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REMARKS ON HARMONIC ANALYSIS AND ITS APPLICATION TO THE ORTHOGONAL AND POINCARE GROUPS

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ABSTRACT: We use the classical construction of the group ring to present a pair of harmonic transform functions as the coordinates of a ring element in two different bases: one, a function $F(g)$ on the group manifold $\mathcal{G}$, the other $\mathfrak{F}(j)$ on the set $\mathcal{S}$ of unitary irreducible representations (UIRs) of the group. The transformation kernel is given by the UIR matrix elements (MEs) of the group $\mathfrak{D}^j(g)$. We develop this mathematical formalism in order to present in a concise fashion several results in relativistic kinematics, Toller's partial wave expansion and some field theories on the Poincaré group. In particular, we prove for the orthogonal groups three theorems which have direct analogues in Toller's work: 1) functions on spaces of cosets lead to a reduction in the expansion basis, 2) harmonic transformation with respect to a group and one of its subgroups yield a re-

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lation between the partial-wave coefficients, familiar from the relation between a Toller pole and its Regge daughter poles, and 3) the (Sciarrino–Toller) factorization of the residues of the former implies a corresponding factorization of the latter.

1. INTRODUCTION

Harmonic Analysis refers to the expansion of an arbitrary function $F(g)$, "well-behaved" in some sense, in terms of a complete and orthogonal set of basis functions $\mathfrak{D}^j(g)$. The carrier space $g \in \mathcal{G}$ on which the function F is defined gives meaning to the completeness and orthogonality of the basis $\{\mathfrak{D}^j\}$ with respect to a definite scalar product. It is usually some subset of the R^n space (R, real field).

When on this space $\mathcal{G}$ we can associate to every ordered pair of points (g_1, g_2) a third point g_3 in $\mathcal{G}$ such that the space has the properties of an n-parameter locally compact Lie group $\mathcal{G}$, we can choose the set of basis functions $\mathfrak{D}^j(g)$ to be the unitary irreducible representation (UIR) matrix elements (MEs) of the group. These constitute, due to the Peter–Weyl theorem[1], with an extension given by Rączka[2] a complete and orthogonal set of functions under the Haar and Plancherel measures. The "partial-wave coefficients" $\mathfrak{F}(j)$ of the expansion can then be considered as a function on a dual space $j \in \mathcal{S}$ of UIRs, harmonic transform to $F(g)$ on $\mathcal{G}$.

Harmonic Analysis on a group manifold refers then, to expansions in which the eigenfunctions are the matrix elements of the regular representations of a group. From the work of Naimark[3] we know that for locally compact topological groups, "enough" eigenfunctions exist, but in general this problem involves rather delicate considerations of topology and functional analysis. For our purposes, however, we shall present, in Section 2, the algebraic aspects of Harmonic Analysis in the language of the group ring[4].

We shall consider a ring element $\boldsymbol{F}$ and its coordinates in two bases, which we shall call the "group" and the "representation" bases. In the first one, the basis vectors are labelled by the elements of the group. The coordinates of the ring element constitute then a function $F(g)$ on the group. The second basis is more difficult to visualize, but it turns out that the basis vectors are characterized by the labels of the Unitary Irreducible Representations (UIRs) of the group and by "row" and "column" indices as the UIR matrices themselves. The coordinates of the ring element $\boldsymbol{F}$ constitute thus a matrix function $\mathfrak{F}(j)$ on the space of UIRs of the group.

The kernel for the transformation between these two sets of coordinates are the UIR matrices.

In Section 3 we introduce among other concepts, kets, the convolution (ordinary) and "coupling" products between elements of the ring, scalar product and the norm. We treat concretely SO_n, the n-dimensional orthogonal groups in Section 4. There we prove in a straightforward manner some of the interesting results given by Toller and Sciarrino[5, 6] for the $O_{3,1}$ group: (a) A reduction in the set of expansion basis for functions on the space of left cosets SO_{n-1} SO_n or the two-sided cosets $SO_{n-1}\backslash SO_n/SO_{n-1}$ (corresponding to the result that only $M=0$ Toller poles contribute in processes involving spinless particles). (b) One partial wave in an SO_n expansion is equivalent to a family of daughter partial waves in a SO_{n-1} expansion (corresponding to the Regge daughter poles of a Toller pole). (c) Factorization of the partial wave coefficients in the former implies their factorization in the latter (corresponding to the same result for the residues of the Toller and Regge poles). Since this work was performed, the corresponding theorems have been proved for $SO_{n,1}$ by C. P. Boyer and F. Ardalan[7].

A group of particular relevance in Physics is the Poincaré group, a spacial case of the ISO_n and $ISO_{n-1,1}$ groups treated in Section 5. Some of the spaces of cosets are of interest since they can be used as carrier spaces[8, 9, 10] for wave functions in order to describe objects with extra degrees of freedom as well as with a mass and spin distribution.

Applications of the present work include several later papers. The kernel reduction technique was used to determine complete sets of functions on homogeneous spaces[11]. The use of the UIRMEs as complete and orthogonal sets of functions labelled by the canonical chain of subgroups has served, when the group manifold is deformed through a transformation generated by operators built out of the group's universal enveloping algebra, to determine the matrix elements of the next higher group with a negative term in the metric, i.e. the $SO_{n,1}$ and $SU_{n,1}$ groups[12]. Finally, the group ring construction has been shown to be useful when applied to the Weyl group, in providing a mathematical framework for the ring of quantum-mechanical operators[13].

2. THE FORMALISM OF HARMONIC ANALYSIS

Let $\mathcal{G}$ be a group, g and element of $\mathcal{G}$, and G the order of the group. Construct a G-dimensional linear space, one of whose linearly independent sets of basis vectors can be chosen to be the group elements g themselves.

The vectors of this space:

$$\mathbf{A} = \underset{g}{\mathrm{S}}\, A(g)\, g\ ,\quad g \in \mathcal{G} \tag{1}$$

constitute the elements of the ring $\mathfrak{R}$. The symbol $\underset{g}{\mathrm{S}}$, $g \in \mathcal{G}$, means Σ_g, $g \in \mathcal{G}$ for finite groups and $\int dg$, where dg is the Haar measure, for continuous groups. The function on the group manifold, $A(g)$ will be called[14] the Group-Spectral Function (GSF) of the ring element $\mathbf{A}$.

The addition of two elements of the ring is associative and commutative, its neutral element $\mathbf{0}$ has $O(g) = 0$ for its GSF. The product law of two ring elements is induced from the product law in $\mathcal{G}$; it is commutative if $\mathcal{G}$ is abelian. Associativity holds and the multiplicative neutral element $\mathbf{1}$ has a GSF $1(g) = \delta_{\mathcal{G}}(g)$, where $\delta_{\mathcal{G}}(g)$ is the group-Dirac distribution, such that for any test function f,

$$\underset{g}{\mathrm{S}}\, f(g)\, \delta_{\mathcal{G}}(g) = f(e)\ ,\quad g \in \mathcal{G}$$

where e is the group neutral element. It is the Kronecker-δ function, $\delta(g, e)$, when the group is finite and can in general be expressed as a Dirac δ-distribution in the parameters of the group if this is continuous.

It is a fundamental theorem[4] that the ring $\mathfrak{R}$ can be decomposed uniquely into minimal two-sided ideals $\mathfrak{I}^j$ generated by their primitive idempotents $\mathbf{e}^j$. For finite groups, we can thus write:

$$\mathfrak{R} = \mathfrak{I}^1 + \mathfrak{I}^2 + \ldots + \mathfrak{I}^r\ ,\quad \mathbf{1} = \sum_{j=1}^{r} \mathbf{e}^j\ .$$

(For continuous compact groups the sum is infinite and for non-compact ones the index j takes a continuum of values as well). Finally, we can decompose each two-sided ideal $\mathfrak{I}^j$ uniquely into its minimal left ideals $\mathfrak{L}^j_m$ generated by their primitive idempotents $\mathbf{e}^j_m$ i.e.

$$\mathfrak{I}^j = \mathfrak{L}^j_1 + \mathfrak{L}^j_2 + \ldots + \mathfrak{L}^j_{d(j)}\ ,\quad \mathbf{1} = \sum_{j=1}^{r} \sum_{m=1}^{d(j)} \mathbf{e}^j_m$$

And thus, for every ring element $\mathbf{A}$ we can write, using the properties of idempotents:

$$\mathbf{A} = \mathbf{1A1} = \sum_{j=1}^{r} \mathbf{e}^{j}\,\mathbf{A}\mathbf{e}^{j} = \sum_{j=1}^{r} \sum_{mm'=1}^{d(j)} \mathbf{A}^{j}_{mm'}$$

where

$$\mathbf{A}^{j}_{mm'} = \mathbf{e}^{j}_{m}\,\mathbf{A}\mathbf{e}^{j}_{m'}$$

maps (by right multiplication) $\mathcal{I}^{j}_{m}$ into $\mathcal{I}^{j}_{m'}$.

One can now show that it is always possible to choose a basis for these left-ideal mappings, $\mathbf{e}^{j}_{mm'}$ such that

$$\mathbf{e}^{j}_{mm'}\,\mathbf{e}^{k}_{nn'} = \mathbf{e}^{j}_{mn'}\,\delta_{j,k}\,\delta_{m',n} \qquad (2)$$

and thus write the ring element as

$$\mathbf{A} = \sum_{j=1}^{r} \sum_{mm'=1}^{d(j)} \mathcal{A}_{mm'}(j)\,\mathbf{e}^{j}_{mm'}\,. \qquad (3)$$

The function $\mathcal{A}_{mm'}(j)$ will be called the Representation-Spectral Function (RSF)[14] of the ring element $\mathbf{A}$.

Clearly, to $\mathbf{0}$ corresponds 0 and to $\mathbf{1}$, $1_{mm'}(J) = \delta_{m,m'}$. Since $\mathbf{A}$ can be written both as (1) and as (3), the next step is to relate the GSF and the RSF, considering the RSF of the basis vectors of the decomposition (1) as has been done in detail in Reference 4. The transformation (for finite groups) from the RSF to the GSF is:

$$A(g) = \sum_{j} \frac{d(j)}{G} \sum_{m,m'} \mathcal{A}_{mm'}(j)\,\mathcal{D}^{j}_{m'm}(g^{-1}) \qquad (4)$$

where $d(j)$ is the dimension of the UIR labelled by j and G is as before, the number of elements in the group.

The transformation kernel $\mathfrak{D}^j_{mm'}(g)$ is a function both on the group space $\mathcal{G}$ and on the representation space $\mathcal{R}$. It is the RSF of the group element g. The group property yields

$$\sum_n \mathfrak{D}^j_{mn}(g_1)\,\mathfrak{D}^j_{nm'}(g_2) = \mathfrak{D}^j_{mm'}(g_1 g_2)$$

which is, of course, the multiplication law for the representation matrices. As the GSF of the left-ideal mapping unit vector $\mathbf{e}^j_{mm'}$ turns out[4] to be proportional to $\mathfrak{D}^j_{m'm}(g^{-1})$, we can write the relation between the function $A(g)$ and the matrix function $\mathfrak{A}(j)$ by:

$$A(g) = \underset{j}{S}\ \mathrm{Tr}\left(\mathfrak{A}(j)\ \mathfrak{D}^j(g^{-1})\right),\ j \in \mathcal{R} \tag{5a}$$

$$\mathfrak{A}(j) = \underset{g}{S}\ A(g)\ \mathfrak{D}^j(g),\ g \in \mathcal{G} \tag{5b}$$

In (5a) Tr means trace and the symbol S_j stands for $\sum_j d(j)/G,\ j \in \mathcal{R}$ when the group is finite and, we can convince ourselves, for $\sum_j d(j)/V,\ j \in \mathcal{R}$ where V is the volume of the group when $\mathcal{G}$ is continuous and compact. If the group is non-compact, both $d(j)$ and V are infinite and new problems appear: the existence of the trace in (5a) and the square integrability of the GSF and the RSF. The cases which have been studied in detail, namely $O_{2,1}$ in ref. (5) and $O_{3,1}$ in refs. (5) and (15), suggest that for square-integrable functions $\mathfrak{A}$, (5a) can still be written, the generalized sum over $\mathcal{R}$ becoming the integral with the appropriate Plancherel measure[16], over the continuum of representations belonging to the principal series, if it exists, a sum over the discrete series.

For this weighted generalized sum it is useful to introduce the representation-Dirac distribution $\delta_{\mathcal{R}}(j, j_0)$ with the property

$$\underset{j}{S}\, f(j)\ \delta_{\mathcal{R}}(j, j_0) = f(j_0),\ j \in \mathcal{R}$$

for any test function f.

For finite groups this means $\delta_{\mathcal{R}}(j, j_0) = \delta_{j, j_0}\, G/d(j)$ while for continuous groups, the group volume V replaces G. The orthogonality and completeness relations of the UIR matrix elements can be obtained from (5). Putting $\mathbf{A} = g$, (5a) gives:

$$\underset{j}{S}\, \mathrm{Tr}\left(\mathbb{D}^{j}(g_1)\, \mathbb{D}^{j}(g_2)\right) = \delta_{\mathcal{G}}(g_1 g_2)\,,\ j \in \mathcal{J}, \tag{6a}$$

while for $\mathbf{A} = \mathbf{e}^{j}_{nm}$, (5b) gives

$$\underset{g}{S}\, \mathbb{D}^{j}_{mn}(g^{-1})\, \mathbb{D}^{j'}_{m'n'}(g) = \delta_{mn'}\, \delta_{nm'}\, \delta_{\mathcal{J}}(j, j')\,,\ g \in \mathcal{G}\,, \tag{6b}$$

which is a method of calculating the Plancherel measure.[13] Using (5) and (6) for a pair of ring elements $\mathbf{A}, \mathbf{B}$ with RSFs $\mathcal{A}$ and $\mathcal{B}$ we obtain using the unitarity of the kernel

$$\mathbb{D}^{j}_{mn}(g^{-1}) = [\, \mathbb{D}^{j}_{nm}(g)\,]^{*}\ ,$$

$$(\mathbf{A}, \mathbf{B}) \equiv \underset{j}{S} \sum_{m,n} \mathcal{A}^{*}_{mn}(j)\, \mathcal{B}_{mn}(j) = \underset{g}{S}\, A^{*}(g)\, B(g)\,,\ j \in \mathcal{J},\ g \in \mathcal{G}\,, \tag{7}$$

which is a mapping of $\mathcal{R} \times \mathcal{R}$ into the field of complex numbers with the properties of the inner product.

For $\mathbf{A} = \mathbf{B}$ (7) gives the familiar Parseval equation and allows us to define a Hilbert-Schmidt norm for the ring as:

$$\| \mathbf{A} \|^{2} \equiv \underset{j}{S} \sum_{mn} |\, \mathcal{A}_{mn}(j)\, |^{2} = \underset{g}{S}\, |\, A(g)\, |^{2},\ j \in \mathcal{J}, g \in \mathcal{G}\,, \tag{8}$$

In order to clarify the connection between the left-ideal indices m, n of the RSF and the chains of subgroups of $\mathcal{G}$, consider a maximal proper subgroup $\mathcal{H}\ \mathcal{G}$. Under right multiplication by the subset $h \in \mathcal{H}$ some of the left-ideals $\mathcal{I}^{j}_{m}$ are no longer mapped into each other and $\mathcal{J}^{j}$ becomes a set of two-sided ideals which can be in turn broken up uniquely into left-ideals and so the index m can be substituted by the pair (j_1), m_1 which label the two-sided and left ideals under the subgroup. When this is done repeatedly, following a chain of subgroups of $\mathcal{G}$ (as the well-known canonical chain in the permutation group $S_n \supset S_{n-1} \supset \cdots \supset S_1$ or similar ones for the unitary, orthogonal and symplectic groups), the left-ideal index m becomes a string of two-sided ideal indices $(j_1, j_2, \ldots, j_{n-1})$ which are the irreducible representation labels for each of the subgroups of the chain. These are the Yamanouchi symbols for the permutation groups[17,18] and the indices of the Gel'fand-Tsetlin kets for the unitary[19], orthogonal[20] and symplectic[21] groups.

Finally, we want to remark that a pair of transforms analogous to (5) between a scalar function on the UIR space $\mathcal{J}$ and another on the manifold of classes of $\mathcal{G}$ can be set up,[22] but does not seem to have found a physical application.

3. CONVOLUTION AND COUPLING

Consider the transformation of the ring elements under left multiplication by a fixed group element $h \in \mathcal{G}$:

$$\mathbf{A} \xrightarrow{h} \mathbf{A}' = h\mathbf{A}$$

It is easy to see from Eqs. (1) and (5b) that this induces on the GSF and RSF the following transformations:

$$A(g) \xrightarrow{h} A'(g) = A(h^{-1}g) \tag{9a}$$

$$\mathcal{A}(j) \xrightarrow{h} \mathcal{A}'(j) = \mathcal{D}^j(h)\, \mathcal{A}(j)\,, \tag{9b}$$

where the group product law is followed, i. e.

$$A(g) \xrightarrow{h_2} A'(g) \xrightarrow{h_1} A''(g) = A'(h_1^{-1}g) = A(h_2^{-1}h_1^{-1}g)\,,$$

while

$$A(g)\, h_1 h_2\; A''(g) = A\big([h_1 h_2]^{-1} g\big)\,.$$

In particular, the ring element $\tilde{\mathbf{e}}^{j_0}_{n_0 m_0}$ whose GSF is $\mathfrak{D}^{j_0}_{m_0 n_0}(g^{-1})$ and RSF is $\delta_{n_0 m}\delta_{m_0 n}\delta_{\mathfrak{g}}(j_0, j)$ has the norm $\delta_{\mathfrak{g}}(j_0, j_0)$ and the familiar transformation properties of the ket $|j_0 n_0>$ i.e.

$$|j_0 n_0> \xrightarrow{h} U(h)|j_0 n_0> = \sum_n |j_0 n> \mathfrak{D}^{j_0}_{nn_0}(h) .$$

Transformations of the ring elements under right multiplication (i.e. $\mathbf{A} \xrightarrow{h} \mathbf{A}h^{-1}$) need not be considered separately since the involution induced on the ring by the correspondence $A(g) \to A(g^{-1})$ makes them equivalent.

Consider now two ring elements $\mathbf{A}$, $\mathbf{B}$ with their GSFs $A(g)$, $B(g)$, and RSFs $\mathfrak{A}(j)$, $\mathfrak{B}(j)$. Then, to the product $\mathbf{C} = \mathbf{AB}$ corresponds, through (3) and (1) the RSF

$$\mathfrak{C}_{mn'}(j) = \sum_r \mathfrak{A}_{mr}(j)\mathfrak{B}_{rn}(j) \tag{10a}$$

(which is just the product of matrices), and the GSF

$$C(g) = \mathop{S}_{h} A(h)B(h^{-1}g) \equiv A \circ B(g), \; h \in \mathcal{G}, \tag{10b}$$

The convolution over the group is associative but not commutative unless the group is abelian. This result is familiar from Fourier analysis[23, 24] where to the product of two functions $\mathfrak{F}(p)\mathfrak{H}(p)$ corresponds the convolution $(F \circ H)(x)$. For this kind of transforms, to the product $F(x)H(x)$ corresponds the convolution of the inverse transforms $\mathfrak{F} \circ \mathfrak{H}(p)$. This, however, is a peculiar property of the T_n-harmonic transform. In general, to the product:

$$C(g) = A(g)B(g) \tag{11a}$$

of two GSFs corresponds what will be called the coupling of two ring elements, $\mathbf{C} = \mathbf{A} \, c \, \mathbf{B}$. The corresponding RSF is

$$\mathrm{C}_{mn}(j) = \underset{j',j''}{S} \sum_{m'm''} \sum_{n'n''} \mathrm{A}_{m'n'}(j')\, \mathrm{B}_{m''n''}(j'') \times$$

$$\times \underset{g}{S}\, \mathcal{D}^{j}_{mn}(g)\, \mathcal{D}^{j'}_{n'm'}(g^{-1})\, \mathcal{D}^{j''}_{n''m''}(g^{-1})\, j, j'' \in \mathcal{S},\, g \in \mathcal{G} \tag{11b}$$

where we have used (5) and exchanged the sums over j', j'' and over g. This product is associative and commutative, it has a unit $A(g) = 1$ and an inverse when $A(g) = 0$. Furthermore, from (8) and (11a),

$$\| \mathbf{A} \circ \mathbf{B} \|^2 \leq \| \mathbf{A} \|^2 \| \mathbf{B} \|^2 .$$

Such a product, symmetric in its factors, can be extended to any number of them.

The harmonic transform of $A(g) = A^{(1)}(g)\, A^{(2)}(g) \ldots A^{(n)}(g)$ is:

$$\mathrm{A}_{mm'}(j) = \underset{j_i}{S} \sum_{m_1 m_1'} \mathrm{A}^{(1)}_{m_1 m_1'}(j_1) \ldots \mathrm{A}^{(n)}_{m_n m_n'}(j_n) \left({j \atop mm'} \middle| {j_1 \atop m_1 m_1'} \cdots {j_n \atop m_n m_n'} \right), j_i \in \mathcal{S},$$

where we have defined

$$\tag{12}$$

$$\left({j \atop mm'} \middle| {j_1 \atop m_1 m_1'} \cdots {j_n \atop m_n m_n'} \right) = \underset{g}{S}\, \mathcal{D}^{j}_{mm'}(g)\, \mathcal{D}^{j_1}_{m_1 m_1'}(g^{-1}) \ldots \mathcal{D}^{j_n}_{m_n m_n'}(g^{-1}),\, g \in \mathcal{G} .$$

$$\tag{13}$$

This integral[25,26] is the one encountered when calculating Clebsch-Gordan coefficients. In fact, when using the formalism for the group SO_3, the Clebsch-Gordan series yields, for the simplest $n = 2$ case:

$$\left({j \atop mm'} \middle| {j_1 \atop m_1 m_1'} ; {j_2 \atop m_2 m_2'} \right) = \frac{8\pi^2}{2j+1}\, C(j_1 j_2 j ; m_1 m_2 m)\, C(j_1 j_2 j ; m_1' m_2' m')^{*} .$$

Notice that when the group $\mathcal{G}$ is continuous and non-compact, the product of two distributions may encounter difficulties.

Recurrence relations can be set up and we can express the n-factor coupling coefficient to the $(n$-1)-factor coefficient involving any of the n-1 j-indices in any order. In (13) however, the coupling order is irrelevant and all factors are taken on the same footing.

This is important, in the use of the Poincaré group for n-body relativistic kinematics. The motivation for (13) here is completely straightforward and requires no auxiliary construction as in Ref. 26.

A coupling coefficient $(g\, g_1, \ldots, g_n)$ dual to (13) can be set up through summing (and integrating) over row-, column- (and representation) indices in a product of $\mathfrak{D}$'s as in (13).

Classical Fourier analysis leads us to expect that if a function satisfies a partial differential equation[27] its transform will satisfy an algebraic equation and will thus turn it into a distribution with support on a lower-dimensional manifold. This is indeed the case, and rather complicated-looking differential equations can be formulated as simple conditions on the transform function[10].

Finally, it must be remarked, many of the properties[28] of the n-dimensional Fourier transform, as behaviour under a general linear transformation $x \to Mx$ of the group space are peculiar to T_n, where the group and representation-space have the same metric.

4. KERNEL REDUCTION, DAUGHTER PARTIAL WAVES AND FACTORIZATION.

Consider the T_n-harmonic transform (ordinary n-dimensional Fourier transform) of a radial function[29]:

$$\mathfrak{F}(p) = \underset{x}{S}\, F(r)\, e^{ip \cdot x}, \quad r = (x^2)^{\frac{1}{2}}, \quad x \in T_n . \tag{14a}$$

Hence $\mathfrak{F}(p)$ will be a function of $s = (p^2)^{\frac{1}{2}}$ only, and we can perform $(n$-1) integrations over the surface of an n-dimensional sphere and reduce it to the form:

$$\mathfrak{F}(s) = 2\pi s^{-\frac{n-2}{2}} \int_0^\infty F(r)\, r^{n/2} J_{\frac{n-2}{2}}(2\pi r s)\, dr . \tag{14b}$$

This example illustrates two points: First, the function $F(x)$ has the same value for all the points on the surface of a sphere in group space and thus the integrations over the sphere can be "factored out" and the kernel in Eq. (14a) reduced to the kernel in Eq. (14b) which is in general (though not here) more tractable. Second, when looking at the transform inverse to (14a) which gives $F(x)$, we may regard $F(x)$ as an even function on a line (diameter) and decide to obtain the T_n-transform (14) or a T_1-transform, $\mathfrak{J}_1(q)$. The term "daughter Regge poles" has been used in this context to denote a family of poles of the $O_{2,1}$-transform of a function whose $O_{3,1}$-transform exhibits a single pole. The position and residue of this ("Toller-Lorentz") pole determines the position and residues of the daughter Regge poles.

We can parametrize the SO_n group manifold[11, 12, 30] by observing that it is the direct product of the SO_{n-1} submanifold and the surface of the n-dimensional unit sphere. Thus take $u_p \in SO_p$ and let $r_{ab}(\theta)$ be a rotation in the a-b plane; enclosing collective variables in curly brackets, we can write:

$$u_n(\{\theta^{(n)}\}) = u_{n-1}(\{\theta^{(n-1)}\}) \times$$

$$\times\, r_{n-1,n}(\theta^{(n)}_{n-1,n}) \ldots r_{23}(\theta^{(n)}_{23})\, r_{12}(\theta^{(n)}_{12}),\; u_2(\theta) = r_{12}(\theta)\,, \qquad (15)$$

and call it the Euler-angle parametrization[31]. For SO_3,

$$u_3(\alpha, \beta, \gamma) = r_{12}(\alpha)\, r_{23}(\beta)\, r_{12}(\gamma);$$

for SO_4,

$$u_4(\alpha, \beta, \gamma\,;\, \zeta, \theta, \phi) = u_3(\alpha, \beta, \gamma)\, r_{34}(\zeta)\, r_{23}(\theta)\, r_{12}(\phi)\,, \text{ etc.}$$

The Haar measure can be split as $du_n = du_{n-1}\, db_n$ where

$$db_n = \sin^{n-2}\theta_{n-1,n}\, d\theta_{n-1,n} \ldots \sin^2\theta_{34}\, d\theta_{34}\, \sin\theta_{23}\, d\theta_{23}\, d\theta_{12}\,. \qquad (16)$$

This will be the measure on the space of $SO_{n-1} \backslash SO_n$ cosets. We shall be specially interested in functions $F(\theta)$ on the space of two-sided cosets:

$$SO_{n-1} \backslash SO_n / SO_{n-1} \qquad (17)$$

i.e. functions of the angle $\theta_{n-1,n}$ only. For SO_3 this is the second Euler angle; for $SO_{2,1}$ or $SO_{3,1}$ it becomes[5] the one "non-compact" parameter.

Let us turn next to a brief description of the transformation kernel $\mathfrak{D}^{j}_{mm'}(u)$, its UIR indices j and "row and column" indices m, m'.

From the work of Gel'fand and Tsetlin[20] we know that the bases for irreducible representations of SO_n classified by the canonical chain $SO_n \supset SO_{n-1} \supset \ldots \supset SO_2$ can be written as:

$$\left| \begin{array}{l} j_{n,1}\, j_{n,2} \cdots\cdots j_{n,[n/2]} \\ \\ j_{n-1,1}\, j_{n-1,2} \cdots j_{n-1,[(n-1)/2]} \\ \vdots \qquad \vdots \\ j_{41} \quad j_{42} \\ j_{31} \\ j_{21} \end{array} \right\rangle ,$$

where $[n]$ is the largest integer smaller or equal to n. This ket transforms as the $\{j_{p,1}, j_{p,2}, \ldots\ldots j_{p,[p/2]}\} \equiv (j_p)$ UIR of SO_p. For convenience, denote by $(\overline{j_q})$ the set $\{(j_q), (j_{q-1}), \ldots, (j_2)\}$ and thus $(\overline{j_{p-1}})$ is the "row" or "column"-label for an SO_p representation. The ranges of the UIR indices are constrained[20] by:

$$j_{n1} \geq j_{n2} \geq \ldots \geq j_{n,[n/2]}$$

For odd n, $n = 2m+1$, $j_{nm} \geq 0$, while for even $n = 2m$, $j_{n,m-1} > |j_{nm}|$. The row-indices satisfy

$$j_{2p+1,k} \geq j_{2p,k} \geq j_{2p+1,k+1} \quad (k = 1, 2, \ldots, p);$$

$$j_{2p,k} \geq j_{2p-1,k} \geq j_{2p,k+1} \qquad (k = 1, 2, \ldots, p-1)$$

and

$$j_{2p+1,p} \geq |j_{2p,p}| .$$

Since a rotation $r_{p-1,p}(\theta)$ involves only coordinates $p-1$ and p, it commutes with all SO_{p-2} transformations and is diagonal (and independent) of the representation labels of $SO_n, SO_{n-1}, \ldots, SO_{p+1}$ we may in principle define the general Wigner d-functions as:

$$d^{(j_p)}_{(j_{p-1})(\overline{j_{p-2}})(j_{p-1})'}(\theta)$$

$$= \langle (j_n)\ldots(j_p)(j_{p-1})(\overline{j_{p-2}}) \,|\, r_{p-1,p}(\theta) \,|\, (j_n)\ldots(j_p)(j_{p-1})'(\overline{j_{p-2}}) \rangle . \quad (18)$$

where the bra and ket have been written horizontally in order to save space.

The expression (18) vanishes if any other bra- and ket-indices are different. No other identification for the d-functions is necessary, since the indices refer it to SO_p.

Instead of writing the expression for the $\mathfrak{D}$-matrix in terms of d-functions through the decomposition (15), let us turn to the familiar cases. For SO_2 we have $\mathfrak{D}^m(\theta) = d^m(\theta) = \exp(-im\theta)$. For SO_3 the familiar $d^j_{mm'}(\theta)$ functions (for rotations around the x-axis) appear in

$$\mathfrak{D}^j_{mm'}(\phi,\theta,\psi) = \mathfrak{D}^m(\phi)\, d^j_{mm'}(\theta)\, d^{m'}(\psi)$$

and a similar expression for $SO_{2,1}$.[5] For SO_4,

$$\mathfrak{D}^{j_1 j_2}_{jm,j'm'}(\alpha\beta\gamma;\zeta\theta\psi) = \sum_{m''} \mathfrak{D}^j_{mm''}(\alpha\beta\gamma)\, d^{j_1 j_2}_{jm''j'}(\zeta)\, d^{j'}_{m''m'}(\theta)\, d^{m'}(\psi),$$

and a similar one for $O_{3,1}$[5].

To have a function on the space of cosets (17) means to have in the last case, a function $F(\zeta)$ independent of the other parameters. We can now construct the integral (5a) and convince ourselves using (6b) that the integration can be performed on all the other parameters yielding Kronecker δs to zero for all row- and column-labels and introducing factors which are the volume of SO_{n-1} and the surface S_{n-1} of the $(n-1)$-dimensional sphere.[11] This means, for SO_3

$$\mathfrak{F}_{00}(j) = (2\pi)^2 \int_0^\pi \sin\theta\, d\theta\, F(\theta)\, d^j_{00}(\theta) \ ,$$

$$F(\theta) = \sum_{j=0}^{\infty} \frac{2j+1}{8\pi^2} \mathfrak{F}_{00}(j) \left[d^j_{00}(\theta) \right]^* \ ;$$

for SO_4

$$\mathfrak{F}_{00,00}(j,0) = 32\pi^3 \int_0^\pi \sin^2\theta\, d\theta\, F(\theta)\, d^{j0}_{000}(\theta) \ ,$$

$$F(\theta) = \sum_{j=0}^{\infty} \frac{j^2}{16\pi^4} \mathfrak{F}_{00,00}(j,0) \left[d^{j0}_{000}(\theta) \right]^* \ .$$

Observe that only the $(j,0)$-partial waves appear since the only SO_4-representations containing the scalar SO_3-representation are those (j_1, j_2) with $j_2 = 0$. For the Lorentz group this means that only Toller's $M = 0$ poles can contribute to the elastic scattering amplitude between spinless particles.[5] One can see that it is sufficient for one of the particles to be spinless for this restriction to hold.

Indeed, for SO_n the transform pair can be seen to be, from (16), (18), and a similar procedure,

$$\mathfrak{F}_0(j) = V(SO_{n-1})\, S_{n-1} \int_0^\pi \sin^{n-2}\theta\, d\theta\, F(\theta)\, d^j_0(\theta) \ , \tag{19a}$$

$$F(\theta) = \sum_{j=0}^{\infty} \frac{d_n(j)}{V(SO_n)} \mathfrak{F}_0(j) \left[d^j_0(\theta) \right]^* \ . \tag{19b}$$

where we have abbreviated $\mathfrak{F}_0(j)$ for $\mathfrak{F}_{\overline{(0)}\,\overline{(0)}}(j, 0, \ldots, 0)$ and $d^j_0(\theta)$ for $d^{(j,0,\ldots,0)}_{(0)(0)(0)}(\theta)$. In (19b)

$$d_n(j) = \sum_{k=0}^{j} d_{n-1}(k)$$

(and $d_3(k) = 2k + 1$) is the dimension of the UIR and

$$V(SO_n) = V(SO_{n-1})\, S_n \quad (\text{and } V(SO_2) = 2\pi)$$

is the volume of the group, where $S_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface of the n-dimensional sphere.

Now, given some function $F(\mathcal{C})$ we may want to consider it as a function on the space of cosets (17) and find its transform (19) for any n. Indeed, this is a special case of the situation we presented at the beginning of this section: a given function $F(v)$ of an SO'_{n-1} sub-manifold of SO_n, not necessarily the canonical SO_{n-1} subgroup, but related to it through a similarity transformation $v = R^{-1}uR$ ($v \in SO'_{n-1}$, $u \in SO_{n-1}$; $R \in SO_n$, fixed).

Considered as a function of the SO'_{n-1} group manifold, it is the transform of:

$$F(v) = \underset{(j_{n-1})}{S} \sum_{(\overline{j_{n-2}})(\overline{j_{n-2}})'} \sum_{(\overline{j_{n-2}})(\overline{j_{n-2}})'} \mathfrak{Z}_{(\overline{j_{n-2}})(\overline{j_{n-2}})'}((j_{n-1}))\, \mathfrak{D}^{(j_{n-1})}_{(\overline{j_{n-2}})'(\overline{j_{n-2}})}(v^{-1}), \tag{20}$$

while considered as a function of a submanifold of SO_n it is the transform of:

$$F(v) = \underset{(j_n)}{S} \sum_{(\overline{j_{n-1}})(\overline{j_{n-1}})'} \sum \mathfrak{Z}_{(\overline{j_{n-1}})(\overline{j_{n-1}})}((j_n))\, \mathfrak{D}^{(j_n)}_{(\overline{j_{n-1}})'(\overline{j_{n-1}})}(v^{-1}). \tag{21}$$

For $v \in SO'_{n-1}$, the transformation kernel in (21) can be written as:

$$\mathfrak{D}^{(j_n)}_{(\overline{j_{n-1}})'(\overline{j_{n-1}})}(v^{-1})$$

$$= \sum_{(j_{n-1})''} \sum_{(j_{n-1})'''} (B^{-1})^{(j_n)}_{(\overline{j_{n-1}})'(\overline{j_{n-1}})''}\, \mathfrak{D}^{(j_n)}_{(\overline{j_{n-1}})''(\overline{j_{n-1}})'''}(u^{-1})\, B^{(j_n)}_{(\overline{j_{n-1}})'''(\overline{j_{n-1}})}, \tag{22}$$

where $B = \mathfrak{D}(R)$. Since $u \in SO_{n-1}$, the decomposition (15) allows us to write:

$$\mathfrak{D}^{(j_n)}_{(\overline{j_{n-1}})''(\overline{j_{n-1}})'''}(u^{-1}) = \delta_{(\overline{j_{n-1}})''(\overline{j_{n-1}})'''}\ \mathfrak{D}^{(j_{n-1})''}_{(\overline{j_{n-2}})''(\overline{j_{n-2}})'''}(u^{-1})\ , \quad (23)$$

where the Kronecker-δ in the representation label (j_{n-1}) is a product of Kronecker-δs in the individual indices, and the use of the symbolic identity $(\overline{j_{n-1}}) \equiv (j_{n-1})(\overline{j_{n-2}})$ is clear.

Equation (23) is independent of the representation label (j_n) as i.e. $\mathfrak{D}^{l}_{mm'}(\phi, 0, 0) = \exp(-im\phi)\ \delta_{mm'} = \mathfrak{D}^{m}(\phi)\ \delta_{mm'}$ is independent of the UIR label l. Replacing (23) in (22), (21) is now expanded in the matrix elements $\mathfrak{D}^{(j_{n-1})}(u)$ of SO_{n-1} and can be compared with (20) where v, being a dummy index and $SO'_{n-1} \simeq SO_{n-1}$, can be replaced by u.

Equality of (20) and (21) is implied by the following relations between the SO_{n-1} and SO_n RFSs, for any one partial wave $(j_n)_0$ of the latter:

$$\frac{d((j_{n-1}))}{V(SO_{n-1})}\ \mathfrak{F}_{(\overline{j_{n-2}})'''(\overline{j_{n-2}})''}((j_{n-1})'') = \frac{d((j_n)_0)}{V(SO_n)}\ \times \quad (24)$$

$$\times \sum_{(\overline{j_{n-1}})(\overline{j_{n-1}})'} \sum B^{(j_n)_0}_{(j_{n-1})''\ (\overline{j_{n-2}})''',\ (\overline{j_{n-1}})}\ \mathfrak{F}_{(\overline{j_{n-1}})(\overline{j_{n-1}})'}((j_n)_0)\ (B^{-1})^{(j_n)_0}_{(\overline{j_{n-1}})',\ (j_{n-1})''(\overline{j_{n-2}})''}\ .$$

We can draw two conclusions from the general form of Eq. (24). First, that to any *one* partial wave $(j_n)_0$ of the RSF in SO_n corresponds a family of daughter partial waves (j_{n-1}) of the RSF in SO'_{n-1} given by (23), i.e. all those (j_{n-1}) contained in $(j_n)_0$. This corresponds in Toller's treatment[5, 6] that any *one* Lorentz pole (in the $SO_{3,1}$-representation manifold) generates a family of daughter Regge poles (in the $SO_{2,1}$-representation manifold).

Second, a general form of factorization holds: if the SO_n-RSF can be factored as:

$$\mathfrak{F}_{(\overline{j_{n-1}})(\overline{j_{n-1}})'}((j_n)) = \rho^{(j_n)}_{(\overline{j_{n-1}})}\ \tau^{(j_n)}_{(\overline{j_{n-1}})}\ , \quad (25)$$

then (24) implies that the SO_{n-1}-RSF can be factorized in a similar fashion with:

$$\rho^{(j_{n-1})}_{(\overline{j_{n-2}})} = K \sum_{(\overline{j_{n-1}})'} B^{(j_n)}_{(\overline{j_{n-1}})(\overline{j_{n-1}})'} \rho^{(j_n)}_{(\overline{j_{n-1}})'} ,$$

$$\tau^{(j_{n-1})}_{(\overline{j_{n-2}})} = K \sum_{(\overline{j_{n-1}})'} \tau^{(j_n)}_{(\overline{j_{n-1}})'} (B^{-1})^{(j_n)}_{(\overline{j_{n-1}})'(\overline{j_{n-1}})} , \tag{26}$$

where

$$K = \left[\frac{V(SO_{n-1})\, d((j_n))}{V(SO_n)\, d((j_{n-1}))} \right]^{\frac{1}{2}} .$$

The physically relevant case[5] which has so far been studied is the expansion of the scattering amplitude of an elastic reaction in terms of the matrix elements of the UIRs of the little group of the momentum transfer. For equal masses, the general little group of the (spacelike) momentum is the $O_{2,1}$ which leaves the X_3-direction invariant while for forward scattering it becomes $O_{3,1}$. In the corresponding SO_4 problem (defining the space coordinates as x_1, x_2, x_3, x_4) the canonical SO_3 subgroup involves coordinates x_1, x_2, and x_3, while the non-canonical SO_3' one would involve coordinates x_1, x_2 and x_4 (leaving x_3 invariant). The rotation R which realizes the equivalence between SO_3 and SO_3' is a rotation by $\pi/2$ in the $x_3 - x_4$ plane and the corresponding B-matrix in Eq. (22) becomes an SO_4 Wigner d-matrix $d^{\lambda M}_{jml}(\pi/2)$ which is diagonal in the SO_2 indices. The $O_{3,1}$ case[6, 32] is clearly more difficult since the $O_{2,1}$ subgroup and the canonical O_3 subgroup are not isomorphic and the rotation R does not belong to $O_{3,1}$, but to its complexification[33]. Furthermore, in the reduction of the $O_{3,1}$ representation (λ, M) into $O_{2,1}$ representations (l), each of the latter $(l = \lambda, \lambda - 1, \ldots)$ appears twice[6, 32, 34], thus, an extra relation[35] between the (residues of the poles of the) transformation matrices B has to be used in order to prove that the factorization of (the residues of the poles of) the RSF in the group implies their factorization in the subgroup.

5. HARMONIC ANALYSIS ON THE POINCARE GROUP

Let $\mathfrak{J}$ and $\mathfrak{H}$ be two groups; $x \in \mathfrak{J}$ and let $f_h(x)$ be an automorphism of $\mathfrak{J}$, determined by an element $h \in \mathfrak{H}$. We define the semi-direct product of $\mathfrak{J}$ and $\mathfrak{H}$, $\mathfrak{G} = \mathfrak{J}(f)\mathfrak{H}$ as a set of all elements $g = (x, h)$ with the product law $(x_2, h_2)(x_1, x_1) = (x_2 \cdot f_{h_2}(x_1), h_2 \cdot h_1)$.

We are especially interested in the cases where $\mathfrak{H}$ is the component of the identity in one of the SO_n or $SO_{n-1,1}$ groups and $\mathfrak{J}$ is T_n, the n-dimensional translation group. The product law takes the form:

$$(x_2, h_2)(x_1, h_1) = (x_2 + h_2 x_1, h_2 h_2) .$$

The usual method[36, 37] of constructing bases for UIRs of the inhomogeneous unimodular orthogonal ISO_n (and pseudo-orthogonal of the type $ISO_{n-1,1}$) groups makes use of the subgroup $\mathfrak{J}$ in order to build a ket $|pm>$, where the n-dimensional vector p labels the UIRs of $\mathfrak{J}$ and m is an auxiliary label (or collection of labels) which will be affected by transformations in $\mathfrak{H}$ which leave the vector p invariant. We call the group of these elements the little group (or stability group) of p.

We define the ***stratum*** of p as the set of all p' which have little groups isomorphic to that of p. For ISO_n we choose the metric of the vector space of $p = (p_1, p_2, \ldots, p_n)$ to be $g_{\mu\mu} = 1$ (all others zero), and there are two strata: for (a) $p \neq 0$ and (b) $p = 0$. Their little groups are SO_{n-1} and the full SO_n respectively. For $ISO_{n-1,1}$ we fix the metric of the $\mathfrak{J}$ carrier space to be $g_{00} \equiv g_{nn} = -g_{11} = \ldots = -g_{n-1,n-1} = 1$. (All other zero). There are four strata: (a) $p^2 > 0$, (b) $p^2 < 0$, (c) $p^2 = 0$ but $p \neq 0$, and (d) $p \equiv 0$, whose little groups are SO_{n-1}, $SO_{n-2,1}$, ISO_{n-2} and the full $SO_{n-1,1}$ respectively. In each stratum we can define a standard vector $\hat{p}$, or a pair $\hat{p}^{(\pm)}$ if the stratum has two disconnected pieces. For ISO_n the standard vector of the first stratum can be taken to be $\hat{p} = (0, 0 \ldots, 0, M)$ where $M^2 = p^2 (M > 0)$. The second is trivially $\hat{p} = 0$. In $ISO_{n-1,1}(n > 2) M^2 = p^2$, the standard vectors in each stratum can be taken to be (a) $\hat{p}^{(\pm)} = (0, 0, \ldots, 0, \pm M)$,(b) $\hat{p} = (0, \ldots, 0, |M|, 0)$ (c) $\hat{p}^{(\pm)} = (0, \ldots, 0, 1, \pm 1)$, while for the $p = 0$ stratum again, trivially $\hat{p} = 0$. (For $SO_{1,1}$ the second stratum has instead two pieces while the third has four). By extension, the vectors in each of these strata will be called timelike, spacelike, lightlike and null.

The construction of the bases for the UIRs now proceeds using the identity $(x, 1)(O, h) = (O, h)(h^{-1}x, 1)$ in order to define the UIR labels and transformation properties of the kets under a general group element (x, h) as:

$$(x, h) \,|\, (MJ), pm >$$

$$= \exp(ix \cdot hp) \sum_{m'} \mathcal{D}^{j}_{m'm}(R_W(p, h)) \,|\, (MJ), pm' > \qquad (27)$$

where we have assumed that the kets with the same p transform irreducibly under the elements of its little group, thus labelled by the UIR index (J). The (generalized) Wigner rotation[36] $R_W(p, h)$ is an element of the little group of the standard vector $\hat{p}$ of the stratum of p and can be written as:

$$R_W(p, h) = L_{hp}^{-1} h L_p \qquad (28)$$

where L_p is an element of $\mathcal{H}$ with the property $L_p \hat{p} = p$ for each stratum. The freedom left in the definition of L_p allows us to ask for the condition[38, 39]

$$R_W(\hat{p}, h) = h \qquad (29)$$

when h belongs to the little group of $\hat{p}$. It is sufficient for (29) to hold that $L_{\hat{p}} = 1$.

Given a function $F(g)$ on the manifold of the group $\mathcal{G}$, we want to construct its harmonic transform. The Haar measure in $\mathcal{G}$ can be seen to be the product of the Haar measures on τ and $\mathcal{H}$: $dg = dx \; dh$.

In order to obtain the transformation kernel we must find the matrix element of the general group element (x, h) between the kets $|(M, J), p, m>$ and for this we make use of (27) and the fact that in T_n, $<p \,|\, p'> = \delta^n(p-p')$.

Consider first the ISO_n groups. Denote by $\mathbf{p}$ the $(n-1)$-dimensional vector $(p_1, p_2, \ldots, p_{n-1})$ and thus:

$$d^n p = dM^2 (2p_n)^{-1} d^{n-1}\mathbf{p} = dM^2 \, d_\Omega p\,,$$

so that we can write:

$$\delta^n(p - p') = \delta(M^2 - M'^2)\, \delta_\Omega(p, p')\,, \qquad (30)$$

where

$$\delta_\Omega(p, p') = 2p_n \delta^{n-1}(\mathbf{p} - \mathbf{p}') \tag{31a}$$

is the Dirac distribution on the $(n-1)$-dimensional sphere $p^2 = M^2$.

No confusion will arise if we use the same symbols for the $ISO_{n-1,1}$ groups: $\mathbf{p}$ will have the same meaning as above, but we choose:

$$d^n p = dM^2 (2p_0)^{-1} d^{n-1}\mathbf{p} = d^n p \, d_\Omega p,$$

so that we can write (30),

$$\delta_\Omega(p, p') = 2p_0 \delta^{n-1}(\mathbf{p} - \mathbf{p}') \tag{31b}$$

being the Dirac distribution on the $(n-1)$ dimensional hyperboloid $p^2 = M^2$.

The UIR matrix elements can thus be written, using (28), as:

$$\mathcal{D}^{MJ}_{p'm', pm}(x, h) = \exp(ix \cdot hp) \, \mathcal{D}^{J}_{m'm}(L_{p'}^{-1} h L_p) \, \delta_\Omega(p', hp) \,, \tag{32}$$

where (31a)(resp. (31b)) is meant for ISO_n (resp. $ISO_{n-1,1}$). For SO_n the form of the UIR label (J) and the row- and column-labels m, m' have been detailed in the last section.

For the $SO_{n-1,1}$ groups, let it suffice to say that the index J_{n1} is replaced by a (complex) index λ which does not obey any of the branching relations, thus $J_{n-1,1}$ only has a lower limit and the representation is infinite dimensional. For details see Refs. (40) and (41).

It is immediate to check that (32) fulfills:

$$\int d_\Omega p'' \sum_{m''} \mathcal{D}^{MJ}_{pm, p''m''}(x_2, h_2) \, \mathcal{D}^{MJ}_{p''m'', p'm'}(x_1, h_1) = \mathcal{D}^{MJ}_{pm, p'm'}(x_2 + h_2 x_1, h_2 h_1), \tag{33}$$

the product law of ISO_n (resp. $ISO_{n-1,1}$), $d_\Omega p$ is the surface element of the sphere (resp. hyperboloid), and the product law of the UIR matrices of the little group of the stratum of p has been used. This holds for any stratum.

The harmonic transform pair (5) can now be written between a GSF $A(x, b)$ and a RSF $\mathcal{A}(M, J)$ of ISO_n (resp. $ISO_{n-1,1}$) as

$$\mathcal{A}(M, J) = \underset{x}{S}\ \underset{b}{S}\ A(x, b)\ \mathcal{D}^{MJ}(x, b),\ x \in \mathfrak{I}, b \in \mathcal{H}\ , \tag{34a}$$

$$A(x, b) = (2\pi)^{-n} \int dM^2\ \underset{J}{S}\ \mathrm{Tr}\ [\mathcal{A}(M, J)\ \mathcal{D}^{MJ}((x, b)^{-1})],\ J \in \mathcal{S}(M), \tag{34b}$$

where the integration range of M^2 is from 0 to $+\infty$ (resp. from $-\infty$ to $+\infty$) and the generalized sum over the little group UIR space refers to that of the corresponding stratum with the kernel (32) and the trace is taken over the row- and column-labels p, m as in (33). Notice that in the integration the strata $M = 0$ have zero measure so that, unless the RSF A has a Dirac δ distribution with support at that point, it may be disregarded. We shall return to this point below. We can check that (34b) is the transform inverse to (34a), replacing (34a) into (34b), exchanging the integrations over group and representation space we obtain an identity if a relation (6a) holds for $\mathcal{G}$. This can be checked directly using (32) and (6a) itself for the little group of every stratum.

As remarked in Section 3, the GSF $A(x, b) = \mathcal{D}^{MJ}_{p'm',\,pm}\ ((x, b)^{-1})$ henceforth denoted by $|(MJ)\,p'm', pm>$ has

$$\mathcal{A}_{q'n',\,qn}(M'J') = \delta_{\mathcal{S}}(M'J', MJ)\ \delta_{\Omega}(q', p)\ \delta_{\Omega}(q, p')\ \delta_{n'm}\delta_{nm'}$$

for its RSF and the transformation properties of the ket $|(MJ)\,pm>$.

The overlap function:

$$<(M_1 J_1)\,p_1' m_1', p_1 m_1\,; \ldots ; (M_n J_n)\,p_n' m_n' p_n m_n |\, (M_1 J_1) \ldots, (M_n J_n)\,; (MJ)\,p'm', pm>\,,$$

can be expanded using (12) or directly through (32) and (34a) in terms of the overlap function in $\mathfrak{I}$, $\delta^n(\Sigma p_j - p)$, and an integral in $\mathcal{H}$. A similar treatment has been presented by Roffmann in Ref. (26) for the Poincaré group, so we shall not repeat it here[42].

It has been Lurçat's idea[9] that, since a description of the state of an object can be given by the (active) transformation g which takes a reference frame R_0 to a frame $R = gR_0$ fixed on the object, a field theory of these objects should involve not merely wave functions $F^J_m(x)$ on Minkowski space, x, but functions $F(g)$ on the Poincaré group manifold $g = (x, \Lambda)$. If the wave $F(g)$ is to represent stable objects of definite mass and spin, momentum and helicity it must satisfy the differential equations (on group space) $P^\mu P_\mu F(g) = M^2 F(g)$ and $W^\mu W_\mu F(g) = -M^2 J(J+1) F(g)$ where W is the Pauli-Lubanski vector[37], $P_\mu F(g) = p_\mu F(g)$ and $S_3(P) F(g) = m F(g)$ where $S(P)$ is the relativistic spin operator[37].

The new feature of this point of view is that we are allowed to construct more general objects with both a mass and spin distribution.

The complementary approach is to make a statement on the functional dependence of the GSF on $\mathcal{G}$ much in the same way as we did in Section 4. As a simplifying analogy in a lower-dimensional space, consider the three-dimensional Euclidean space, where $g = (\mathbf{x}, r(\phi, \theta, \psi)) \in ISO_3$ transforms a reference frame R_0 to a frame R fixed on the object. In 3-space we can have three kinds of objects: (a) Pointlike, i.e. invariant under rotations $r(\phi, \theta, \psi)$. The wave function $F(g)$ will thus be a function on the space T_3 and reduces to an $F(\mathbf{x})$. (b) Axially-symmetric, i.e. invariant under rotations ψ around the body axis. The wave function $F(g)$ will thus be a function on the space of cosets ISO_3/SO_2 and thus, the function reduces to an $F(\mathbf{x}, \phi, \theta)$. (c) Completely assymetric, whereupon its description needs the full $F(\mathbf{x}, \phi, \theta, \psi)$. We will not consider objects with discrete symmetry groups nor infinitely-extended objects, i.e. $F(g)$ as a function on coset spaces of the type $T_3 \backslash ISO_3 \simeq SO_3$. We can thus count the different cases as the homogeneous spaces[11] of the group: all the coset spaces of ISO_3 by the (continuous) subgroups of SO_3 and the identity, namely $ISO_3/SO_3 \simeq T_3$, ISO_3/SO_2 and ISO_3 itself.

A similar treatment has been made for the Poincaré group, both from the geometrical[43] and the group-theoretic[8, 44] point of view. There are clearly more cases since, for instance, the axis-like objects may have a time-, space- or light-like axis. It turns out that there are eleven possible distinct homogeneous spaces. Among them, using Finkelstein's[43] notation, we can have a function on the whole Poincaré space [6], $F(x, \Lambda)$, a function on the Minkowski space [0], $F(x)$. Four other cases are interesting: those corresponding to the cosets by the subgroups SO_3, $SO_{2,1}$ and ISO_2, cases [3], [3′] and [3_0] respectively; and [4], by the nilpotent subgroup N in the Iwasawa decomposition. This last case, motivated by the geometry of electromagnetism, has been investigated by J. Nilsson and A. Kihlberg[8,44,45]

We write out (34a) with the kernel (32) for the element $g = (x, \Lambda)$ of the Poincaré group:

$$\mathfrak{Z}_{p'm',pm}(MJ) = \int d^4x\, d^6\Lambda\, F(x, \Lambda F(x,\Lambda) \exp(ix\cdot\Lambda p) \times$$

$$\times\ \mathfrak{D}^J_{m'm}(L^{-1}_{p'}\Lambda L_p)\, \delta_\Omega(p', \Lambda p)\,. \tag{35}$$

The $SO_{3,1}$ group element Λ can be written[46] as $\Lambda = RH_k$ where, for each stratum, R belongs to the little group of the standard vector $\hat{p}$ of the stratum of k, and $H_k = L_k$, where $L_k\hat{p} = k$ is such that (29) holds. The Haar measure over $SO_{3,1}$ can then be written[46] as $d^6\Lambda = d^3R\, d_\Omega k$.

Furthermore, in (35) the δ-distribution reduces the set of $\Lambda = RH_k$ as the subset $L_{p'}R'L_p^{-1}$ (R' belongs to the little group of $\hat{p}$). The argument of $\mathfrak{D}$ is now R' and Eq. (35) simplifies to

$$\mathfrak{Z}_{p'm',pm}(MJ) = \int d^4x\, d^3R'\, F(x, L_{p'}R'L_p^{-1}) \exp(ix\cdot p')\, \mathfrak{D}^J_{m'm}(R')\,, \tag{36}$$

where we have used the invariance of the Haar measure.

The RSF indices and variables may have support on several strata, and all of them are needed in order to reconstruct $F(x,\Lambda)$ through (34b), but for each stratum Eq. (36) holds with the corresponding decomposition of Λ.

One can point to the unconventional feature of having to interpret two "momentum" labels, p' and p. This is inevitable in (this basis of) the Poincaré partial-wave expansion, since the labels are imposed by those of the transformation kernel.

When the GSF is a function on one of the 7-parameter homogeneous spaces [3], [3′] or [3_0] : $F(x,\Lambda) = F(x, \Lambda R)$, the kernel (32) does not appear to reduce simply, as was the case for the orthogonal groups in the last section. We would like to point out, however, that for the particular column $p = \hat{p}$ (timelike), since $L_{\hat{p}} = 1$, the integration of the right-hand side of Eq. (36) over the (compact) divisor group can be performed, yielding

$$\mathfrak{Z}_{p'm',\hat{p}m}(MJ) = \delta_{m',0}\,\delta_{m,0}\,\delta_{\mathfrak{H}_D}(J,0) \int d^4x\, \exp(ix\cdot p')\, F(x, H_{p'})\,, \tag{37}$$

where $\mathfrak{H}_D$ is the representation space of the divisor group.

The treatment of the 8-dimensional homogeneous space [4] requires[8] the Iwasawa decomposition of $SO_{3,1}$, so that the above formalism does not yield a simple kernel reduction. Furthermore, the homogeneous spaces with non-compact divisors present more fundamental difficulties[11] due to the fact that wave functions are then not square-integrable on the Poincaré group manifold.

FOOTNOTES AND REFERENCES

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RESUMEN

Usamos la construcción clásica del anillo de un grupo para presentar un par de funciones, transformadas armónicas la una de la otra, como las coordenadas de un elemento del anillo en dos bases diferentes: una, una función $F(g)$ sobre la variedad del grupo $\mathcal{G}$, la otra $\mathfrak{F}(j)$ sobre el conjunto $\mathcal{J}$ de representaciones unitarias irreducibles (RUIs) del grupo. El Kernel de transformación está dado por los elementos de matriz de las RUIs $\mathfrak{D}^j(g)$.

Desarrollamos este formalismo matemático con objeto de presentar en forma concisa varios resultados de cinemática relativista, desarrollo en ondas parciales de Toller y algunas teorías de campo sobre el grupo de Poincaré. En particular, probamos para los grupos ortogonales tres teoremas que tienen análogos directos en el trabajo de Toller: 1) funciones sobre espacios de coclases se desarrollan en términos de una base reducida, 2) las transformadas armónicas respecto a un grupo y a uno de sus subgrupos dan una relación entre sus coeficientes de ondas parciales, conocidas de las relaciones entre los polos de Toller y Regge y, 3) la factorización (de Sciarrino y Toller) de los residuos del primero implica una factorización correspondiente de los residuos del segundo.