

Due to (vi), when dealing with an anti-unitary operator and an orthonormal basis $\{e_i\}_i$, up to a unitary transformation we can identify θ with the conjugation w.r.t. $\{e_i\}_i$. So that $\theta e_i = e_i$ and for $u = \sum_i u_i e_i$, $\theta u = \sum_i \bar{u}_i e_i$.

As we have seen, one needs to be careful when operating with anti-unitary operators, since their behaviour can be rather counter-intuitive. Some final properties that are found to be useful in practice are:

(i) $\theta|e_i\rangle\langle e_j|\theta = |\theta e_i\rangle\langle\theta e_j| = |e_i\rangle\langle e_j|$.

(ii) $\theta|e_i\rangle\langle e_j| = |e_i\rangle\langle e_j|\theta$.

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