

RECENT DEVELOPMENTS IN RANDOM-MATRIX THEORY*

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ABSTRACT :

The aim of random-matrix theory is to provide a coherent account of the statistical properties of complex quantum systems; its most important applications have been to nuclear physics. The present paper attempts to review the work that has been carried out in recent years in this field, after providing a brief description of the basic ideas derived from earlier work. Attention is paid to the comparison with experimental data.

1. INTRODUCTION

In the last two decades much progress has been made in the understanding of the detailed characteristics of low-lying states in nuclei¹. But the situation is different when dealing with highly excited states lying, say, 8 to 10 MeV above the ground state in medium or heavy nuclei. No nuclear model exists that is capable of describing adequately these states, and even

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if there were one, the number of levels in this region of excitation is such that any description would be uninteresting. As in the analogous situation in statistical mechanics, physically significant results are obtained from averaging in some way over many levels rather than from the study of the positions or decay widths, say, of individual levels. In order to interpret the experimental results one must then provide theoretical models for such average properties of quantum-mechanical systems having a large number of energy levels in a given region.

For nuclear physics, such average properties fall roughly into two classes:

- a) global properties, examined over a large energy region; the level density (the number of levels per unit energy interval) or the strength function (the sum of the decay widths of all levels lying in a unit energy interval) are examples;
- b) local properties, examined over a narrow range in which the global properties are constant; of this type are the fluctuation distributions, for instance the distribution of nearest-neighbour spacings (the distances between adjacent levels), or that of the level widths.

For the global properties, simple models were proposed already many years ago. Thus the level density for a considerable energy range can be fit by a model which considers the nucleons to be particles of a Fermi gas²; this yields a density function

$$\rho(E) = \frac{C}{E^2} \exp(E^{1/2}). \quad (1.1)$$

A somewhat more detailed model³ gives a density of the form

$$\rho(E) = \frac{C}{(E - \Delta)^{5/4}} \exp(2a\sqrt{E - \Delta}) \quad (1.2)$$

where the constant a and Δ depend on the specific nucleus considered. Similarly, the optical model - in which the many-body problem is reduced to a single-particle one - is able to account for the energy variation of the strength function⁴.

As regards the spectrum fluctuations, the first account of the distribution of nearest-neighbour spacings is due to Wigner⁵. He realized, in the

first place, that there should be no correlation between levels of different spin or parity; secondly, he introduced the concept of level repulsion, that is to say, the idea that there is a correlation among levels of the same spin and parity such that the smaller the spacing the less frequently it occurs, and the probability for levels to coincide is 0. On this basis he proposed what is now called the "Wigner surmise": if we write s for the spacings and D for their average, then the spacing distribution function should be

$$p_W(0; x) = \frac{\pi}{4} x \exp(-\pi x^2/4) \quad \text{where } x = s/D. \quad (1.3)$$

This distribution is consistent with the experimental data^{6, 23} obtained from the scattering of very slow neutrons from heavy nuclei, where the orbital angular momentum of the incoming particles is almost always 0, so that only levels of the same spin and parity are excited.

It has been shown⁷ that when sufficient level distributions of the type (1.3) for different spins and parities are superimposed, then the correlations between the levels of the same spin and parity are obscured and the total spacing distribution approaches the limiting Poisson distribution

$$p_P(0; x) = e^{-x}. \quad (1.4)$$

Concerning the decay widths Γ in a given channel (for instance, neutron emission), Porter and Thomas⁸ suggested that the structure of the states might be sufficiently complex to make the central-limit theorem of statistics applicable; this then means that the reduced amplitudes γ (except for irrelevant factors, $\Gamma = \gamma^2$), for which the contributions from the component configurations sum, will have a Gaussian distribution; hence

$$p(\Gamma) = \frac{1}{2\sqrt{\Gamma}} \frac{\exp[-\Gamma/2\bar{\Gamma}]}{\sqrt{2\pi\bar{\Gamma}}}, \quad (1.5)$$

where $\bar{\Gamma}$ is the mean width; this distribution is in reasonable agreement with experiment⁶.

Thus different models are used to explain different features of the energy spectrum. The basic ideas of random-matrix theory, whose aim it is to satisfy the ensuing need for a unified approach to the statistical properties of spectra, were introduced in 1959⁹; this theory is to some extent modelled

on statistical mechanics, where one is also interested in the overall properties of a system time averages of dynamical quantities in this case. Now in statistical mechanics such time averages are never calculated explicitly: instead one introduces an ensemble of systems and replaces the time averages by averages over the ensemble; that this is justified is the aim of ergodic theory to prove. In a similar fashion, instead of treating the statistical properties of the eigenstates and levels of a given (and, in the nuclear case, not well known) Hamiltonian, an ensemble of Hamiltonians is introduced and ensemble averages are studied. The ergodicity of such ensembles (i. e. that the ensemble averages are equal to spectrum averages) has so far to be assumed.

In order to have objects of known mathematical structure, the Hamiltonian operators forming the ensemble are replaced by their matrix representations; the continuum part of their spectra is ignored, and the dimension of the matrices is allowed to tend to infinity. Moreover, each such Hamiltonian matrix is taken to represent only states of fixed angular momentum and parity; as mentioned above, a mixture of different angular momenta and parities tends to a distribution of independent states (for which the nearest-neighbour spacings, for instance, simply have a Poisson distribution) and thus presents no particular interest.

An extensive discussion, together with reprints of the more essential papers published on the subject before 1964, can be found in the book by Porter¹⁰; the mathematical properties known up to 1966 are extensively discussed in Mehta's book¹¹. The purpose of the present paper is to review some results obtained since then.

In section 2 we shall review some new results for the best known ensemble of random matrices proposed, the Gaussian orthogonal ensemble; its properties are by now rather well known, and as we shall see in section 2, it predicts local properties rather well but fails, for instance, for the level density. This, together with the fact that the physical conceptions on which the Gaussian orthogonal ensemble rests are not very firmly based, has led in the last few years to a search for more satisfactory ensembles. Reviewing these attempts is the purpose of section 3.

2. THE GAUSSIAN ORTHOGONAL ENSEMBLE

2.1 Definition

We consider the submatrix in the matrix representation of the Hamiltonian which corresponds to a given value of J and π ; we take this submatrix to be

real and symmetric (this can always be achieved if H is invariant under time-reversal¹¹). The ensemble will then consist of $N \times N$ matrices whose elements are real random numbers. The Gaussian orthogonal ensemble (GOE) is then defined by making the following two assumptions¹² which establish the joint distribution of these random numbers:

(i) *Orthogonal invariance.* This distribution - and hence all the statistical properties of the ensemble - should remain invariant when the same orthogonal transformation is carried out on all matrices belonging to it. This corresponds to treating all possible bases in Hilbert space as equivalent. Wigner's argument¹³ for this assumption was that there does not appear to exist a coordinate system in Hilbert space which plays a preferred role - except that in which the physical Hamiltonian is diagonal; every transformation of the base will, of course, diagonalize some of the matrices in the ensemble, but certainly not all of them.

(ii) *Statistical independence.* With no other justification than simplicity one requires that the matrix elements be distributed independently.

Assumption (i) implies that the joint distribution of the matrix elements H_{ij} ,

$$P(H) = P(H_{11}, H_{12}, \dots, H_{NN}) \tag{2.1}$$

is a function¹² only of the invariants of H , i.e. $\text{Tr } H, \text{Tr } H^2, \dots, \text{Tr } H^N$. Assumption (ii) then leads to

$$P(H) \propto \exp \left[-\frac{1}{2} \text{Tr } H^2 \right] = \prod_i \exp \left[-\frac{H_{ii}^2}{2\sigma^2} \right] \prod_{i > j} \exp \left[-\frac{H_{ij}^2}{\sigma^2} \right] \tag{2.2}$$

where we have chosen the origin of the energy scale such that the ensemble average of $\text{Tr } H$ is zero; assumption (ii) then implies that the ensemble average of each diagonal matrix element is also zero. In the last form in (2.2) we have eliminated the variables $H_{ji}, i < j$, since they are equal to the H_{ij} ; their standard deviation is then $1/\sqrt{2}$ times that of the diagonal elements.

If in (2.2) we change the variables from the H_{ij} to the eigenvalues E_i and $\frac{1}{2}N(N-1)$ variables¹², we obtain the distribution of the eigenvalues, known in this context as the Wishart distribution:

$$P(E) \equiv P(E_1, E_2, \dots, E_N) = \prod_{i < j} |E_i - E_j| \exp \left[-\sum_i E_i^2 / 2\sigma^2 \right] \tag{2.3}$$

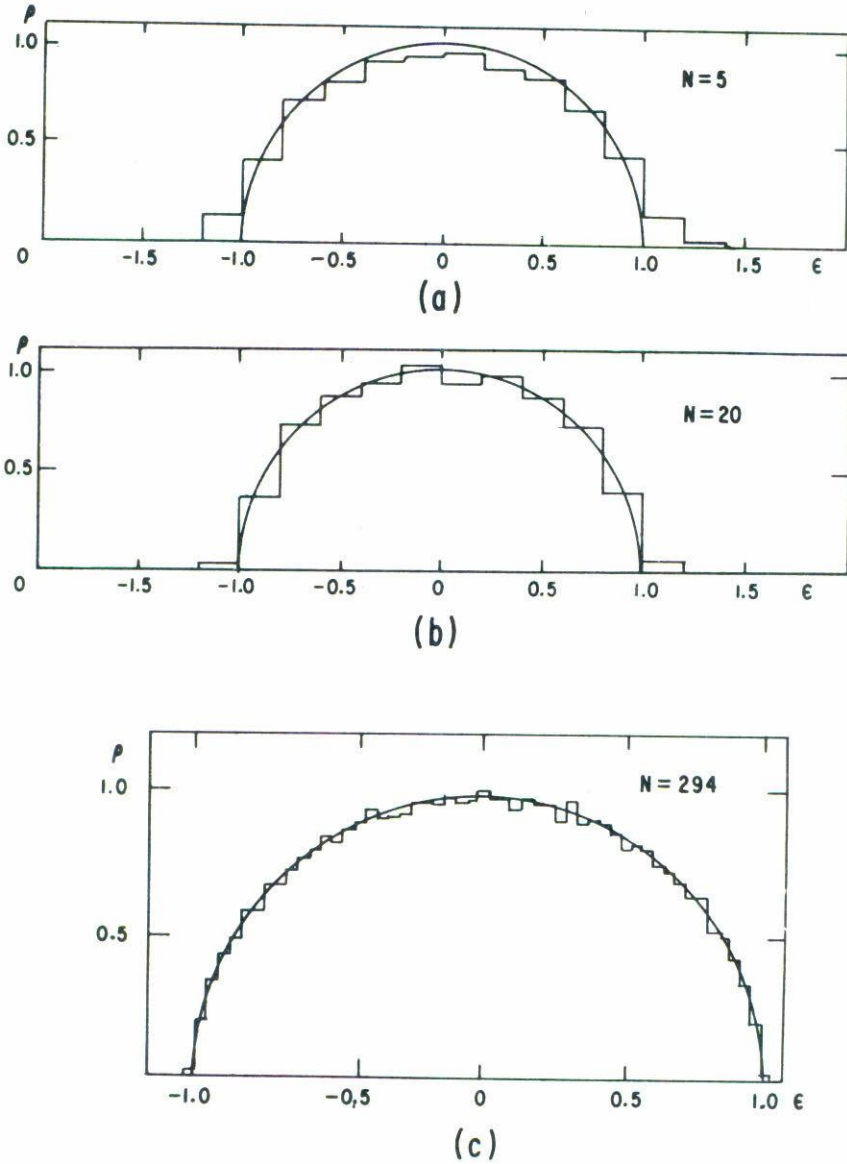


Fig. 1 Plots of the level density for the GOE. The histograms arise from Monte-Carlo calculations and the full line is Wigner's semicircle law $\rho \propto \sqrt{1-e^2}$, $e = E/(\sigma\sqrt{2N})$. Figs. 1a and 1b were taken from ref. 12 and Fig. 1c from ref. 28.

In this equation the level repulsion is already evident, since $P(E) = 0$ when any two eigenvalues coincide.

2.2 Global and local properties: theoretical results

The level density predicted by the GOE¹¹ follows the so-called "semicircle law", shown in Fig. 1 for the limit $N \rightarrow \infty$; the units in the abscissa are $E/(\sigma\sqrt{2N})$.

With what sort of data should such a prediction be compared? Before answering this question, it is important to note that a finite matrix must have a density function that decreases to 0 at the upper energy limit and to this extent is unphysical: experimentally, the level density increases until the continuum is reached. Thus in any comparison we must disregard the decreasing part of the density function predicted by a stochastic ensemble.

If, then, we compare the left half of the semicircle law with experimental data, as frequently proposed, we immediately face disagreement: the curvature of the semicircle is negative, while the experimental data (as may be seen from semi-empirical relationships like (1.2)) curve upwards.

It may seem paradoxical that a very simple scheme like the independent-particle model can yield a fair approximation to the level-density variations, while the GOE will not. But the GOE does not incorporate any of the specific features of nuclear interactions, not even those described by the single-particle model; it is thus not surprising that it should fail for the level density; the surprise is rather that it should describe many local properties quite well, as we shall see.

One might argue that the GOE should therefore not be compared directly to experiment; its matrices should be taken as representing the residual interactions when the valence nucleons are restricted to moving within one shell. And since we cannot analyse the experimental level density into the contributions of the individual shells, the comparison should be made with appropriate shell-model calculations. Such a comparison has been made by Ratcliff¹⁴ and also failed.

Therefore a global property like the level density is not correctly predicted by the GOE.

Of the local properties, the nearest-neighbour spacings for the GOE have been studied by Mehta and Gaudin¹⁵, who were able to obtain the exact analytical distribution function. Although it has a very complicated algebraic structure, its graph turns out to be almost indistinguishable from Wigner's surmise (1.3), as may be seen in fig. 2. The w statistic, developed by one of the authors as a measure of the level repulsion in nearest-neighbour spacing

distributions¹⁶, is exactly 1 for the Wigner surmise and 0.9525 for Mehta and Gaudin's exact result.

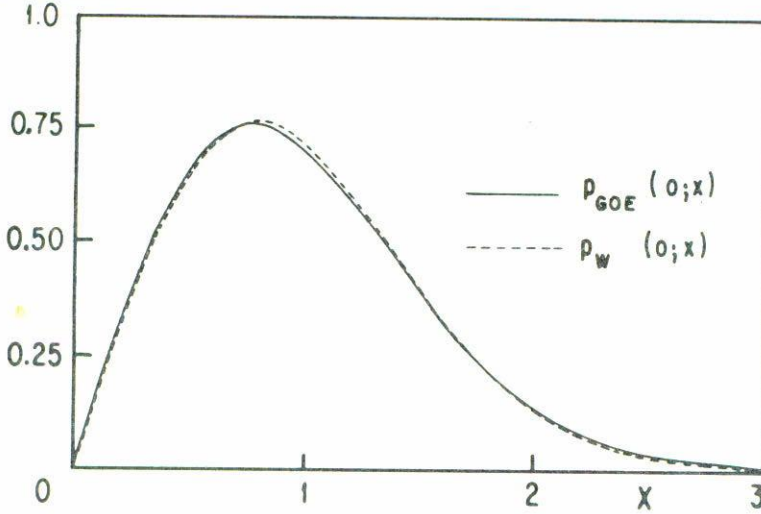


Fig. 2 Nearest neighbour spacing distribution p_{GOE} predicted by GOE (solid line) compared with Wigner's surmise (dashed line). (Taken from Ref. 11).

Results have also been obtained for the k^{th} -neighbour spacings

$$x_i^{(k)} = \frac{s_i^{(k)}}{D} = \frac{E_{i+k+1} - E_i}{D} \quad (2.4)$$

where D is the mean spacing $s_i^{(0)}$. The distribution function $p(k; x) dx$ is then the probability that a spacing $x_i^{(k)}$ lies between x and $x + dx$. For $p(1; x)$, Kahn¹⁷ obtained a closed expression in 1963. More recently, Mehta and Des Cloizeaux¹⁸ have obtained the general expression for $p(k; x)$ in terms of spheroidal functions, but numerical values are available only for $k \leq 3$. Fig. 3 shows the spacing distributions for $k \leq 10$, obtained from Monte Carlo calculations by Bohigas and Flores⁹, who diagonalized 344 matrices of dimension 61. The figure also shows the theoretical distributions (solid lines) for $k \leq 3$. It will be seen that the distributions $p(k; x)$ are asymmetric with respect to their centroid for small k , but become more and more symmetric as k increases. The dotted lines in the figure are normal distributions with centre $k + 1$ and standard deviation $\sigma(k)$ equal to the width of the corresponding Monte Carlo result. Clearly with increasing k the Gaussian approximation becomes better and better.

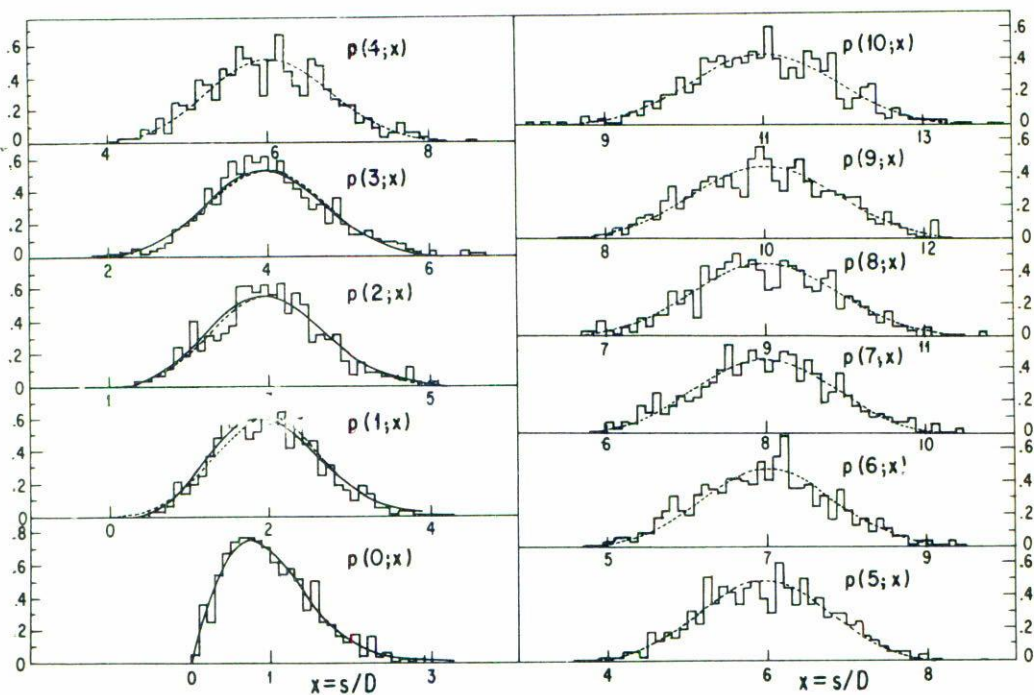


Fig. 3 Spacing distributions for the Gaussian orthogonal ensemble. (Taken from Ref. 19).

A further theoretical result of interest concerns the joint probability distribution of two adjacent nearest-neighbour spacings. This function was found by Mehta²⁰, it predicts a linear correlation coefficient²¹ between the two spacings of -0.27 .

A local property of a different kind, since it deals with the statistical properties of eigenvectors, is the distribution of the amplitudes of transitions leading to a reaction channel (for instance, the emission of neutrons); these amplitudes are proportional to the overlap of the eigenvector and the channel state; if we take the channel state to be parallel to one of the basis vectors of the matrices, then the distribution of the amplitudes (and the derived distribution of the widths, which are the squares of the amplitudes) is obtained from that of a single component of the eigenvectors¹². For the GOE, this distribution is that of a component of a unit vector distributed at random with equal probabilities assigned to equal areas on the N dimensional hypersphere's surface on which the unit vectors end; this is a consequence of postulate (i) in section 2.1. Explicit calculation gives, in the limit of $N \rightarrow \infty$, a Gaussian¹²; this gives for the widths the distribution proposed by Porter and Thomas⁸.

2.3 Comparison with experimental data

Before considering how the GOE compares with experiment, some general comments seem in order. The greater part of the experimental data comes from neutron resonance studies with heavy even-even nuclei; the experiments are difficult to carry out, since it is important to observe as far as possible a complete series of levels of the same spin and parity. Narrow resonances, below a limit set by the experimental arrangement, will not be seen, nor is it always easy to exclude levels belonging to another series. The resulting crude data must be corrected for these two effects; this implies lengthy and complex calculations, the effect of which on the comparison with theory is by no means easy to assess²³.

In fig. 4 we present the distribution for nearest-neighbour spacings taken from an extensive experimental series²³. The corrections for the two effects of missing and extraneous levels would affect some 6% of the levels in the worst case. The results are compatible with the Wigner surmise (1.3); the authors of the experimental paper used as a criterion the Δ statistic developed for this purpose by Dyson and Mehta²², rather than the usual χ^2 test, but the visual comparison also makes the agreement clear. Moreover, the correlation coefficient between adjacent spacings, given for several nuclei in Table I, is in quite good agreement with the theoretical value -0.27 .

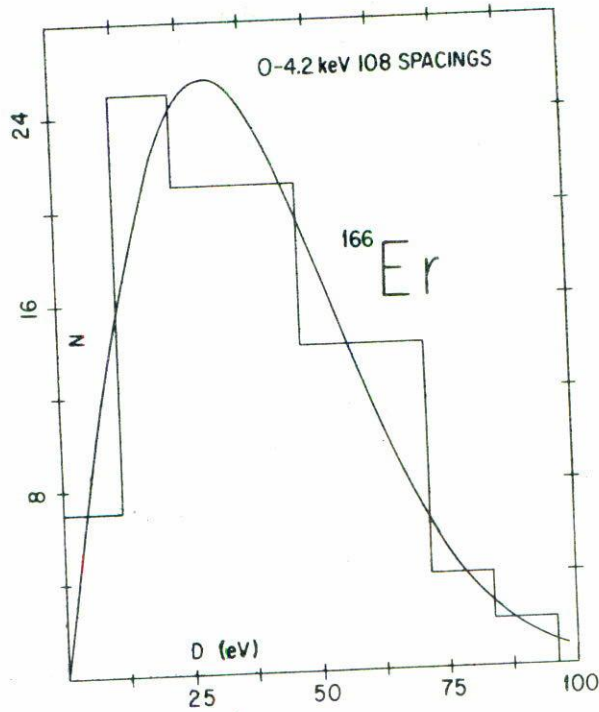


Fig. 4 Histogram for nearest-neighbour spacings for ^{166}Er , compared with Wigner's surmise. (Taken from Ref. 23).

TABLE I

Correlation coefficient for adjacent spacings. (Taken from Ref. 13, p. 210)

	^{166}Er	^{168}Er	^{182}W	^{184}W	^{152}Sm	^{172}Yb
N	109	50	41	30	70	55
E_{max}	4200	4700	2607	2621	3665	3900
$\text{cov}(s_j, s_{j+1})$	-0.22	-0.29	-0.37	-0.28	-0.26	-0.24
	± 0.08	± 0.14	± 0.15	± 0.18	± 0.11	± 0.13

N = number of levels

E_{max} = upper limit of energy interval (in eV) containing the N levels

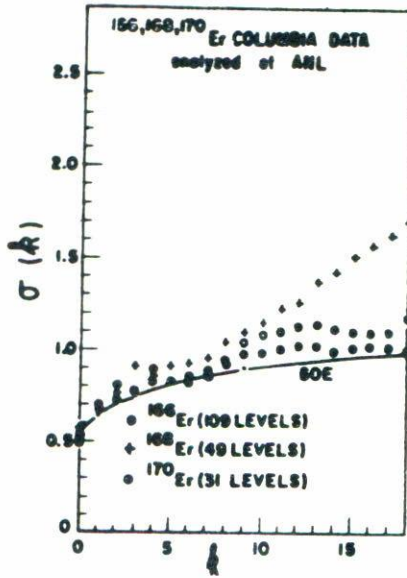


Fig. 5 Prediction of the GOE for $\sigma(k)$, compared with experimental data. (Taken from Ref. 13, p. 204).

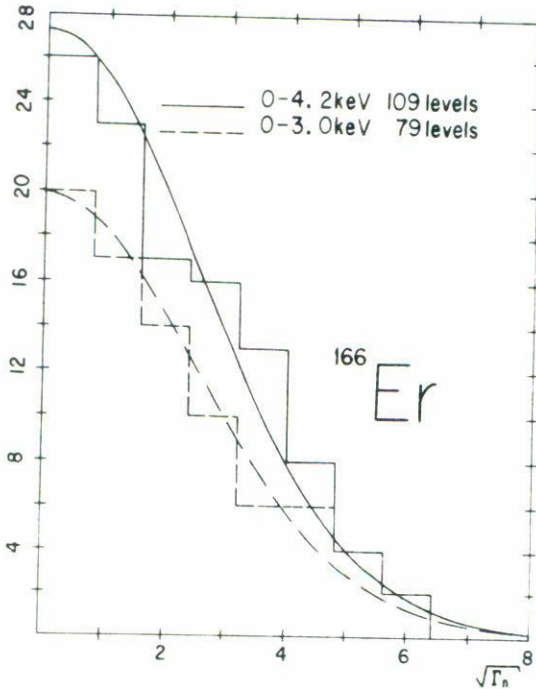


Fig. 6 Plot of the distribution of amplitudes for ^{166}Er from 0 to 3 and from 0 to 4.2 keV. The Porter-Thomas curves are shown for comparison. (Taken from Ref. 23).

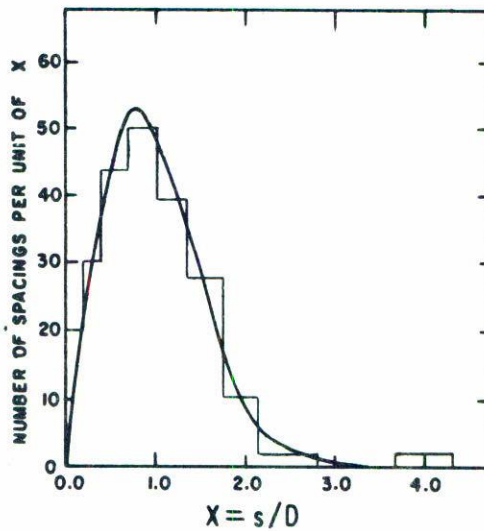


Fig. 7 Nearest neighbour spacing distribution for levels near an analogue state in ^{49}V . (Taken from Ref. 24).

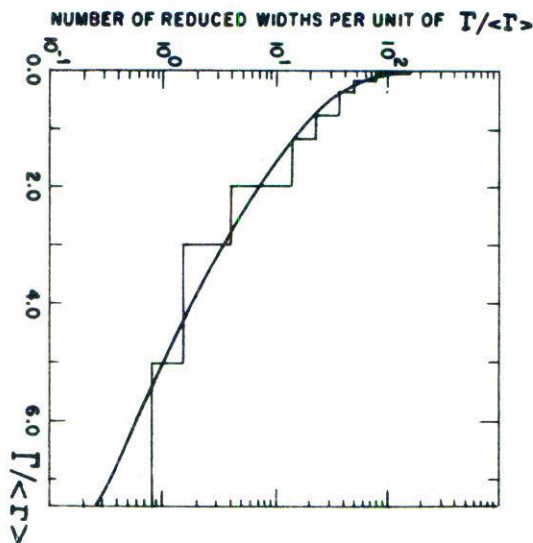


Fig. 8 Distribution of widths for levels near an analogue state in ^{49}V . (Taken from Ref. 24).

For the higher-order spacings the data are scarce. In fig. 5 we show the theoretical values of the standard deviations of the k^{th} -neighbour spacing distributions, obtained from Monte Carlo calculations¹⁹ when $k > 3$, and compare these with some experimental results. The fit is reasonable.

Fig. 6 shows the distribution of the neutron emission widths²³ for ^{166}Er , and for comparison the Porter-Thomas distribution; the mean of the theoretical curve is taken from the experimental points. Again experiment and theory agree quite reasonably.

The data we have discussed, together with most previous results, come from neutron scattering experiments. Recently, however, Bilpuch et al.²⁴ have been able to extract statistical information from an analysis of the resonances seen around an isobaric analogue state in ^{49}V . We present their results for the nearest-neighbour spacings and the proton-scattering widths in figs. 7 and 8, respectively. The data of fig. 7 have been corrected for the variation of the level density, which changes in this region by a factor of 2; it is only after this correction that the fit becomes good. (The Wigner law (1.3) supposes that the level density is constant.) In fig. 8 no such correction has been applied, yet the fit is surprisingly good: the authors explain this as due to the fact that the variation in the level density is compensated for by a similar but opposite variation in the strength function (i. e. the local average of the reaction widths divided by the average local spacing, or, equivalently, the total reaction width per unit energy interval)

The experimental results discussed here are typical of many other nuclear data; one may therefore conclude that the local statistical properties are well accounted for by the GOE, whereas the level density it predicts is in clear disagreement with nuclear data.

2.4 Ergodic properties

As was mentioned in the introduction, the aim of random matrix theory is to explain the statistical properties of a nucleus as averages over an ensemble of Hamiltonians. For this idea to work the ensemble must have two properties:

(i) The vast majority of members in the ensemble must have the same statistical properties, which will then be the ensemble averages: this we shall call the *internal ergodic property*. It is clear that this is only a consistency requirement; without it the ensemble averages are not meaningful.

(ii) The ensemble averages of the statistical properties must represent adequately those of the physical system under consideration, - the given

nucleus, in this case. We will call this the *external ergodicity* of the ensemble; its physical meaning is obvious, but in practice it is far more difficult to establish than internal ergodicity, since the lack of sufficiently stringent conditions on nuclear models usually means that it can only be shown to be valid piecemeal, by comparison with experimental results.

Some results for internal ergodicity have been obtained. Olson and Uppuluri²⁵ have succeeded in showing for an ensemble which is somewhat more general than the GOE* that almost all matrices in it, except a set of zero measure, have a level density which follows the semicircle law of fig. 1. (The corresponding external ergodic property does not hold, as we have seen from the experimental evidence.)

Another result is due to Brody and Mello²⁶, who prove that for an ensemble satisfying the orthogonal-invariance condition of section 2.1, almost all matrices, again except for a set of zero measure, yield the Porter-Thomas distribution for the widths. In this case the experimental data do not conflict with the validity of external ergodicity.

The problem of the internal ergodicity of the spacing distributions has not been solved; experimentally there is no reason to doubt the external property, as we have seen. Some theoreticians suspect that the local properties of a large class of ensembles may be independent of the kind of ensemble (see e. g. Dyson²⁷, French and Wong²⁸): if this should turn out to be true, then the good agreement between the Wigner surmise and experimental results might be of considerable general interest, though it would lose much of its usefulness in nuclear spectroscopy.

3. RECENT DEVELOPMENTS

It will be clear from the discussion of section 2 that the GOE does not constitute a satisfactory solution to the problem of building a suitable ensemble for the representation of nuclear properties. In order to provide a background for the description of more recent attempts to overcome some of the limitations of the GOE, it is perhaps worth while setting out explicitly the requirements that an ideal ensemble should satisfy:

* Their ensemble consists of all real symmetric matrices whose elements are independently distributed with zero mean, the same variance for all off-diagonal elements and all moments of the distribution finite.

(i) The definition of the ensemble should be based on the principles underlying the dynamics of the systems to be described; in our case this means that the dynamics of the nuclear force should provide the framework, or at least many-body quantum theory should do so. In the case of classical statistical mechanics, in analogy to which random-matrix theory is built up, this has proved possible: Newtonian mechanics yields, via Liouville's theorem, a measure over the possible ensembles; used together with the Hamiltonian function of the system, this allows one to define a suitable ensemble.

(ii) The ensemble averages should reproduce the general behaviour of both the local and the global statistical properties as observed experimentally.

(iii) Any parameters in the definition of the ensemble should be linked to (non-statistical) properties of the nuclei which can be obtained from other sources. This will then allow of adapting the general model to a specific nucleus, just as in statistical mechanics the mass of the individual molecules, their number, and so on, are fixed from other considerations and then determine the statistical properties deduced from the ensemble theory.

(iv) Internal ergodicity for the ensemble should be satisfied. This, as mentioned above, merely implies the internal consistency of the mathematical model.

(v) It would be convenient if the ensemble were mathematically tractable, so that its properties could be deduced in a rigorous fashion, for comparison with experiment.

Unfortunately we are very far from the ideal situation; in fact none of these five requirements can at present be satisfied. It is not yet possible to derive a suitable measure for matrix ensembles from our still inadequate knowledge of nuclear forces; we cannot even delimit the general class of ensembles within which one might search for one that has the necessary properties. This helps to understand why all the recent attempts to create more suitable ensembles have simply tried to remedy one particular aspect that was felt to be unsatisfactory, in the hope that the other properties of the new ensemble would not be significantly worsened. Moreover, many properties of these proposed ensembles have been obtained from Monte Carlo studies, since they do not satisfy requirement (v).

In what follows we give a brief description of three different directions in which such attempts have been made in recent years. In the first of these, additional features are introduced in the definition of the ensemble, in order to improve the level density. The second direction defines an ensemble in

such a way that the experimental level density may be directly incorporated in it. The third tendency attacks the problem of satisfying requirement (i) by taking into account some, at least, of the known features of nuclear forces.

3.1 Positive definite ensembles

In the SIAM Review, Wigner²⁹ pointed out that nowhere in the GOE is the condition imposed that the energy spectrum must have a lower bound, and he suggested that remedying this might also improve the level-density predictions. He proposed an ensemble of positive definite matrices of the form

$$H = A^2 \quad (3.1)$$

where the A 's are generated according to the GOE, so that the H 's still satisfy the orthogonal-invariance requirement (i) of section 2.1. The resulting level density, however, has the form³⁰

$$\rho(E) \sim \frac{1}{E} \sqrt{2N - E^2} \quad (3.2)$$

which diverges at the origin and so cannot represent the nuclear situation.

A second proposal for a positive definite ensemble, also due to Wigner³¹, consists of symmetric matrices of the form

$$H = \text{Re } A^T A \quad (3.3)$$

where the A 's are complex matrices, not in general symmetric, whose elements have real and imaginary parts distributed independently according to Gaussian distributions with the same variance. The density for this ensemble is predicted to be³²

$$\rho(E) \sim \frac{1}{E} \sqrt{\frac{3}{N} E - 1 - \frac{E^2}{4N^2}} \quad (3.4)$$

This is shown as the continuous curve of fig. 9, together with the results of a Monte Carlo calculation³⁰. The level density is now finite at the lower

bound, but its curvature is still not consistent with experiment for the nuclear case. Some local properties of this ensemble have been investigated numerically³⁰; in spite of the unsatisfactory result for the level density, the nearest-neighbour spacings correspond rather well to the Wigner surmise and so agree with experiment.

So far, then, the goal of getting a correct nuclear level density has not been achieved with these two attempts; on the other hand, one cannot exclude the possibility that the second ensemble will prove useful for certain other quantum systems.

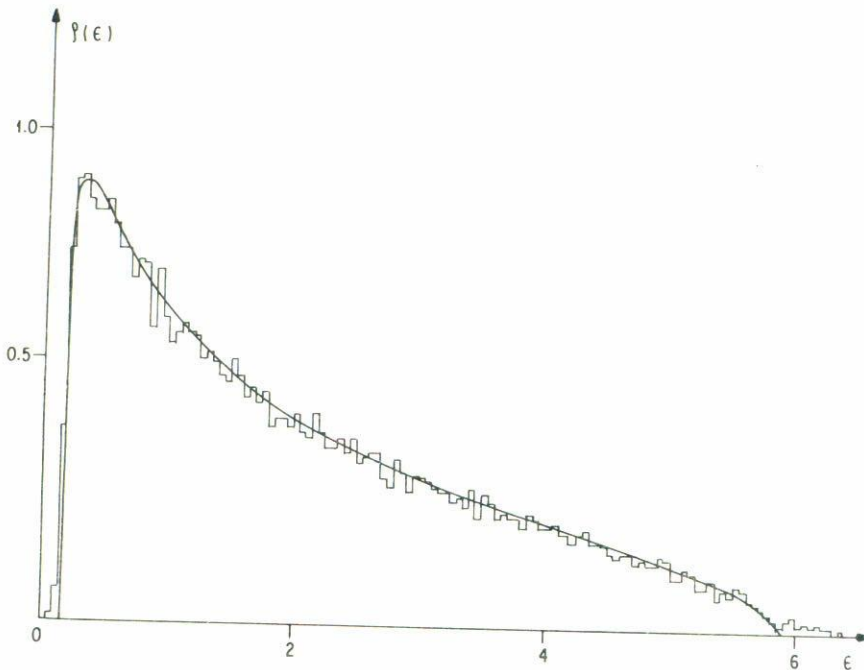


Fig. 9 Level density histogram for a set of 800 matrices with $N = 30$, generated according to the distribution (3.3), compared to the theoretical result (3.4). (Taken from Ref. 30).

3.2 Brownian-motion ensembles

These ensembles were developed by Dyson²⁷ in order to allow the specification of the level density in an (almost) arbitrary fashion. He noticed³³ that the joint distribution of the eigenvalues for the GOE may be written as

$$P(E_1, E_2, \dots, E_N) = C \exp(-W) \tag{3.5}$$

where

$$W(E_1, \dots, E_N) = - \sum_{i < j} \ln |E_i - E_j| + \sum_i (E_i^2 / 2\sigma_i^2) \tag{3.6}$$

But if we interpret the E_i as the coordinates along a line of N infinite wires with equal electric charges, the first term of (3.6) becomes the potential energy between them; the second term, a harmonic-oscillator force, can be taken as necessary to confine the wires to a finite region. If now (3.5) is to be the probability distribution of the positions, we are supposing the existence of an ensemble of such systems, as in classical statistical mechanics, for which the equilibrium position is given by the minimum of the function W .

Such an ensemble could be constructed in many ways. Dyson chooses to consider that the wires constantly interchange energy with a stochastic medium through frictional forces, so that the E_i now fluctuate. In order to get (3.5) as the stationary solution of the equations for the resulting Brownian motion, the wires must feel an external field

$$F(E_i) = - \frac{\partial W}{\partial E_i} = \sum_{j \neq i} \frac{1}{E_i - E_j} - \frac{E_i}{\sigma^2} \tag{3.7}$$

in addition to the stochastic forces.

So far we merely have a physical analogue for the GOE. But it can clearly be generalized; this was done by Dyson in a more recent paper²⁷, where he keeps the two-dimensional Coulomb term which produces the repulsion, but uses a more general form of the confining potential

$$\int \frac{r(\lambda)}{\lambda - E_i} d\lambda. \tag{3.8}$$

Here $r(\lambda)$ is the density of the charged wires along the line. Dyson now shows that in the stationary state this density distribution is obtained, whatever the initial conditions. Since $r(\lambda)$ may be any continuous finite function, we can clearly construct in this way an ensemble to yield any given density function. Of course GOE can easily be seen to be a particular case, where the density is the semi-circle law.

Dyson also obtains exact analytic expressions for the local correlation functions, which give the probability for finding n eigenvalues round the positions E_1, E_2, \dots, E_N , regardless of the positions of the remaining $N - n$ eigenvalues. For the particular type of ensemble where complex Hermitian matrices are used instead of real symmetric ones, these expressions are valid for the stationary state to which the Brownian-motion ensembles tend, and for N large enough do not depend on $r(\lambda)$; Dyson makes it plausible that for the ensembles of real symmetric matrices his expression will also tend to a limit independent of $r(\lambda)$. This conjecture is of considerable importance: since the local properties of the eigenvalue distributions depend only on these correlation functions, it implies that these local properties are independent of a global property such as the level density (though they do depend, of course, on the type of ensemble, and for example are different for the real symmetric and complex Hermitian cases). Thus in particular the spacing distributions of the ensembles of interest in nuclear physics would be those already known from the study of the GOE.

The Brownian-motion ensembles, at the very least, offer a convenient phenomenological way of building ensembles with a given level density. Moreover, Dyson's conjecture concerning the independence of local and global properties, if confirmed, would be a remarkable result in more than one sense. However, the "charged-wire model" merely functions as an analogy meant to stimulate the imagination, and nowhere in it is there a way of incorporating any of the characteristic features of the nuclear many-body problem. Also, one would like to have a model which instead of adapting itself to any given level density, the unreasonable ones as well as the experimentally found ones, would allow one to *deduce* a level density which approaches the experimental one from fairly generally accepted physical principles.

3.3 The two-body random ensemble

Such arguments lead one to consider a completely different type of ensemble of random matrices, namely those which explicitly take into account certain known features of the nuclear forces. Such ensembles were introduced independently by French and Wong³⁴ and by Bohigas and Flores³⁵. The feature to be taken into account is the two-body character of the force, within the general framework of the shell model. One pays a price for this more "physical" approach: the orthogonal-invariance condition of section 2.1, which all ensembles discussed so far satisfy, must be abandoned.

The background for the idea of a random-matrix ensemble with shell-model characteristics may be found in calculations such as those of Ratcliff¹⁴

or Zuker³⁶; they studied the level distributions obtained from large realistic shell-model matrices and found that the level density has a roughly Gaussian shape - very different from the GOE's semi-circle law and of the right curvature at the lower end (as mentioned, only this part is expected to have any physical relevance).

Some physical insight into the behaviour of random-matrix ensembles based on the shell model comes from analytic studies carried out by A. Gervois, using the perturbation theory of statistical mechanics and taking the two-body matrix elements of the interactions as independent random numbers. She considers the limit where the size of the shell goes to infinity, and finds two special cases of great interest: (i) if the number of particles in the shell also goes to infinity in such a way that the fraction of the shell filled remains finite, then the level density is Gaussian in shape, and (ii) if the number of particles remains finite, the level density tends to the semi-circle form. The second result might have been expected: if the number of possible states open to a finite number of particles grows sufficiently large, then the restrictions imposed by the specific shell model features and even by the Pauli principle lose their force, so that the correlations between the matrix elements become unimportant and we approach a limit very like the GOE. The first case, however, gives strong support to the idea of a matrix ensemble within the shell-model framework as a possible alternative to GOE.

This ensemble is constructed as follows:

Let us consider p particles moving in a set of subshells labelled by the usual single-particle quantum numbers j_i (the angular momentum) and m_i (its z projection). In this (finite-dimensional) shell-model space, the Hamiltonian of a two-particle interaction is completely defined by its matrix elements with respect to two-particle states, which we write as

$$\langle j_1 j_2 JM | V | j'_1 j'_2 JM \rangle \quad (3.9)$$

Here V is the interaction. J and M , the total two-particle angular momentum and its projection, are the same in bra and ket, reflecting the invariance of the Hamiltonian under the rotation group. Once the brackets (3.9) are known for all possible values of the quantum numbers, any matrix element of the Hamiltonian for $p > 2$ can be calculated by standard shell-model techniques.

The two-body random ensemble is created by taking the matrix elements (3.9) as independent random numbers; the distribution from which they are sampled is usually taken to be Gaussian, but there is some evidence³⁴ that its choice is not critical. The elements of the p -particle matrix will then be

random numbers in their turn, but the shell model's selection rules will often impose a considerable number of zeros in the matrix, and among the remaining elements there will be strong correlations. In fact, the number of independent variables is often surprisingly small; thus for the $(J=3, T=1)$ matrix in the $(ds)^{12}$ shell the dimensionality is 6706, so that there are some 2×10^7 matrix elements, of which only 67 are independent²⁸.

An ensemble of such matrices cannot, of course, obey the orthogonal-invariance condition of section 2.1: for instance, all matrices in the ensemble have the same zeros.

No exact analytical results are available for any of the properties of such an ensemble. Fairly extensive Monte Carlo calculations have, however, been made. For the level density the results approach a Gaussian distribution quite closely when p increases, as is shown in fig. 10 for a particular case³⁵.

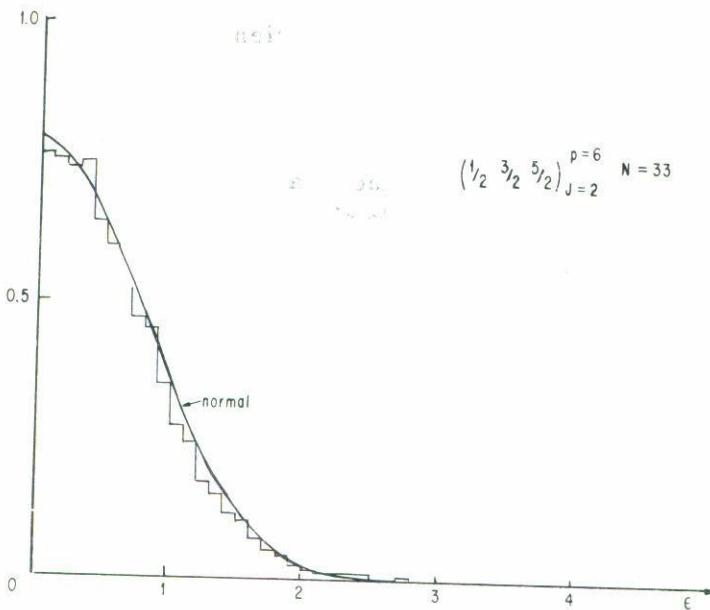


Fig. 10 Histogram for the level density for the two-body random ensemble, compared with a Gaussian distribution. (Taken from Ref. 35).

Such a variation of the level density, obtained from a single shell, has been interpreted by French and his collaborators as a *partial* density^{23, 38}; they show that the total variation, of the form (1.2), say, can arise as the sum of such terms whose parameters are derived from the shell model. This interpretation clearly implies a general model made up of finite-sized and non-interacting shells. Two considerations are then relevant. In the first place, this model is essentially phenomenological in its combination of shell model features and random-matrix theory, and in fact the Gaussian partial densities can be derived from quite different theoretical formulations. However adequate such an approach may be for nuclear physics, the question of whether a satisfactory random-matrix theory can be built remains open. A second point arises from the neglect of the interactions between shells. This is known to be physically unsatisfactory. These interactions may have some influence on the level density; more importantly, the superposition of statistically independent level distributions destroys, as we have mentioned before (see discussion with eq. (1.4)), the level repulsion which experimentally is observed; thus improving the model for the level densities spoils it for the spacing distributions. (This theoretical result⁷ is essentially valid for the GOE; Monte Carlo calculations for the superposition of partial densities from the two-body random ensemble are in progress.)

A more promising way of looking at the level densities might then be to take into account the remaining interactions between separate shells by considering them as component parts of a "supershell", the size of which could grow very large. No calculations on such a model have yet been made.

As for the level spacings, the picture presented by the two-body random ensemble is not much happier. No analytical results are known here. Bohigas and Flores¹⁹ carried out Monte Carlo calculations for the spacings and found that, while the nearest-neighbour spacings have a distribution very similar to that for the GOE, the higher-order spacings fluctuate more; the curve of the standard deviations of their distributions as a function of the spacing order k slopes more steeply than for the GOE. In these calculations, only a few spacings were taken from each of the matrices in the (necessarily finite) set diagonalised, paralleling what is done both analytically and numerically for the GOE. But the validity of this procedure presupposes internal ergodicity of the ensemble; or rather, that for the size of matrices used, the ensemble is close enough to internal ergodicity so that the random density fluctuations between matrices may be safely ignored.

French and Wong²⁸ questioned this point. They observed that the density function of the individual matrices in the ensemble for quite large dimensionalities (~ 300) fluctuates rather severely both in the position of the

centroid and in the width; moreover, the results vary considerably with the dimensionality. One cannot, of course, conclude that the ensemble lacks internal ergodicity; but at least the approach to ergodicity is too slow, so that results such as those of Bohigas and Flores¹⁹ may not be meaningful. French and Wong introduce a correction procedure, where many spacings are taken from each matrix and corrected for the average density of that matrix; with this method the widths of the distributions coincide with those of the GOE, to within statistical error limits.*

Since the results predicted by the GOE are quite close to the experimental data for the level spacings, the French-Wong method seems very well justified. It is an unhappy fact that at the same time it raises the question of the internal ergodicity of the ensemble, which we still lack the theoretical tools to tackle.

4. CONCLUSIONS

In conclusion it may be said that while a great deal of work has been done in the last few years and our understanding of the details of random-matrix theory and its applications in nuclear physics is now much more complete, several fundamental problems remain outstanding.

For the level-spacing distributions, the much more extensive experimental data now available show a quite detailed agreement with the predictions of the GOE; in particular, the higher-order level spacings, that had not previously received much attention. The study of several new attempts at creating ensembles suitable for nuclear physics gives some support to Dyson's conjecture²⁷ that local properties are to a large extent independent of the choice of ensemble; it could be this which explains the success of the GOE for the level spacings, since this ensemble clearly has very little detailed physical foundation.

If we accept the validity of this conjecture, we must go to the global properties to decide among different ensembles. Here none of the recent formulations of new ensembles have yielded results in agreement with experiment, and in fact there is still some discussion on just what kind of comparison

* French and Wong also give results for another procedure, which is equivalent to an ensemble of two-body random matrices normalised so that the squares of all matrices have constant trace. This procedure yields the same results when the spacing distributions are computed according to the "classical" method used by Bohigas and Flores.

should be made. However, a phenomenological approach, in which random-matrix theory provides one element, offers a way out of the difficulty.

The problem is, at root, that none of the recent attempts at specifying new ensembles comes close to satisfying the five requirements outlined at the beginning of section 3. We are thus as far as ever from having a consistent theoretical construction able to explain a wide-ranging body of experimental results.

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RESUMEN

El objeto de la teoría de matrices estocásticas es lograr una explicación coherente de las propiedades estadísticas de sistemas cuánticos complejos; su aplicación más importante hasta ahora ha sido en la física nuclear. En este trabajo se presenta una revisión de lo que se ha logrado en este campo en los últimos años. También se discuten brevemente las ideas básicas de trabajos anteriores. Se hace énfasis en la comparación de los resultados teóricos con datos experimentales.