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THE WEYL GROUP — A SURVEY

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ABSTRACT: We survey some of the properties and representations of the Weyl group, the three-parameter Lie group generated by the (Weyl) algebra realized in Quantum Mechanics by the coordinate, momentum and unity operators. We consider one-dimensional coset spaces as homogeneous spaces under the group and, taking plane-wave and harmonic oscillator eigenfunction bases, we construct all the unitary irreducible representation matrices of the group. We add some previously published material on the Weyl group as a deformation of a semisimple group, on non-Schrödinger realizations, on some of its enveloping algebras and place it in the context of the general nilpotent groups.

1. INTRODUCTION

Consider the Weyl algebra $\mathfrak{W}$ whose elements $\boldsymbol{Q}$, $\boldsymbol{P}$ and $\boldsymbol{H}$ satisfy the commutation relations

$$[\boldsymbol{Q},\boldsymbol{P}] = i\boldsymbol{H}\,,\ [\boldsymbol{Q},\boldsymbol{H}] = 0\,,\ [\boldsymbol{P},\boldsymbol{H}] = 0\ . \tag{1.1}$$

This is the Weyl algebra familiar from Quantum Mechanics[1], where $\boldsymbol{Q}$ and $\boldsymbol{P}$ are identified with the position and momentum operators and $\boldsymbol{H}$, being the center of the algebra, is replaced by a multiple $\lambda\boldsymbol{1}$ of the identity operator. The number λ is taken to be $\hbar$, Planck's constant divided by 2π.

The Weyl group W is the three-parameter nilpotent Lie group generated by $\mathfrak{W}$. It is the purpose of this article to survey some of its properties and present in a unified fashion some related approaches[2, 3, 4].

The Weyl group does not appear as a group of geometrical invariance in any known physical system. Its algebra, however, lies at the very root of Quantum Mechanics: the Weyl universal enveloping algebra $\overline{\mathfrak{W}}$ represents the set of all quantum-mechanical observables (spin disregarded), that is to say, the generators and homogeneous space of the group of all canonical transformations.[5] In line with some recent work by Moshinsky et al.[6], it appears to be of considerable interest to extend the study of physically relevant Lie algebras (symmetry and dynamical) to finite transformations in the generated Lie groups. Furthermore, as we have shown in a former publication[7], harmonic analysis on the Weyl group, which reduces essentially to Fourier analysis, whose powerful results can be immediately used, provides a well-grounded scheme for the formalism of Quantum Mechanics. It is therefore odd that the Weyl group, being to the family of nilpotent groups[8] (Appendix A) what the three-rotation group is to the family of semisimple groups, is relatively little studied by physicists.

In Section 2 we develop some of the results on the Weyl group manifold, homogeneous spaces and representations given in Reference 7, which deal with the chains of groups $W \supset W_Q$ and $W \supset W_P$ where W_Q and W_P are the one-dimensional subgroups of W generated by $\boldsymbol{Q}$ and $\boldsymbol{P}$ respectively. The unitary irreducible representations (UIRs) of W are completely classified by their eigenvalue λ with respect to $\boldsymbol{H}$.

Out of the enveloping algebra $\overline{\mathfrak{W}}$ of $\mathfrak{W}$ we can choose the operator

$$\Phi = \frac{1}{2}\,(\boldsymbol{P}^2 + \boldsymbol{Q}^2)\ . \tag{1.2}$$

Diagonalization of (1.2) (instead of $\boldsymbol{Q}$ or $\boldsymbol{P}$) leads to the Harmonic Oscillator wave functions to be used as representation basis. This is developed in Section 3, the method being akin to W. Miller's treatment[2] of an enlarged four-parameter group which has (1.2) for one of its generators.

Section 4 is devoted to place the developed material in a wider context: the Weyl algebra and group as contractions[3, 9] of the three-parameter semi-simple algebra and groups, the non-Schrödinger realizations[10] of $\mathfrak{W}$ which generate multiplier representations of W and some covering algebras.[11]

In three appendices, we give some properties of the general nilpotent algebras and groups[8], a realization of an n-dimensional version of the Weyl algebra and group, and a generating function for associated Laguerre polynomials.

2. THE WEYL GROUP IN THE CHAIN $W \supset W_Q$

2.1 *Construction of the Group*

Consider the matrix representation of the Weyl algebra $\mathfrak{W}$ in (1.1) given by

$$\mathbf{Q} \longleftrightarrow Q = \begin{bmatrix} 0 & -i & -i \\ i & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \tag{2.1a}$$

$$\mathbf{P} \longleftrightarrow P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \tag{2.1b}$$

$$\mathbf{H} \longleftrightarrow H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \end{bmatrix}. \tag{2.1c}$$

We parametrize the generated Lie group W when we construct the general group element $g(x, y, z)$ as

$$g(x, y, z) = \exp i\,[xQ + yP + zH]$$

$$= \begin{bmatrix} 1 & x+iy & x+iy \\ -x+iy & 1+2iz-\frac{1}{2}(x^2+y^2) & 2iz-\frac{1}{2}(x^2+y^2) \\ x-iy & -2iz+\frac{1}{2}(x^2+y^2) & 1-2iz+\frac{1}{2}(x^2+y^2) \end{bmatrix} \tag{2.2}$$

(see footnote 12), the exponentiation of $xQ + yP + zH$ is easy since all its powers higher than two are identically zero.

The group multiplication law follows[13]

$$g(x_1, {}_1, z_1)\, g(x_2, y_2, z_2) = g(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}[y_1 x_2 - x_1 y_2]) . \tag{2.3}$$

The group identity is $e = g(0,0,0)$ and the inverse $g(x,y,z)^{-1} = g(-x,-y,-z)$. All parameters range over the whole real lir ... manifold of W is therefore non-compact and simply conne
ated by H, is the subgroup of eleme

2.2 *Generators as Differen'*

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and action from the left

$$f(g) \xrightarrow{g'(L)} f^L_{g'}(g) = f(g'^{-1} g) . \tag{2.4b}$$

In terms of the coordinates (2.2), the transformation Jacobians of (2.4) can be found from (2.3) to be, at the origin

$$\partial(gg')/\partial(g)\Big|_{g'=e} = \partial(g'^{-1}g)\Big|_{g'=e} = 1$$

hence the right and left invariant Haar measure for W is

$$d\mu(g) = dx\ dy\ dz . \tag{2.5}$$

The expressions for the generators of W as differential operators in the parameters (2.2) can also be found from (2.4) when we consider transformations by group elements $\delta g' = g(\delta x', \delta y', \delta z')$ in the neighborhood of the identity[14]. Through action from the right,

$$f(x,y,z) \xrightarrow{\delta g'(R)} f(x+\delta x', y+\delta y', z+\delta z' + \frac{1}{2}[y\,\delta x' - x\,\delta y'])$$

$$= \left(1 + \delta x'\left\{\frac{\partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial z}\right\} + \delta y'\left\{\frac{\partial}{\partial y} - \frac{1}{2} x \frac{\partial}{\partial z}\right\} + \delta z' \frac{\partial}{\partial z} + \mathbb{O}(\delta^2 g')\right) f(x,y,z)$$

$$= \left(1 + i\,[\delta x'\, \boldsymbol{Q}^R + \delta y'\, \boldsymbol{P}^R + \delta z'\, \boldsymbol{H}^R] + \mathbb{O}(\delta^2 g')\right) f(x,y,z), \tag{2.6}$$

and similarly for action from the left. We thus obtain

$$\boldsymbol{Q}^R = -i\left(\frac{\partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial z}\right), \quad \boldsymbol{Q}^L = i\left(\frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}\right), \tag{2.7a}$$

$$\boldsymbol{P}^R = -i\left(\frac{\partial}{\partial y} - \frac{1}{2} x \frac{\partial}{\partial z}\right), \quad \boldsymbol{P}^L = i\left(\frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z}\right); \tag{2.7b}$$

$$\boldsymbol{H}^R = -\boldsymbol{H}^L = -i\,\frac{\partial}{\partial z}\,. \tag{2.7c}$$

As a check, it can be verified that the generators of rigth transformations follow the commutation relations (1.1) and so do those of left transformations. Furthermore, one set commutes with the other.

2.3 *Invariant Spaces of Functions and Functions on Homogeneous Spaces*

We can construct subspaces of functions on W which will transform among themselves under the actions (2.4) of the group: those functions $f(g(x,y,z))$ whose z-dependence is given by

$$f^{\lambda}(g(x,y,z)) = F^{\lambda}(x,y)\, e^{i\lambda z}, \quad \lambda \in (-\infty, \infty), \tag{2.8a}$$

which can be seen to preserve their form under (2.3)–(2.4) and which are clearly eigenfunctions of $\boldsymbol{H}$ in (2.7c):

$$H^R f^\lambda(g) = -H^L f^\lambda(g) = \lambda f^\lambda(g). \tag{2.8b}$$

Furthermore, as any function (within a wide class for which the Fourier exists) can be expanded as

$$f(g(x,y,z)) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} d\lambda\, F^\lambda(x,y)\, e^{i\lambda z} \tag{2.9}$$

we can say that we have expanded the space of functions on W as a direct integral of subspaces invariant under the action of W labelled by an index λ, their eigenvalue with respect to the generator $\boldsymbol{H}$.

In each of these subspaces we can write the generators (2.7) as

$$\boldsymbol{Q}^R = -i\frac{\partial}{\partial x} + \frac{1}{2}\lambda y\,,\ \boldsymbol{P}^R = -i\frac{\partial}{\partial y} - \frac{1}{2}\lambda x\,,\ \boldsymbol{H}^R = \lambda \boldsymbol{1}\,,$$

$$\boldsymbol{Q}^L = i\frac{\partial}{\partial x} + \frac{1}{2}\lambda y\,,\ \boldsymbol{P}^L = i\frac{\partial}{\partial y} - \frac{1}{2}\lambda x\,,\ \boldsymbol{H}^L = -\lambda \boldsymbol{1}\,, \tag{2.10a}$$

acting on the functions $F^\lambda(x,y)$ in (2.8a). This function is effectively a function on the homogeneous space $W/W_H = W_H\backslash W$. The commutation relations (1.1) hold for (2.10a) and (2.10b). Again, one set commutes with the other.

Consider now the decomposition of an arbitrary element in W as

$$g(x,y,z) = g(0,y,0)\, g(x,0,\omega),\ \omega = z - \frac{1}{2}xy\,, \tag{2.11}$$

where $g(0,y,0) \in W_P$, the one-parameter subgroup generated by $\boldsymbol{P}$. The set $W_P\, g(x,0,\omega)$, for fixed x and ω, is a left coset of W by W_P and $g(x,0,\omega)$ its representative. The space of cosets $W_P\backslash W$ is the set of all representatives $g(x,0,\omega)$. As the Jacobian $\partial(x,y,z)/\partial(x,y,\omega) = 1$, the Haar measure (2.5) can be written as

$$d\mu(g) = d\mu_P(y)\ dc(x,\omega) = dy\, dx\, d\omega\,, \tag{2.12}$$

i.e. as the product of the measure $d\mu_P$ on W_P and the measure dc on the space of left cosets $W_P\backslash W$. The decomposition (2.11) is also unique and hence we have decomposed the group manifold of W as a direct product of the manifolds of W_P and $W_P \backslash W$.

A function on the space of cosets is a function of the parameter x and ω in (2.11). If such a function is further taken to lie in one of the H-eigenspaces labelled by λ, it must be of the form

$$f^{\lambda}(g(x,y,z)) = \phi^{\lambda}(x)\, e^{i\lambda\omega}. \tag{2.13}$$

The transformation of $\phi^{\lambda}(x)$, function on $W_P \backslash W$ in the eigenspace λ under the right mapping (2.4a) is thus described by

$$\phi^{\lambda}(x) \xrightarrow{g'(R)} \phi^{\lambda}(x+x') \exp\left[i\lambda(z'-y'x-\tfrac{1}{2}x'y')\right] \tag{2.14}$$

$$= U(g')\,\phi^{\lambda}(x),$$

(See footnote 15). Equation (2.14) is a multiplier representation of W on a space of functions of one variable. Proceeding as in subsection 2.2, we can find[14] the generators of the transformation (2.14) as

$$\mathbf{Q} = -i\frac{\partial}{\partial x},\ \mathbf{P} = -\lambda x,\ \mathbf{H} = \lambda \mathbf{1}, \tag{2.15}$$

which, again, fulfill (1.1).

2.4 *A Hilbert Space and a Basis*

Out of the space of functions $\phi^{\lambda}(x)$ on which W acts through the multiplier representation (2.14) we can build a Hilbert space $\mathcal{H}^{\lambda}$ by introducing the scalar product

$$(\phi_1, \phi_2) = \int_{-\infty}^{\infty} dx\, \phi_1(x)^* \phi_2(x) \tag{2.16}$$

and considering the subset $\mathcal{L}^2(-\infty,\infty)$ of Lebesgue square-integrable functions[16] of x.

If we now choose a complete, orthonormal basis $\{\psi_n^{\lambda}\}$, $n \in N$, where N

is some index set, for $\mathcal{H}^{\lambda}$ we can define the representation matrices of W through the mapping (2.14) as

$$D^{\lambda}_{nn'}(g') = (\psi^{\lambda}_{n}, U(g')\psi^{\lambda}_{n'}) \tag{2.17}$$

and be assured that they follow the group multiplication law. Under the scalar product (2.16), the operators (2.15) are hermitean, for real λ. The corresponding representation matrices (2.2) will be unitary.

A basis consisting of eigenfunctions of (1.2), esssentially the harmonic oscillator wave-functions, will be described in the next section. Here we will choose an eigenbasis of $\boldsymbol{Q}$ i. e.,

$$\boldsymbol{Q}\chi^{\lambda}_{q} = q\chi^{\lambda}_{q} \tag{2.18a}$$

which, represented by the operator (2.15) is

$$\chi^{\lambda}_{q}(x) = (2\pi)^{-\frac{1}{2}} e^{iqx}, \quad q\in(-\infty,\infty) \tag{2.18b}$$

which, though outside $\mathcal{H}^{\lambda}$ (they are in C^{∞}, but are not square-integrable. See footnote 16), are orthonormal in the sense

$$(\chi^{\lambda}_{q}, \chi^{\lambda}_{q'}) = \delta(q-q') \tag{2.19a}$$

and complete under the Plancherel measure dq i. e.,

$$\int_{-\infty}^{\infty} dq\, \chi^{\lambda}_{q}(x)^{*}\, \chi^{\lambda}_{q}(x') = \delta(x-x'), \tag{2.19b}$$

when placed inside an integral over x, as is known from the theory of Fourier transforms.

An eigenbasis of $\boldsymbol{P}$, i. e.,

$$\boldsymbol{P}\tilde{\chi}^{\lambda}_{p} = p\tilde{\chi}^{\lambda}_{p} \tag{2.20a}$$

can be built, for the representation (2.15) as

$$\tilde{\chi}_p^{\lambda}(x) = |\lambda|^{\frac{1}{2}} \delta(p+\lambda x), \; p \in (-\infty, \infty) \tag{2.20b}$$

which is also outside $\mathcal{H}^{\lambda}$ (in $\mathcal{D}'$ the space of continuous linear functionals, see footnote 16), also orthonormal in the sense (2.19a), i.e.,

$$(\tilde{\chi}_p^{\lambda}, \tilde{\chi}_{p'}^{\lambda}) = \delta(p-p') \tag{2.21a}$$

and complete

$$\int_{-\infty}^{\infty} dp \, \tilde{\chi}_p^{\lambda}(x)^* \, \tilde{\chi}_p^{\lambda}(x') = \delta(x-x') \, . \tag{2.21b}$$

As no subspace of either (2.18) or (2.20) is left invariant under the mapping (2.14), the representations thus obtained will be unitary, irreducible representations.

2.5 *The Unitary Irreducible Representation Matrices*

The matrix representations thus obtained have rows and columns labelled by continuous indices, that is, they act as integral kernels. From (2.14), (2.16), (2.17) and (2.18) we find

$$D_{qq'}^{\lambda}(g(x,y,z)) = \delta(\lambda y - [q'-q]) \exp i \, (\lambda z + \frac{1}{2} [q+q'] \, x) \, , \tag{2.22}$$

which can be checked[17] directly to follow the group composition law (2.3) under multiplication

$$\int dq' \, D_{qq'}^{\lambda}(g_1) \, D_{q'q''}^{\lambda}(g_2) = D_{qq''}^{\lambda}(g_1 g_2) \, , \tag{2.23a}$$

and be unitary,

$$D_{qq'}^{\lambda}(g^{-1}) = D_{q'q}^{\lambda}(g)^* \, . \tag{2.23b}$$

As functions on the W manifold, they are eigenfunctions of the commuting operators in (2.7)

$$\mathbf{Q}^R D^{\lambda}_{qq'}(g) = q' D^{\lambda}_{qq'}(g) \tag{2.24a}$$

$$\mathbf{Q}^L D^{\lambda}_{qq'}(g) = -q D^{\lambda}_{qq'}(g) \tag{2.24b}$$

$$\mathbf{H}^R D^{\lambda}_{qq'}(g) = -\mathbf{H}^L D^{\lambda}_{qq'}(g) = \lambda D^{\lambda}_{qq'}(g)\,, \tag{2.24c}$$

indeed, in Reference 7 they were constructed in this way. The D's (2.22) are thus classified by the chain $W \supset W_Q$.

Further, as UIRs, they are orthogonal on W as we can verify, using (2.5) and integrating over W

$$\int_W d\mu(g)\, D^{\lambda_1}_{q_1 q_1'}(g)^* D^{\lambda_2}_{q_2 q_2'}(g) = \frac{4\pi^2}{|\lambda_1|}\delta(\lambda_1-\lambda_2)\,\delta(q_1-q_2)\,\delta(q_1'-q_2')\,. \tag{2.25}$$

They are, moreover, all representations of W as they form a complete set of functions on W: they are orthonormal in the sense (2.19b)–(2.21b) on the space $\hat{W}$ of UIRs of W, isomorphic to the real line, with the Plancherel measure

$$d\hat{\mu}(\lambda) = \frac{|\lambda|}{4\pi^2}\,d\lambda \tag{2.26a}$$

whose weight function as its inverse in the right-hand side of (2.25), i.e.,

$$\int_{\hat{W}} d\hat{\mu}(\lambda) \int dq \int dq'\, D^{\lambda}_{qq'}(g(x_1 y_1 z_1))^* D^{\lambda}_{qq'}(g(x_2 y_2 z_2))$$
$$= \delta(x_1-x_2)\,\delta(y_1-y_2)\,\delta(z_1-z_2)\,. \tag{2.26b}$$

The D functions are thus unitary transformation kernels between functions on the W manifold, and their harmonic transforms, matrix functions on the dual $\hat{W}$ manifold. This is developed in reference 7.

A parallel procedure for the basis (2.20) yields

$$D^{\lambda}_{pp'}(g(x,y,z)) = \delta(\lambda x - [p - p']) \exp i\,(\lambda z + \frac{1}{2}[p + p']\,y), \quad (2.27)$$

with the properties (2.23) of the group UIRs, eigenfunctions of $\mathbf{P}^R$ and $\mathbf{P}^L$ with eigenvalues p' and $-p$, analogous to (2.24), orthogonal (2.25) and complete (2.26).

They are the double Fourier transforms of the former, i.e.,

$$D^{\lambda}_{pp'}(g) = (2\pi|\lambda|)^{-1} \int dq \int dq' \exp[-ipq/\lambda]\, D^{\lambda}_{qq'}(g) \exp[ipq/\lambda]\,. \quad (2.28)$$

Clearly, in the limit $\lambda \to 0$, the Weyl group UIRs become the UIRs of the two-dimensional abelian algebra generated by (commuting) $\mathbf{P}$ and $\mathbf{Q}$. This is regarded[1,7] as the group-theoretical meaning of the "classical limit" of Quantum Mechanics.

2.6 *The Coordinate Basis*

The realization (2.15) of the Weyl group generators as operators in the Weyl group coset manifold is not the usual one in Quantum Mechanics. There, the eigenvalue q of $\mathbf{Q}$ is regarded as the physical configuration-space coordinate. A function of the coordinates can be built as the functional

$$f^{\lambda}(q) = (\chi^{\lambda}_q, \phi^{\lambda}) = (2\pi)^{-\frac{1}{2}} \int dx \exp[-iqx]\, \phi^{\lambda}(x), \quad (2.29)$$

i.e., as the Fourier transform of the functions considered.

In this space, we have the Schrödinger realization[1] of the Weyl algebra

$$\mathbf{Q} f^{\wedge}(q) = (\chi^{\lambda}_q, \mathbf{Q}\phi^{\lambda}) = q f^{\lambda}(q) \quad (2.30a)$$

$$\mathbf{P} f^{\lambda}(q) = (\chi^{\lambda}_q, \mathbf{P}\phi^{\lambda}) = -i\lambda \frac{\partial}{\partial a} f^{\lambda}(q) \quad (2.30b)$$

As the Fourier transformation $f^\lambda \leftrightarrow \phi^\lambda$ is a unitary mapping of $\mathcal{H}^\lambda$ on itself, we have the functionals $f^\lambda(q)$ constituting a Hilbert space with a scalar product identical to (2.16) with $q \leftrightarrow x$ i.e.,

$$(f_1^\lambda, f_2^\lambda) = \int_{-\infty}^{\infty} dq\, f_1^\lambda(q)^* f_2^\lambda(q), \tag{2.31}$$

which, under the action of the group W transforms, from (2.14), as

$$f^\lambda(q) \xrightarrow{g(x,y,z)} U(g(x,y,z)) f^\lambda(q) = f^\lambda(q+\lambda y) \exp i\left[\lambda\left(z + \frac{1}{2}xy\right) + qx\right], \tag{2.32a}$$

that is, the D-matrix (2.22) acts as an integral kernel

$$U(g(x,y,z))\, f^\lambda(q) = \int dq'\, D^\lambda_{qq'}(g)\, f^\lambda(q') . \tag{2.32b}$$

The representations in momentum space follow suit.

3. HARMONIC OSCILLATOR WAVE FUNCTIONS AND THE WEYL GROUP REPRESENTATIONS

3.1 *The Harmonic Oscillator Basis*

In order to find the Weyl group representations (2.17) we can employ any basis of functions dense in $\mathcal{H}^\lambda$ the Hilbert space of square-integrable functions with scalar product (2.16) which transform under the action of W through the multiplier (2.14) or (2.32). Two non-denumerable basis were used in the last section. In this section we shall use the denumerable orthogonal basis $\{\psi_n^\lambda(q)\}$ provided by the eigenfunctions of (1.2), i.e., in the realization (2.30) by the operator

$$\Phi = -\frac{\lambda^2}{2}\frac{\partial^2}{\partial q^2} + \frac{1}{2}q^2 . \tag{3.1}$$

As (3.1) is identical with the quantum oscillator Hamiltonian of frequency unity[18], unit mass and λ for $\hbar$, its eigenfunctions are well known and can be written as

$$\psi_n^\lambda(q) = (2^n n!)^{-\frac{1}{2}} (\pi\lambda)^{-\frac{1}{4}} e^{-q^2/2\lambda} H_n(q/\sqrt{\lambda}) \, , \; n = 0, 1, 2, \ldots \qquad (3.2a)$$

and are the square-integrable solutions of the Schrödinger equation

$$\Phi\psi_n^\lambda = \lambda\left(n + \frac{1}{2}\right)\psi_n^\lambda \qquad (3.2b)$$

for $\lambda > 0$ and n a non-negative integer (see footnote 19).

It is meaningless to consider the case $\lambda < 0$ in this context as (3.1) is the same differential equation, but (3.2) is non-square integrable and presents multivaluation problems. From eqs. (2.22) and (2.27), however, we see that, irrespective of the basis,

$$D^{-\lambda}(g(x, y, z)) = D^\lambda(g(x, -y, -z)) = D^\lambda(g(-x, -y, z))^* \, . \qquad (3.3)$$

This seems to be extendable to any basis[20].

3.2 *The Representation Matrices*

The representation matrix (2.17) can thus be built in the basis (3.2) as

$$D_{mn}^\lambda(g(x, y, z)) = (\psi_m^\lambda, U(g(x, y, z))\,\psi_n^\lambda)$$

$$= [2^{m+n} m!\, n!\, \pi\lambda]^{-\frac{1}{2}} \int_{-\infty}^{\infty} dq \exp[-q^2/2\lambda]\, H_m(q/\sqrt{\lambda}) \times \qquad (3.4a)$$

$$\exp[-(q+\lambda y)^2/2\lambda]\, H_n([q+\lambda y]/\sqrt{\lambda}) \exp i\left[\lambda\left(z + \frac{1}{2}xy\right) + qx\right] .$$

$$\sum_{n,m=0}^{\infty} [2^{m+n} m!\,n!]^{\frac{1}{2}} D^{\lambda}_{mn}(g(x,y,z)) \frac{t^m s^n}{m!\,n!}$$

$$= (\pi\lambda)^{-\frac{1}{2}} \exp\left[i\lambda\left(z+\frac{1}{2}xy\right)\right] \exp\left[-(t^2+s^2)-\frac{1}{2}\lambda y^2 + 2sy\sqrt{\lambda}\right] \times$$

$$\times \int_{-\infty}^{\infty} dq \exp\left[-q^2/\lambda + q(2\{t+s\}/\sqrt{\lambda} - y + ix)\right] =$$

$$= \exp\lambda\left[iz+\frac{1}{4}(x^2+y^2)\right] \exp\left[2ts + s\sqrt{\lambda}(y+ix) + t\sqrt{\lambda}(-y+ix)\right] . \tag{3.4b}$$

A generating function of the associated Laguerre polynomials is (see Appendix C)

$$\exp\left[ab+ac-c^* b\right] = \sum_{n,m=0}^{\infty} (m!)^{-1} L_n^{(m-n)}(c^* c)\, a^m b^n c^{m-n} . \tag{3.5}$$

Using (3.5) in (3.4b) for $a=\sqrt{2}t$, $b=\sqrt{2}s$, $c=(\lambda/2)^{\frac{1}{2}}(-y+ix)$, collecting the coefficients of terms $t^m s^n$, we obtain

$$D^{\lambda}_{mn}(g(x,y,z)) = \exp\left[\lambda\left(iz+\frac{1}{4}\{x^2+y^2\}\right)\right] \times$$

$$\times (n!/m!)^{\frac{1}{2}} (\lambda/2)^{(m-n)/2)} (-y+ix)^{m-n} L_n^{(m-n)}(\lambda[x^2+y^2]/2) , \tag{3.6a}$$

which is valid, for the definition of the associated Laguerre polynomial, for $m \geqslant n$. Similarly, setting $a=\sqrt{2}s$, $b=\sqrt{2}t$, $c=(\lambda/2)^{\frac{1}{2}}(y+ix)$ in (3.4), the same procedure yields, replacing the dummy indices $m \longleftrightarrow n$ in the right-hand side of (3.4b), the equivalent formula[23]

$$D^{\lambda}_{mn}(g(x,y,z)) = \exp\left[\lambda\left(iz+\frac{1}{4}\{x^2+y^2\}\right)\right] \times$$

$$\times (m!/n!)^{\frac{1}{2}} (\lambda/2)^{(n-m)/2} (y+ix)^{n-m} L_m^{(n-m)}(\lambda[x^2+y^2]/2) , \tag{3.6b}$$

valid for $m \leqslant n$.

It is easy to check unitarity of the representation

$$D^{\lambda}_{mn}(g(-x,-y,-z)) = D^{\lambda}_{nm}(g(x,y,z))^{*}\,, \tag{3.7}$$

comparing (3.6a) for the right-hand side member and (3.6b) for the left-hand side one.

The proof for irreducibility is parallel to that given by W. Miller[24], while the representation property follows from construction and yields summation formulae for the associated Laguerre polynomials.

4. OTHER REALIZATIONS AND IMBEDDINGS OF THE WEYL ALGEBRA

4.1 *Contractions to the Weyl Algebra*

The Weyl algebra $\mathbb{W}$ is the contraction[9] of the $\mathfrak{SO}(3)$ algebra whose generators J_1, J_2, J_3 satisfy

$$[J_1\,,\,J_2] = iJ_3\,,\ \ [J_3\,,\,J_1] = iJ_2\,,\ \ [J_2\,,\,J_3] = iJ_1\,. \tag{4.1}$$

Upon the replacement

$$\boldsymbol{Q}_{\epsilon} = \epsilon^{\frac{1}{2}} J_1\,,\ \boldsymbol{P}_{\epsilon} = \epsilon^{\frac{1}{2}} J_2\,,\ \boldsymbol{H}_{\epsilon} = \epsilon J_3\,, \tag{4.2}$$

eqs. (4.1) read

$$[\,\boldsymbol{Q}_{\epsilon}\,,\,\boldsymbol{P}_{\epsilon}] = i\boldsymbol{H}_{\epsilon}\,,\ [\boldsymbol{Q}_{\epsilon}\,,\,\boldsymbol{H}_{\epsilon}] = -\,i\epsilon\boldsymbol{P}_{\epsilon}\,,\ [\boldsymbol{P}_{\epsilon}\,,\,\boldsymbol{H}_{\epsilon}] = i\epsilon\boldsymbol{Q}_{\epsilon}\,. \tag{4.3}$$

In the limit $\epsilon \to 0$ these become the Weyl commutation relations (1.1). As in that limit the matrix elements of given (finite) representations of $\mathfrak{SO}(3)$ become zero through (4.2), it is suggested by the behaviour of the operators $\epsilon J^2 - \epsilon^2 J_3^2 = \boldsymbol{Q}_{\epsilon}^2 + \boldsymbol{P}_{\epsilon}^2$ that we take the limit in such a way that $\epsilon l(l+1)$ remain finite, i.e., $l \sim \epsilon^{-\frac{1}{2}}$ as $\epsilon \to 0$. If we now fix our attention on the lower diagonal corner of the $\mathfrak{SO}(3)$ algebra representation matrices[26] and let them extend downwards, we can write down the familiar[27] representation of $\mathfrak{W}$ by infinite-dimensional hermitean matrices

$$\lim_{\epsilon \to 0} Q_\epsilon = \sqrt{\frac{\lambda}{2}} \begin{bmatrix} 0 & \sqrt{1} & . & . & . \\ \sqrt{1} & 0 & \sqrt{2} & . & . \\ . & \sqrt{2} & 0 & \sqrt{3} & . \\ . & . & \sqrt{3} & 0 & . \\ . & . & . & . & . \end{bmatrix}, \tag{4.4}$$

$$\lim_{\epsilon \to 0} P_\epsilon = -i\sqrt{\frac{\lambda}{2}} \begin{bmatrix} 0 & \sqrt{1} & . & . & . \\ -\sqrt{1} & 0 & \sqrt{2} & . & . \\ . & -\sqrt{2} & 0 & \sqrt{3} & . \\ . & . & -\sqrt{3} & 0 & . \\ . & . & . & . & . \end{bmatrix},$$

$$\lim_{\epsilon \to 0} H_\epsilon = \lambda 1 .$$

In fact, this result can also be obtained from (3.6) considering elements $g(x, y, z)$ in a neighborhood of the identity.

A closely related contraction is given by

$$\tilde{Q}_\epsilon = \epsilon^{\frac{1}{2}} J_3 , \ \tilde{P}_\epsilon = -\epsilon^{\frac{1}{2}} J_2 , \ \tilde{H}_\epsilon = \epsilon J_1 , \tag{4.5}$$

which also yield commutation relations of the type (4.3), which in the limit $\epsilon \to 0$ become those of the Weyl algebra.

The difference between the contractions (4.2) and (4.5), however, resides in the following: if we start from the familiar $\mathfrak{so}(3)$ basis where J^2 and J_3 are diagonal, the operator H_ϵ remains diagonal. The rows and columns of the matrix remain discrete and, as was done by J.D. Talman[3], the matrices (3.6) are obtained from the $SO(3)$ group representation matrices in the neighborhood of the identity. In the contraction (4.5), on the other hand, the matrix rows and columns are classified by their eigenvalues under $\tilde{Q}_\epsilon$ and the representations as the eigenvalue under $\epsilon^2 J^2 = H_\epsilon^2 + \epsilon(P_\epsilon^2 + Q_\epsilon^2)$. Thus, from the group representations of $SO(3)$ in the neighborhood of the identity, one expects to regain the representation (2.22)[28]

4.2 *Non-Schrödinger Realizations*

In finding operators which exhibit the $\mathbb{W}$ commutation relations (1.1) we can count on the Stone-von Neumann theorem[29] to know that we are able to perform a unitary equivalence transformation from any one-dimensional homogeneous space realization, to obtain the Schrödinger realization (2.30).

It remains true, however, that the realization

$$Q f^\lambda(q) = q f^\lambda(q), \tag{4.6a}$$

$$P^{(\alpha)} f^\lambda(q) = [-i\lambda \frac{\partial}{\partial q} + \sigma\alpha(q)]\, f^\lambda(q),\ \sigma \text{ real}, \tag{4.6b}$$

$$H f^\lambda(q) = \lambda f^\lambda(q), \tag{4.6c}$$

satisfies (1.1). It may be that the form (4.6) is forced upon us by the measure of the homogeneous space in the scalar product (2.31) not being unity, but some $d\mu(g) = \omega(q)\, dq$ in which case we require in (4.6b) that $\alpha(q) = \lambda\omega'(q)/2i\omega(q)$ (the prime indicates derivation with respect to the argument) if the operator (4.6b) is to remain hermitean.

In ordinary quantum mechanics, however, where the measure is simply $d\mu(q) = dq$, the addition of any real function $\alpha(q)$ in (4.6b) preserves the hermiticity of the momentum operator, but changes the representation of the finite transformations (3.2) of the Weyl group to a multiplier representation[11]. Indeed, whereas (2.32a) for $g(0,y,0)$ is seen simply to translate the homogeneous space q in $\alpha(q)$, the realization (4.6b) brings in a multiplier function μ:

$$U^{(\alpha)}(g(0,y,0))\, f^\lambda(q) = \mu(g(0,y,0),\, q)\, f^\lambda(q + \lambda y), \tag{4.7a}$$

where, as in (2.32a), general group elements $g(x,y,z)$ will not change the multiplier:

$$\mu(g(\boldsymbol{x}, y, \boldsymbol{z}), q) = \mu(y, q). \tag{4.7b}$$

The usual multiplier properties obtained from the group multiplication properties here read

$$\mu(y_1, q)\mu(y_2, q + \lambda y_1) = \mu(y_1 + y_2, q),$$

$$\mu(0, q) = 1 \quad \text{and} \quad \mu(-y, q) = \mu(y, q - \lambda y)^{-1}.$$

These relations[30] suggest we write

$$\mu(y, q) = [\rho(q + \lambda y)/\rho(q)]^{\sigma} \tag{4.7c}$$

where $\rho(q)$ is any non-zero differentiable function of q.

If we replace (4.7c) in (4.7a) and consider group elements in the one-parameter subgroup generated by $\boldsymbol{P}$ in the neighborhood of the identity, we have

$$(\boldsymbol{1} + i\delta y\boldsymbol{P} + \mathbb{O}(\delta y^2))\, f^{\lambda}(q) = \left(\boldsymbol{1} + \delta y\lambda\left\{\frac{\sigma}{\rho}\frac{\partial\rho}{\partial q} + \frac{\partial}{\partial q}\right\} + \mathbb{O}(\delta y^2)\right) f^{\lambda}(q)$$

whereupon we can identify the multiplier-generating function $\alpha(q)$ in (4.6b) as

$$\alpha(q) = -i\lambda\frac{\partial}{\partial q}\ln\rho(q), \tag{4.8a}$$

which means that, for real $\alpha(q)$, the multiplier (4.7c) will be a ratio of imaginary exponentials, i.e., a phase

$$\mu(y, q) = \exp\left[\frac{i\sigma}{\lambda}\{\nu(q + \lambda y) - \nu(q)\}\right], \; \alpha(g) = \frac{\partial\nu(q)}{\partial q}, \tag{4.8b}$$

which clearly follows the group property. A unitary transformation $\mu(y, q)\,\boldsymbol{P}^{(\alpha)}\,\mu(y, q)^{-1}$ brings $\boldsymbol{P}^{(\alpha)}$ back to the Schrodinger representation $\boldsymbol{P}^0$.

This is one of the key steps in proving the Stone-von Neumann theorem[29].

4.3 *Subalgebras and Covering Algebras*

Consider the elements of the vector space $\mathbb{W}$ given by

$$\overline{\mathbf{Q}} = a\mathbf{Q} + b\mathbf{P} + e\mathbf{H}$$
$$\overline{\mathbf{P}} = c\mathbf{Q} + d\mathbf{P} + f\mathbf{H} \tag{4.9}$$

where $a, b, \ldots, f$ are complex numbers subject to the condition $ab - cd = 1$, i.e., the transformation (4.9) is a complex inhomogeneous symplectic one. As $\overline{\mathbf{Q}}$, $\overline{\mathbf{P}}$ and $\mathbf{H}$ also exhibit the commutation relations (1.1), the transformation (4.9) is said to be canonical. It induces a corresponding transformation in the one-parameter groups generated by $\mathbf{Q}$ and $\mathbf{P}$ and it constitutes thus a mapping where the W group manifold is the homogeneous space of a group of complex inhomogeneous symplectic transformations. This has been used by C. Itzykson[31] in order to find the latter's irreducible representations. In particular, for real a, b, c, d and $e, f \equiv 0$, the representation matrices (2.22) or (2.27) diagonalized with respect to the subgroups generated by $\overline{\mathbf{Q}}$ and $\overline{\mathbf{P}}$ (which are hermitean if $\mathbf{Q}$ and $\mathbf{P}$ are) can be obtained through the integral transformation kernel

$$K^{\lambda}(q, \overline{q}') = K^{\lambda}(\overline{q}', q)^{*} = (\chi^{\lambda}_{q}, \overline{\chi}^{\lambda}_{q'}), \tag{4.10a}$$

where

$$\overline{\mathbf{Q}}\chi^{\lambda}_{q} = q\chi^{\lambda}_{q}, \quad \overline{\mathbf{Q}}\,\overline{\chi}^{\lambda}_{q'} = q'\overline{\chi}^{\lambda}_{q'} \tag{4.10b}$$

which have been calculated in Reference 6, and which carry thus a unitary representation (since $K^{\lambda}(q, \overline{q}') = K^{\lambda}(\overline{q}', q)^{*}$) of the symplectic transformation (4.9).

A related though distinct transformation is

$$K_{\pm} = 2^{-\frac{1}{2}}(P \pm iQ), \tag{4.11}$$

for which

$$[K_-, K_+] = H, \ [K_\pm, H] = 0 . \tag{4.12}$$

We can now define, out of the universal enveloping algebra of the Weyl algebra, the operator

$$K_0 = H^{-1}(K_- K_+ - \tfrac{1}{2}H) = H^{-1}\Phi , \tag{4.13}$$

which is properly defined (in the sense of its representations), since H commutes with all other operators. It can be checked directly that

$$[K_0, K_\pm] = \pm K_\pm, \ [K_0, H] = 0 \tag{4.14}$$

and

$$[K_0, Q] = -iP, \ [K_0, P] = iQ . \tag{4.15}$$

The operators K_0, $K_\pm$ and H can now be considered the generators of a four-parameter algebra[32] with the commutation relations (4.12)-(4.14). This is the algebra $\mathcal{G}(0,1)$ considered by W. Miller[2] which is then realized as first-order differential operators on one- and two-dimensional homogeneous spaces.

Further, we can consider the generators[6]

$$J_1 = \tfrac{1}{4} H^{-1}(P^2 - Q^2), \tag{4.15a}$$

$$J_2 = \tfrac{1}{4} H^{-1}(QP + PQ), \tag{4.15b}$$

$$J_3 = \tfrac{1}{2} K_0 = \tfrac{1}{2} H^{-1}\Phi = \tfrac{1}{4} H^{-1}(P^2 + Q^2) , \tag{4.15c}$$

and check that they close under commutation as the generators of an $\mathcal{S}p(2) \cong \mathcal{S}U(1,1) \cong \mathcal{S}O(2,1)$ algebra. If we add the set $K_\pm$, H or Q, P and

$\boldsymbol{H}$, we obtain an algebra which we can call Weyl-symplectic $\mathfrak{W}\mathfrak{s}\mathfrak{p}(2)$. Acting on $\boldsymbol{Q}$ and $\boldsymbol{P}$ through the commutator, they generate the six-parameter group of canonical inhomogeneous symplectic transformations (4.9).

APPENDIX A: THE NILPOTENT ALGEBRAS AND GROUPS

The purpose of this Appendix is to define the general nilpotent Lie algebras and their corresponding Lie groups.

DEFINITION 1. A Lie algebra $\mathfrak{n}$ is called nilpotent if we can find a sequence of sets

$$\mathfrak{n} = \mathfrak{n}_n \supset \mathfrak{n}_{n-1} \supset \ldots \supset \mathfrak{n}_1 \supset \mathfrak{n}_0 = 0 \ , \tag{A.1}$$

such that

$$[\mathfrak{n}, \mathfrak{n}_k] \subset \mathfrak{n}_{k-1}, \ (k = 1, \ldots, n) \ . \tag{A.2}$$

Since (A.2) implies that $[\mathfrak{n}_k, \mathfrak{n}_k] \subset \mathfrak{n}_k$ it is clear that $\mathfrak{n}_k (k = 0, 1, \ldots, n)$ are subalgebras of $\mathfrak{n}$. As sets, they are also ideals of $\mathfrak{n}$ under the bracket operation (recall that $\mathfrak{I}$ is an ideal of $\mathfrak{n}$ if $[\mathfrak{n}, \mathfrak{I}] \subset \mathfrak{I}$).

Equation (A.2) also tells us that for any subset $\mathfrak{n}' \subset \mathfrak{n}$, $[\mathfrak{n}', \mathfrak{n}']$ is properly contained in $\mathfrak{n}'$ and hence, if we repeat the commutator a sufficient number of times, we get zero, i.e.,

$$[\mathfrak{n}', [\ldots, [\mathfrak{n}', \mathfrak{n}'] \ldots]] = 0 \ .$$

We can characterize a nilpotent Lie group N as that generated by a nilpotent algebra $\mathfrak{n}$. We shall now explore some of its properties. Notice first that each subalgebra $\mathfrak{n}_k$ of $\mathfrak{n} (k = 0, 1, \ldots, n)$ will generate a subgroup N_k of N as in the sequence (A.1), with $N_0 = e$, the trivial group consisting of the identity.

Recall that[33] if $H \ni h$ is a normal subgroup of $G \ni g$, then $g h g^{-1} \equiv Ad_g h \in H$ and hence $h^{-1} g^{-1} h g \in H$. Now, if $h = \exp(h_\mu X^\mu)$, ($X^\mu \in \mathfrak{H}$ generating H) and $g = \exp(g_\mu X^\mu)$, ($X^\mu \in \mathfrak{G}$ generating G), where h_μ and g_μ are the coordinates of h and g, then, as sets $[\mathfrak{G}, \mathfrak{H}] \subset \mathfrak{H}$. Hence the structure constants $C_{\cdot}^{\cdot\cdot}$ in $[X^\mu, X^\nu] = c_\rho^{\ \mu\nu} X^\rho$ have the property

$$C_{\mathfrak{G}, \mathfrak{H}}^{\mathfrak{G} - \mathfrak{H}} = 0 \tag{A.3}$$

where the algebra labels $\mathcal{H}$ and $\mathcal{G}$ are used to indicate the set of indices μ such that $\boldsymbol{X}^{\mu} \in \mathcal{H}$ and $\boldsymbol{X}^{\mu} \in \mathcal{G}$, and $\mathcal{G} - \mathcal{H}$ is the complement of $\mathcal{H}$ in $\mathcal{G}$.

From the property (A.3) of (A.2) we conclude that N_{k-1} is a normal subgroup of N_k $(k = 1, 2, \ldots, n)$. The condition (A.3), however, is weaker than (A.2) since it would also be true if (A.2) were written with $\mathfrak{n}_k$ in the right-hand side. Nilpotent groups have, thus, a further property.

Consider the decomposition of an element g of N into an element h in N_k times an element c in N, representative of a coset in the space of cosets N/N_k. This space of cosets is generated by $\mathfrak{n}$ modulo $\mathfrak{n}_k$, that is, by the sets $\{\boldsymbol{X}\} = \{\boldsymbol{X} + \boldsymbol{J}, \boldsymbol{J} \in \mathfrak{n}_k\}$, elements of $\mathfrak{n}/\mathfrak{n}_k$, and the group element c lies on a one-dimensional group generated by some $\boldsymbol{X} \equiv \{\boldsymbol{X}\} \bmod \mathfrak{n}_k$. Since $\mathfrak{n}_k$ is an ideal of $\mathfrak{n}$, a bracket operation can be defined for elements $\{\boldsymbol{X}\}$ and $\{\boldsymbol{Y}\}$ in $\mathfrak{n}/\mathfrak{n}_k$:

$$[\{\boldsymbol{X}\}, \{\boldsymbol{Y}\}] = \{[\boldsymbol{X} + \boldsymbol{J}, \boldsymbol{Y} + \boldsymbol{K}], \boldsymbol{J}, \boldsymbol{K} \in \mathfrak{n}_k\} =$$

$$= \{[\boldsymbol{X}, \boldsymbol{Y}] + \boldsymbol{L}, \boldsymbol{L} \in \mathfrak{n}_k\} = \{[\boldsymbol{X}, \boldsymbol{Y}]\}.$$

If in (A.2) we introduce $\mathfrak{n}_{k-1}$ as the divisor algebra,

$$[\mathfrak{n}/\mathfrak{n}_{k-1}, \mathfrak{n}_k/\mathfrak{n}_{k-1}] \subset \mathfrak{n}_{k-1}/\mathfrak{n}_{k-1} \equiv \boldsymbol{0} \bmod \mathfrak{n}_k .$$

For the generated Lie groups we have therefore, that the representatives of the elements of the coset spaces N/N_{k-1} and N_k/N_{k-1} commute, as $\boldsymbol{0}$ generates the group identity element. This means that N_k/N_{k-1} belongs to the centre of N/N_{k-1}. As the argument may also be followed backwards, we can give a definition of the Lie group N generated by a nilpotent Lie algebra $\mathfrak{n}$ as

DEFINITION 2. A Lie group N is called nilpotent if there exists a sequence of subgroups

$$N = N_n \supset N_{n-1} \supset \ldots \supset N_1 \supset N_0 = \boldsymbol{e}, \qquad (A.4)$$

where N_{k-1} is normal subgroup of N_k and N_k/N_{k-1} is contained in the centre of N/N_k $(k = 1, 2, \ldots n)$.

For simply connected Lie groups N and their Lie algebras $\mathfrak{n}$ the two definitions are equivalent.

APPENDIX B: AN n-DIMENSIONAL WEYL GROUP

The group W_n of $(n+2)\times(n+2)$ matrices

$$g(x, y, z) = \begin{bmatrix} 1 & \xi & \xi \\ -\xi^+ & 1 + 2iz - \tfrac{1}{2}|\xi|^2 & 2iz - \tfrac{1}{2}|\xi|^2 \\ \xi^+ & -2iz + \tfrac{1}{2}|\xi|^2 & 1 - 2iz + \tfrac{1}{2}|\xi|^2 \end{bmatrix}, \tag{B.1a}$$

where ξ is a complex vector

$$\xi_j = x_j + iy_j\,, \quad (j = 1, 2, \ldots, n), \tag{B.1b}$$

ξ^+ its transpose conjugate, and z real, has the followiiig group composition law

$$g(\xi_1, z_1)\, g(\xi_2, z_2) = g(\xi_1 + \xi_2, z_1 + z_2 - \tfrac{1}{2}\,\mathrm{Im}\, \xi_1^+ \xi_2)\,. \tag{B.2}$$

It is an n-dimensional Weyl group in the sense that the generating matrices defined through (B.1) as

$$g(x, y, z) = \exp i\, [\sum_j x_j \mathbf{Q}_j + \sum_j y_j \mathbf{P}_j + z\mathbf{H}] \tag{B.3}$$

have the following commutation relations

$$[\mathbf{Q}_i, \mathbf{Q}_j] = 0\,, \quad [\mathbf{P}_i, \mathbf{P}_j] = 0 \tag{B.4a}$$

$$[\mathbf{Q}_i, \mathbf{P}_j] = i\delta_{ij}\mathbf{H}\,, \quad [\mathbf{Q}_i, \mathbf{H}] = 0\,, \quad [\mathbf{P}_i, \mathbf{H}] = 0\,, \tag{B.4b}$$

i.e., that of the quantum-mechanical operators of position and momentum in an n-dimensional space. Clearly,

$$W_1 = W\,.$$

APPENDIX C: A GENERATING FUNCTION FOR THE ASSOCIATED LAGUERRE POLYNOMIALS

One of the known generating functions of the associated Laguerre polynomials is[34]

$$e^{-uv}(1+v)^m = \sum_{n=0}^{\infty} L_n^{(m-n)}(u)\, v^n \tag{C.1}$$

Set $u = cd$, $v = b/c$; multiplying both sides of (C.1) by

$$c^m a^m / m!$$

and summing over all non-negative m

$$\exp\left[ab + ac - db\right] = \sum_{n,m=0}^{\infty} \frac{1}{m!} L_n^{(m-n)}(cd)\, a^m b^n c^{m-n} . \tag{C.2}$$

Equation (C.2) generalizes a related generating function given by W. Miller[35] for $c = d$, while putting $c^* = d$ we obtain eq. (3.4). Expression (C.2) seems to be new.

FOOTNOTES AND REFERENCES

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2. W. Miller *Lie Theory and Special Functions*, (Academic Press 1968), Chapter 4.
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8. A.A. Kirillov, Russ. Math. Surv. 17 (1962) 53.
9. Reference 4, Chapter 11.
10. Reference 1, Section 22.
11. K. B. Wolf *Point Transformations in Quantum Mechanics I. General Formalism*, Comunicaciones Técnicas C.I.M.A.S.S. 3 (1972) Nº. 7, Subsection 4.2.

12. A different but equivalent matrix realization of $\mathbb{W}$ (and thus W) has been given in Reference 7. There (eq. (2.2))

$$g(x,y,z) = \begin{bmatrix} 1 & 0 & x \\ -y & 1 & -z-\frac{1}{2}xy \\ 0 & 0 & 1 \end{bmatrix} ,$$

which may be easier to work with. In our case, eq. (2.2) generalizes rather easily to an n-dimensional Weyl group, as is done in Appendix B.

13. Parametrizations other than (2.2) are used: W. Miller (Reference 2, Sections 4-1 and 4-11) defines

$$g\,\{a,b,c\} = \exp ia\boldsymbol{H} \exp ib\boldsymbol{P} \exp ic\boldsymbol{Q} ,$$

with $\mathcal{J}^- \leftrightarrow i\boldsymbol{Q}$, $\mathcal{J}^+ \leftrightarrow -i\boldsymbol{P}$, $E \leftrightarrow i\boldsymbol{H}$ and $\pi \equiv 0$ which relates with ours as

$$g(x,y,z) = g\{z - \tfrac{1}{2}xy, -y, x\},\ g\{a,b,c\} = g(c,-b,a-\tfrac{1}{2}bc).$$

The connection with his group S_4 in (4.117), Ref. 2, our (2.14) is achieved by $\alpha \equiv 0$, $x' \leftrightarrow \omega^*/2$ $y' \leftrightarrow -1\omega/2$, $z' \leftrightarrow \delta, \lambda \leftrightarrow \mu$.
J.D. Talman's parametrization (Reference 3) agrees with ours except for a change in sign of z.

14. In order to do this we must restrict our space of functions to the subspace C^∞ of functions having a Taylor expansion i.e., analytic or infinitely-differentiable functions.
15. Compare with W. Miller (Reference 2, equation (4.19)) once the appropriate change in variables, as given in footnote 12, is made.
16. The generators $\boldsymbol{Q}$ and $\boldsymbol{P}$ have then their representation (2.15) only in the subspaces of infinitely differentiable (see footnote 14) functions of, for instance, compact domain or of fast decrease. These subspaces, being dense in $\mathcal{L}^2(-\infty,\infty)$ and in $\mathcal{D}'$, the space of continuous linear functionals, are "good enough" to write the representation (2.15) as valid for their closure. In this connection see L. O'Raifeartaigh *Unitary Representations of Lie Groups in Quantum Mechanics* in *Group Representations in Mathematics and Physics*, Lecture notes in Physics 6, Battelle Seattle Rencontres, edited by V. Bargmann, Springer Verlag (1970), Section 2.
17. Compare with equation (2.9) in Reference 7.
18. If the realization (2.15) is employed, essentially its momentum-space transform, we have a harmonic oscillator of frequency $|\lambda|$, while its mass and the Heisenberg constant ($\hbar$) are put equal to unity.
19. Notice that any of a variety of Hamiltonians yielding a complete and orthogonal basis, including, of course, the free-particle Hamiltonian $\tfrac{1}{2}\boldsymbol{P}^2$ whose eigenbasis can be taken to be (2.20) can be used to classify the rows and columns of a representation. When working with n-dimensional Weyl groups (Appendix B), coordinate systems other than cartesian may be employed.
20. This relation is borne by W. Miller (Reference 2), compare Sections 4.14 on D^λ and 4.16 on $D^{-\lambda}$. The correspondences are (Miller's) $l \leftrightarrow$*(our)* λ, (Miller's) $\lambda \equiv 0$, and a change of sign in the parameters of the one-dimensional subgroups generated by $\boldsymbol{P}$ and $\boldsymbol{H}$.
21. I.S. Gradshtein, I.M. Ryzhik *Tablitsy Integralov, Summ, Ryadov i Proizvedenii*, Izd. Nauka, Moscow (5th edition 1971), Equation 8.957.1.
22. Using also eq. 3.323.2, Reference 21.
23. This is to be compared with W. Miller (Reference 2, eq. (4.123)) once the replacements of footnote 13 have been implemented, and with J.T. Talman

(Reference 3, eq. (13.16)) through the replacements of footnote 13 and $\lambda \equiv 1$.

24. Reference 2, Lemma 4.5 with z replacing α.
25. Reference 2, equation (4.127).
26. M. E. Rose *Elementary Theory of Angular Momentum*, (John Wiley & Sons, 1957), equation (2.28).
27. P. Roman *Advanced Quantum Theory*, Addison Wesley Publ. Co. Inc., (1965), eqs. (1.34); L. O'Raifeartaigh, op. cit. ref. 16 eqs. (2.3). Note the change in units of $\lambda^{\frac{1}{2}}$.
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32. Which is *not* nilpotent due to (4.14).
33. M. Hamermesh *Group Theory and its Application to Physical Problems*, Addison Wesley Publ. Co. Inc., (1962), page 303.
34. Reference 21, equation 8.978.2.
35. Reference 2, equation 4.124.

RESUMEN

Exploramos algunas de las propiedades y representaciones del grupo de Weyl, el grupo de Lie de tres parámetros generado por el álgebra (de Weyl) realizada por los operadores cuánticos de posición, momento y el operador unidad. Consideramos espacios de coclases de una dimensión como espacios homogéneos bajo la acción del grupo y, tomando bases de ondas planas y funciones de onda del oscilador armónico, construimos todas las matrices representaciones unitarias irreducibles del grupo. Agregamos algo de material ya publicado sobre realizaciones no Schrödingerianas, sobre las álgebras de cubrimiento, y lo colocamos en el contexto de los grupos nilpotentes generales.