Suplemento de la Revista Mexicana de Física **6** 011312 (2025) 1–7

Possible dynamical paths towards the constrained optimization, and other fundamental aspects, of the LMC family of statistical measures of complexity

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Received 14 March 2025; accepted 21 May 2025

The celebrated LMC measure of complexity, advanced by López-Ruiz, Mancini and Calbet thirty years ago, is based on the idea that scenarios exhibiting large amounts of order, or large amounts of disorder, are characterized by low or vanishing amounts of complexity. According to this idea, complexity adopts its maximum value at some intermediate regime between extreme order and extreme disorder. Following on the LMC steps, researchers have introduced several other statistical measures of complexity, akin to the original LMC one, that also comply with the aforementioned requirements. These measures, which we collectively refer to as "LMC-measures", are defined as products of information or entropic-like quantities. The LMC measures have been applied by scientists to the study of diverse systems or processes in physics, chemistry, and other fields, leading to a research literature of respectable size. In spite of the intriguing results yielded by those investigations, various fundamental issues concerning the LMC measures remain unaddressed. It seems timely, thirty years after the original LMC proposal, to reconsider its foundations. We shall discuss various basic aspects of the LMC measures, including some exploratory steps regarding possible dynamical mechanisms leading to probability densities optimizing the measures under suitable constraints.

Keywords: LMC statistical measures of complexity; composability; expansibility; constrained optimization; nonlinear Fokker-Planck equations.

DOI: https://doi.org/10.31349/SuplRevMexFis.6.011312

1. Introduction

In 1995 Lopez-Ruiz, Mancini, and Calbet (LMC) entertained the idea of defining a quantitative measure of complexity for discrete probability distributions, or for continuous probability densities [1]. The LMC proposal is based on the intuition that systems exhibiting high order or high disorder, have low or vanishing complexity. Most scientists studying complexity in Nature agree with that intuition. When considering discrete probability distributions, maximum order can be identified with certainty and maximum disorder with equiprobability. In accordance with these identifications, the LMC measure is defined in such a way that it vanishes in two extreme situations: when the probability distribution has one probability equal to one and the rest are equal to zero (certainty), and when all the probabilities are equal (equiprobability). The LMC measure, on the other hand, adopts its maximum value at some intermediate situation between certainty and equiprobability [2].

In their original paper from 1995, Lopez-Ruiz, Mancini, and Calbet defined their statistical measure of complexity, and provided an illustrative application to the logistic map, for which the measure exhibits its maximum at the edge of chaos [1]. The first work devoted to investigate the general properties of the LMC measure, and to determine the form of the probability distributions that optimize the measure, was done by Anteneodo and Plastino in 1996 [2]. In subsequent years the LMC measure attracted the attention of some leading researchers in the physics of complex systems [3]. Even though some of those early investigations on the LMC measure were rather critical about the measure's merits, the LMC measure, together with various other complexity measures, played a valuable role as testing grounds for exploring basic issues concerning the idea of assessing quantitatively the amount of complexity exhibited by a system or process [4]. In this regard, it is worth mentioning that the LMC proposal and its generalizations constitute only a small subset of all the approaches that have been advanced by researchers, over the years, in order to measure complexity in natural phenomena. In the present contribution, we are going to consider only the LMC proposal and its extensions. That is, complexity measures are defined as products of entropic or information quantities, evaluated on probability distributions or densities. In the rest of this work we shall refer to the idea behind this type of measures as the "LMC proposal", and to the study of this kind of measures, and their applications, as the "LMC approach". Towards the late 90s, the researchers interested in the LMC measure and its various extensions and generalizations (here referred to as "LMC measures") gradually shifted their efforts from the foundations of the LMC approach, to its applications. There is already a vast research literature on applications of the LMC measures to physics, chemistry, biology, and other fields. There are some remarkable works indicating that, within some scenarios, the LMC measures yield valuable insights into the properties of complex systems. It is impossible to review here all the work that has been done on applications of the LMC proposal and its various extensions and generalizations. As a few notable examples, we can mention the impressive body of research that Esquivel and collaborators have conducted on applications in quantum chemistry [4,5], the interesting works by Ribeiro and collaborators on the classification of music styles [6] and by Guisande and Montani on applications to neuroscience [7], and a recent remarkable application to planetary systems [8].

Despite the considerable effort that has been devoted over the years to applications of the LMC measures, the fundamental reasons behind the phenomenological successes of these measures are still poorly understood. There still are basic issues that have to be addressed. Fundamental features, such as composability and expansibility, have been largely overlooked. Other basic subjects that remain largely unexplored are the dynamical mechanisms that lead to the constrained optimization of the LMC measure. Here we consider some of these issues. In particular, as an exploratory step towards elucidating the last-mentioned point, we advance, and investigate the basic features of a Fokker-Planck-like nonlinear equation that satisfies an H-theorem related to the LMC measure.

2. The original LMC statistical measure of complexity

In its original form, the LMC statistical measure of complexity of a discrete probability distribution $(p_1, p_2, \ldots, p_N)$, is defined as

$$C = S \cdot D, \tag{1}$$

where

$$S = -\sum_{i=1}^{N} p_i \ln p_i, \tag{2}$$

is the Shannon entropy of the probability distribution $\{p_i\}$, and

$$D = \sum_{i=1}^{N} \left[p_i - \frac{1}{N} \right]^2, \tag{3}$$

is the "disequilibrium" of the probability distribution, which provides a quantitative indication of how much does the probability distribution $\{p_i\}$ differ from the equiprobable distribution $\{p_i^{(e)} = 1/N\}$. The entropy S vanishes in the case of certainty, while the disequilibrium D vanishes in the case of equiprobability. Therefore, the product structure of the measure C implies that it vanishes both for *certainty* and for *equiprobability*, satisfying the intuitive "boundary conditions" desired for a complexity measure. It is obvious, however, that there are plenty of alternative choices for the factors entering the definition of C, instead of S and D, that also yield measures satisfying the desired boundary conditions. For example, instead of the Shannon logarithmic entropy S, one can consider other entropic or information measures, such as Tsallis entropy. By the same token, instead of the disequilibrium D, one can consider other quantities indicating the deviation of the probability distribution $\{p'_i\}$ from the equiprobable one. We shall refer to the original LMC measure as the "LMC measure" (singular). On the other hand, we shall refer, collectively, to the family of measures defined by products of quantities complying with appropriate boundary requirements, as the "LMC measures" (plural).

The LMC proposal, which constitutes arguably the most simple and straightforward possible way of defining a measure of complexity, had a considerable impact on the scientific community (the 1995 work where the LMC measure was advanced has more than 1000 Google Scholar citations). As already mentioned, the LMC measures provided a valuable testing ground for exploring basic issues concerning the concept of a quantitative measure of complexity. Work on the LMC family of complexity measures was an important stimulus for research into complex systems in various parts of the world. In fact, several physicists (coming mostly, but not only, from statistical physics) entered the field of complex systems via the exploration of the LMC measures and their applications.

In spite of the several applications of the LMC measures of complexity that have been investigated so far, the basic meaning of the measures still raises conceptual issues that need further examination. Even the very motion of assigning a quantitative amount of complexity to a probability distribution might be problematic. The LMC measures might be "shadows" of more elaborate measures that involve not just a probability distribution, but also other specific structures associated with the systems or processes under consideration. These hypothetical, context-dependent, more elaborate measures might be required in order to achieve a full understanding of the phenomenological successes of the LMC measures. Besides, they might provide an answer to the basic question made by Feldman and Crutchfield in Ref. [3]: "*What exactly is the statistical complexity measuring?*". In order to shed some light on these issues, and to understand the fundamental reasons for the phenomenological successes of the LMC measures, it would presumably be useful to find operational interpretations, and axiomatic characterizations of the LMC measures. We believe that the points discussed in the following sections may constitute useful steps towards those goals.

3. Other LMC-like measures of complexity

As already mentioned, Anteneodo and Plastino (AP), in Ref. [2], were the first to explore the main properties of the original LMC measure, including the probability distributions that optimize the measure. AP discovered that the original LMC measure of statistical complexity exhibits some features that are at odds with what one would expect from a reasonable measure of complexity. In particular, AP showed that the original LMC measure lacks two fundamental invariance properties: it is neither invariant under scale changes nor under replication. The lack of invariance under changes of scale means that the continuous version of the measure, evaluated on a continuous probability density, changes when the probability density is modified according to a change of scale. This is an undesirable property, because one expects that a simple re-scaling should not affect the amount of complexity associated with a probability density. The lack of invariance under replication means that a system consisting of two identical, and statistically independent copies of an original system, has an amount of complexity different from that of the original single copy of the system. This feature violates a principle proposed by Lloyd and Pagels [9], according to which complexity should not change under replication. By pointing out the above fundamental defects that afflict the original LMC proposal, the AP-1996 paper contributed to establish the agenda regarding research on the fundamental properties that LMC-like measures of complexity should have. Other researchers (including some of the authors of the original LMC proposal) addressed these issues and proposed new versions of the LMC measure. An important example is given by a modification of the original LMC measure for continuous systems, advanced by Catalán, Garay, and López-Ruiz (CGL) in 2002 [10]. The new version of the LMC measure reads

$$\mathcal{C}[f] = \mathcal{D}[f]\exp(\mathcal{H}[f]), \tag{4}$$

with

$$\mathcal{H}[f] = -\int f(x)\ln(f(x))dx, \tag{5}$$

and

$$\mathcal{D}[f] = \int f(x)^2 dx, \tag{6}$$

where $f(x)$ is a normalized probability density ($\int f(x)dx = 1$), and we regard x and f as dimensionless quantities. The modification (4) of the LMC measure cured some of the deficiencies that afflicted the original LMC measure. The new measure is invariant under re-scaling transformations and under (a particular interpretation of) replication. It is remarkable that a simple modification in the form of the LMC measure is enough to obtain a measure that satisfies the desired properties lacked by the original LMC measure. The new measure adopts its minimum value, equal to 1, for rectangular-like probability densities. A rectangular density has, for some range of x-values having a total length L, the value $1/L$, and is equal to zero for x-values outside the alluded range. Interestingly, the new LMC measure can be expressed in terms of the Renyi entropies. Indeed, we have that

$$\mathcal{C}[f] \;=\; \exp\left(R^{(1)}[f] - R^{(2)}[f]\right), \tag{7}$$

where

$$R^{(\alpha)}[f] \;=\; \frac{1}{1-\alpha}\;\ln\left(\int f(x)^{\alpha}\,dx\right), \quad 0<\alpha, \tag{8}$$

is the Renyi entropy of order α. For $\alpha \to 1$, one recovers the Shannon entropy. The basic properties of the LMC-like measure $\mathcal{C}$ were studied in Ref. [10], and a generalization, based on the Renyi entropies, was proposed in Ref. [11]. The formulation in terms of Renyi entropies makes the validity of some of the basic properties of the measure (4) specially transparent. We want to emphasize, moreover, that the measure (4), and its Renyi-based generalizations are particularly interesting members of the LMC-like statistical measures of complexity, because they comply with another fundamental property that has been largely overlooked in the literature concerning the LMC measures: *composability*.

4. Composability of entropies and complexity measures

Composability is an important notion that has been investigated mostly in connection with entropic and information measures. As already mentioned, we want here to emphasize that composability should also be regarded as essential for complexity measures. In order to clarify this point, it is instructive to briefly review first the idea of composability of entropies.

4.1. Composability of entropies

The basic idea of composability of entropies is that, for two statistically independent systems A and B, the total entropy, when one considers them to be a single system $A+B$, has to depend only on the individual entropies of A and B, and not on any other specific features of these systems (see [12,13] and references therein). The property of composability can be encapsulated in the equation

$$S(A+B) = \Phi(S(A), S(B)), \tag{9}$$

where $S(A)$, $S(B)$, and $S(A+B)$ are the entropies of systems A, B, and of the composite $A+B$, and $\Phi(.,.)$ is a function describing the form of the composability law. Composability plays an important role in connection with entropic measures. It imposes strong constraints on the allowed forms for entropies. Let us consider the general family of entropic functionals

$$S[p] \;=\; G\left(\sum_i h(p_i)\right), \tag{10}$$

where $G(x)$ and $h(x)$ are functions that comply with $h(0) = G(h(1)) = 0$, and are also typically assumed to satisfy appropriate monotonicity and concavity properties [13]. Of special relevance, among the above family of entropies, are those with $G(x) = x$, which are known as *trace form entropies*. Entropies of the form

$$S[p] \;=\; G\left(\sum_i p_i^{\alpha}\right), \tag{11}$$

are the only entropies of the general family (10) that comply with composability. Sometimes it is stated that, formally, the most general form for composable entropies within the family (10) is $G\left(\sum_i (c_1 p_i + c_2 p_i^{\alpha})\right)$, with $c_{1,2}$ appropriate constants. But, if the probability distribution $\{p_i\}$ is normalized, this more general form can always be re-cast under the guise of (11), by recourse to an appropriate re-definition of the function G. Paradigmatic examples of entropies complying with composability are the Renyi and the Tsallis entropies. The Renyi entropies, which are parameterized by a real parameter q, are given by

$$S_q^{(R)} = \frac{1}{1-q} \ln \left(\sum_i p_i^q \right). \tag{12}$$

Notice that the above entropies are the discrete versions of the entropies defined in (8) (the parameter q corresponds to the parameter α in (8)). We use here a slightly different notation, in order to highlight the connection between the Renyi entropies and the Tsallis entropies, which we shall define later. The Renyi entropies are the most general additive entropies. Additivity, of course, is a particular form of composability, with a composability law of the form

$$S_q^{(R)}(A+B) = S_q^{(R)}(A) + S_q^{(R)}(B). \tag{13}$$

The Tsallis entropies, also parameterized by a real parameter q, are given by

$$S_q^{(T)} = \frac{\sum_i (p_i - p_i^q)}{q-1} = \frac{1}{q-1}\left(1 - \sum_i p_i^q\right), \tag{14}$$

and are also important examples of entropies satisfying composability. Indeed, they are the most general trace-form entropies complying with composability (notice that the Tsallis entropies can be written in two equivalent forms, one of which is trace-form, but the other is not). The well-known composability law for the Tsallis entropies is

$$S_q^{(T)}(A+B) = S_q^{(T)}(A) + S_q^{(T)}(B) + (1-q) S_q^{(T)}(A) S_q^{(T)}(B). \tag{15}$$

The Shannon entropy is a particular instance $(q=1)$ of the above two families of entropies.

In summary, composability is nowadays regarded as a basic, fundamental property that physically sensible entropic measures must satisfy. Shannon, Renyi, and Tsallis entropies constitute important examples of composable entropies.

4.2. Composability of complexity measures

The composability requirement for measures of complexity is similar to the one for entropies. We require that, for two statistically independent systems A and B, the total complexity $C[A+B]$, when one regards A and B as a single system $A+B$, has to depend only on the individual complexities, $C[A]$ and $C[B]$, of A and B. The composability property is expressed by the equation

$$C[A+B] = \Phi(C[A], C[B]), \tag{16}$$

where $\Phi(x,y)$ is a function characterizing the composability law. The composability of a complexity measure can be illustrated by the following example. If one has a system A here on Earth, and another totally independent system B far away in Andromeda, one should be able to determine the total complexity of $A+B$ just from knowing the complexity of A and the complexity of B.

Weird things happen when a complexity measure is not composable. For example, two independent systems with zero complexity each can, jointly, constitute a composite system with finite complexity. For instance, the original LMC complexity measure, $C = (-\sum_i p_i \ln p_i)(\sum_i (p_i - (1/N))^2)$, is not composable. Consider the following pair of independent systems, which individually have vanishing complexity. On the one hand, one has the system A_1, with N_1 states, and a uniform probability distribution. One has $C(A_1) = 0$. On the other hand, one has the system A_2, with N_2 states, and one state with probability 1. One has $C(A_2) = 0$. However, the composite system $A+B$, with $N_1 \times N_2$ states, regarded as a single entity, has a non-vanishing amount of complexity. That is, $C(A+B) > 0$. In other words, the non-composability of C leads to the existence of independent systems that individually have zero complexity, but jointly have a finite amount of complexity. No sensible measure of complexity should allow for this type of undesired situation.

Interestingly, the modified version of the LMC measure given by Eq. (4) satisfies the composability requirement. In fact, let us consider two statistically independent systems A and B, described by the probability densities $f^{(A)}(x_1)$ and $f^{(B)}(x_2)$, where the variables x_1 and x_2 correspond to the state spaces associated with systems A and B. The joint composite system $A+B$ is then described by the factorizable joint probability density $f^{(A+B)}(x_1, x_2) = f^{(A)}(x_1)\, f^{(B)}(x_2)$. Then one has,

$$\mathcal{C}[A+B] = \mathcal{C}[A] \cdot \mathcal{C}[B]. \tag{17}$$

In particular, if the two systems A and B have the minimum possible complexity, $\mathcal{C}[A] = \mathcal{C}[B] = 1$, then the composite system $A+B$ also has minimum complexity, $\mathcal{C}[A+B] = 1$, thus avoiding the aforementioned kind of paradoxical situations.

The above state of affairs strongly suggests that, if one wants to consider an LMC-like measure for discrete systems, the discrete version of (4), given by

$$\mathcal{C}^{(\text{discrete})}[p] = \left(\sum_i p_i^2\right) \cdot \exp\left(-\sum_i p_i \ln p_i\right), \tag{18}$$

which *is composable*, is superior to the original LMC measure (1). Notice that the minimum value of (18) is 1. Besides adopting its minimum value $\mathcal{C}^{(\text{discrete})} = 1$ for *certainty* and for *equiprobability*, the LMC-like complexity measure (18) also adopts its minimum value for probability densities of the more general form $p_i = 1/k,\ i = 1, \ldots, k;\ p_i = 0,\ i > k$, which are discrete versions of the rectangular probability densities that we already mentioned in connection with the continuous measure (4). The complexity measure (18) can be expressed as the difference of two Renyi entropies, $\mathcal{C}^{(\text{discrete})} = \exp(S_1^{(R)} - S_2^{(R)})$ (in this case, one of the entropies happens to be Shannon's), similarly to what occurs in the continuous case. A notion of complexity based on the difference $S_1^{(R)} - S_2^{(R)}$ has been studied in connection with various quantum mechanical systems or processes (see [14] and references therein), including the spread of wave-packets in a tight-binding lattice [15]. It is straightforward to generalize the measure $\mathcal{C}^{(\text{discrete})}$, along the same lines followed in Ref. [16], and define a bi-parametric family of discrete measures, expressed in terms of the difference between Renyi entropies, given by

$$\mathcal{C}_{q_1,q_2}^{(\text{discrete})} = \exp\left(S_{q_1}^{(R)} - S_{q_2}^{(R)}\right), \quad 0 < q_1 < q_2, \qquad (19)$$

which constitute discrete versions of the measures studied in Ref. [18]. All the above complexity measures, defined in terms of the difference between two Renyi entropies, comply with composability. The conceptual superiority of complexity measures based on the difference between Renyi entropies, such as the LMC-like one (18), over the original LMC measure (1), does not necessarily mean that, when used as practical tools for the classification of patterns, these different measures will not lead to similar results [14]. However, for studying the theoretical foundations of the phenomenon of complexity, measures such as (18) (or its generalizations (19)) may lead to deeper insights than the original LMC one (1).

There is another property that a reasonable LMC-like measure of complexity should satisfy, which is *expansibility*. The property of expansibility means that adding a new state of zero probability does not change the complexity of a system. The original LMC measure (1) does not comply with expansibility, but the modified version (18) does. The form of the probability distributions corresponding to its minimum value, together with the property of expansibility, lead to an intuitive interpretation of the meaning of the measure (18): it measures how different a probability distribution is from an "even" distribution, which is a distribution that is uniform over those states with non-vanishing probability. Said differently, the statistical measure of complexity (18) provides a quantitative assessment of the degree of "unevenness" exhibited by a probability distribution. This is another sense in which the measure (18) is superior to the original LMC: for (18) one can give an answer to the question: what does this quantity actually measure?

Summing up, in this Section we have considered the notions of composability and expansibility as applied to LMC-like measures of statistical complexity. We pointed out that LMC measures that are not composable lead to some counter-intuitive weird situations. On the other hand, we emphasized the fact that the family of LMC measures for continuous probability densities, constructed on the basis of the difference between two Renyi entropies, complies with composability. Moreover, the discrete version of these measures also satisfies the important property of expansibility. It would be interesting to re-visit, in connection with the notion of composability, other LMC-like measures that have been proposed in the literature (see, for instance, [18,19,20]).

5. A possible dynamical path to probability densities optimizing the measure $\mathcal{C}$ under suitable constraints

As already mentioned, many applications of the LMC-like statistical measures of complexity, to physics and other fields, have been considered by researchers. In many of these works, the LMC-like measures proved to be useful tools for investigating the phenomenology of various systems and processes. To understand the basic reasons behind these phenomenological successes, it would be convenient to have operational interpretations, and axiomatic characterizations of the LMC-like measures. We believe that the considerations made in the previous sections may constitute useful guides towards the achievement of those goals. On the other hand, it may also be enlightening to explore the dynamical aspects of the LMC measures. In particular, it would be interesting to explore dynamical mechanisms that are consubstantial with the LMC measures, naturally leading to probability densities that optimize the measures under suitable constraints. In order to take some exploratory steps in that direction, we shall investigate a possible Fokker-Planck-like evolution equation satisfying an H-theorem based on the modified LMC measure (4) (an equation of this kind, based on the original LMC measure, was proposed in Ref. [14]). We consider the nonlinear evolution equation

$$\frac{\partial f}{\partial t} = \mathcal{D}_1 \frac{\partial^2 f}{\partial x^2} - \mathcal{D}_2 \frac{\partial^2 \left(f^2\right)}{\partial x^2} + \frac{\partial}{\partial x}\left(f \frac{\partial V}{\partial x}\right), \qquad (20)$$

where $f(x,t)$ is a time-dependent probability density and $V(x)$ is a potential energy function. The quantities $\mathcal{D}_1[f]$ and $\mathcal{D}_2[f]$ are effective diffusion coefficients given by

$$\mathcal{D}_1[f] = \delta\, \mathcal{D}[f]\, \exp(\mathcal{H}[f]),$$
$$\mathcal{D}_2[f] = \delta\, \exp(\mathcal{H}[f]), \qquad (21)$$

where δ is a diffusion constant, and $\mathcal{H}[f]$ and $\mathcal{D}[f]$ are given by (5) and (6). Notice that the effective diffusion coefficients $\mathcal{D}_1[f]$ and $\mathcal{D}_2[f]$, even though they depend upon the density f, are just numbers, not functions of the spatial coordinate x. They depend on global features of f. The nonlinear evolution equation (20) complies with an H-like theorem based on the modified LMC measure (4). We define the functional

$$\mathcal{F}[f] = \left(\int f(x)\, V(x)\, dx\right) - \delta\, \mathcal{C} = \langle V \rangle - \delta\, \mathcal{C}, \qquad (22)$$

akin to a free-energy, where $\langle V \rangle$ is the mean value of the potential $V(x)$, and $\mathcal{C}[f] = \mathcal{D}[f] \cdot \exp(\mathcal{H}[f])$ is the modified version of the LMC measure (4). It is possible to prove that the time derivative of the functional $\mathcal{F}$ is always non-positive,

$$\frac{d\mathcal{F}}{dt} = -\int dx\ f(x) \left[\frac{\partial V}{\partial x} - 2\,\delta\,\frac{\partial f}{\partial x}\,\exp(\mathcal{H}) + \delta\,\mathcal{D}\,\exp(\mathcal{H}) \left(\frac{1}{f}\frac{\partial f}{\partial x} \right) \right]^2 dx \leq 0. \qquad (23)$$

The probability densities $f(x)$ that make the time derivative $d\mathcal{H}/dt$ equal to zero have to comply with the differential equation

$$\frac{\partial V}{\partial x} - 2\,\delta\,\frac{\partial f}{\partial x}\,\exp(\mathcal{H}) + \delta\,\mathcal{D}\,\exp(\mathcal{H}) \left(\frac{1}{f}\frac{\partial f}{\partial x} \right) = 0. \qquad (24)$$

It is not evident that there exist densities satisfying the Eq. (24) that are normalized stationary solutions of Eq. (20). This issue deserves further investigation.

The nonlinear evolution Eq. (20), related to the measure (4), has some important features akin to those exhibited by other evolution equations that have been previously discussed in the research literature. In particular, it exhibits interesting connections with Fokker-Planck equations endowed with nonlinear diffusion terms [15]. Evolution equations of this type have proved, in recent years, to be useful tools for the study of complex systems (see, for instance [16]). As is the case with this kind of nonlinear Fokker-Planck equations (see [17] and references therein), the evolution equation (20) admits an H-theorem in terms of a free-energy-like quantity based on a non-standard entropy (or entropy-related) functional. The evolution equation (20) has global regulation. That is, the partial time derivative of a probability density $f(x,t)$ that evolves according to (20), depends on global features f, given by the functionals $\mathcal{D}$ and $\mathcal{H}$. Something similar occurs, for example, with the reaction-diffusion evolution equation investigated by Troncoso *et al.* in Ref. [18]. Finally, the nonlinear evolution Eq. (20) exhibits a combination of two different types of diffusion: one with a standard, linear Laplacian term, and one with a Laplacian term acting upon the square of f. A similar situation happens with an evolution equation, advanced by Andrade Jr *et al.* in Ref. [19], which provides a thermostatistical, effective mean-field description of systems of confined interacting particles moving in the overdamped-motion regime.

To recapitulate, in this Section, we introduced and investigated the basic properties of the nonlinear Fokker-Planck evolution equation (20), which exhibits a dynamics closely linked to the LMC-like measure (4). This equation satisfies an H-theorem involving a functional, formally similar to a free energy, related to the measure (4). The evolution equation (20) exhibits some features that are intriguingly similar to those exhibited by other nonlinear evolution equations that have been investigated in the literature, mostly in connection with the study of complex systems. We studied equation (20) as an exploratory step toward elucidating the kind of dynamics that leads to the constrained optimization of LMC-like measures.

6. Conclusions

The LMC statistical measure of complexity, and its variants, extensions, and generalizations, have had some intriguing phenomenological successes in various particular applications. This state of affairs indicates that the LMC proposal and its applications deserve further theoretical and conceptual examination. In this regard, the LMC-like measures still constitute useful testing grounds for exploring some basic notions concerning complexity. It is plain that further theoretical work is needed in order to explain the basic reasons for the aforementioned phenomenological successes. Besides, a deeper understanding of the LMC proposal may help to elucidate if there exist any interesting connections between the LMC idea and more general theoretical approaches to the phenomenon of complexity [20]. In this regard, it is intriguing that, in spite of all the literature devoted to applications of the LMC measures, the LMC treatment seems, to some extent, to be a kind of isolated island, rather disconnected from other prominent venues of current theoretical research on complexity. In the present work we have discussed some fundamental features of the LMC-like measures, highlighting the important role played by *composability*. This notion, which is now recognized as an essential property of entropic and information measures [12], should also play a central role in connection with the LMC family of complexity measures. Here we have discussed composability only in connection with the original LMC measure, and with the modified LMC measure proposed in Ref. [10]. It would be interesting to consider composability in relation to other LMC measures, defined as products of entropic or information quantities. In the case of discrete probability distributions, it seems that all measures of this type that refer explicitly to the total number N of states of the system under consideration, and to the corresponding equiprobable distribution, comply neither with composability nor with expansibility.

In this work we have also investigated a possible dynamical setting leading to the constrained optimization of the CGL modified version of the LMC measure of complexity. We advanced an example of a nonlinear Fokker-Planck-like equation satisfying an H-theorem based on the CGL measure. We hope that our present considerations will stimulate further research into the theoretical foundations of the LMC family of complexity measures, and, in particular, into the physical meaning, origin, and properties of the probability densities yielded by the constrained optimization of the LMC measures. The illustrative evolution equation advanced in the present work admits some direct generalizations. It would be interesting to explore suitable analytical approximations to, or to conduct a numerical study of, the time-dependent solutions of this family of equations. Any advances along these or related lines of inquiry will certainly be very welcome.

Acknowledgments

We acknowledge support from the Brazilian funding agencies: Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro (FAPERJ) and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES). We are grateful for the kind hospitality of the Centro Brasileiro de Pesquisas Físicas (CBPF), where part of this research was conducted.

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