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CONVERGENCE OF THE SCATTERING TRANSPORT PROCESS TO BROWNIAN MOTION

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ABSTRACT: It is shown that the convergence in distribution of the multidimensional isotropic scattering transport process to Brownian motion, is a consequence of a general invariance principle for a class of random motions.

The isotropic scattering transport process with constant speed in d-dimensional Euclidean space R^d, is defined by the trajectory of a particle under the following motion. Originating at a point in R^d, the particle moves in a fixed direction with constant speed c until it suffers a collision, whereupon it starts afresh from the collision point, independently of the past, in a direction which is isotropically chosen. The intercollision times are exponentially distributed with parameter b.

This process converges in distribution to a d-dimensional Brownian motion process when c and b tend to infinity under the condition $c^2/b \to d/2$. The one-dimensional case was proved by N. Ikeda and H. Nomoto[6] (see also M. A. Pinsky[9]), the two-dimensional case by To. Watanabe[10] (see also A. S. Monin[8]), and the general d-dimensional case by Sh. Watanabe and To. Watanabe[11].

The author's work[4] (see also[3,5]) extended the convergence theorem of ref. (10) and a theorem of M. Kac[7], for more general distributions of intercollision times and direction changes, and from R^2 to R^d for any d. The purpose of this note is to point out that the invariance principle of ref. (4) covers also the convergence theorem of ref. (11) for d-dimensional isotropic scattering. The invariance principle will be stated here not in its full generality, but only insofar as is relevant to the process on hand.

The isotropic scattering transport process in R^d may be described as follows. Assuming the particle starts out from the origin, its position at time $t \geqslant 0$ is given by the column-vector

$$X(t) = c \sum_{i=1}^{N(t)} T_i(b) A_{i-1} \dots A_0 x + c\left(t - \sum_{i=1}^{N(t)} T_i(b)\right) A_{N(t)} \dots A_0 x \, ,$$

where the intercollision times $T_1(b)$, $T_2(b)$, ... are independent random variables having the exponential distribution with parameter b (i.e. Prob $[T_1(b) \geqslant t] = e^{-bt}$),

$$N(t) = \max \left\{ n \colon \sum_{i=1}^{n} T_i(b) \leqslant t \right\}$$

(i.e. the number of collisions up to the time t), the direction changes $A_1, A_2, \dots$ are independent random orthogonal matrices having the Haar distribution on the proper orthogonal group $O^+(d)$, $A_0 = I$ (the identity matrix), $\{T_i(b)\}$ and $\{A_i\}$ are independent sets, and x is the initial (random) direction.

The intercollision times $T_i(b)$ may be viewed as T_i/b, where T_i is exponentially distributed with parameter 1. Let us instead assume only that $ET_1^a < \infty$ for some $a > 3$, where E denotes expectation.

Instead of possessing the Haar distribution on $O^+(d)$ (the isotropy condition), let us assume that the matrices A_i satisfy the much weaker requirement of being irreducible in the following sense. A random matrix (endomorphism of R^d) is called *irreducible* when it has no nontrivial subspace which is invariant with probability one, for $d > 2$, and when it is not nonnegative with probability one, for $d = 1$. This seems to be the weakest condition which prevents the process X from concentrating on a proper subspace of R^d, for $d > 2$, or from not being able to change direction in R^1. It is a fact that if A is an irreducible random contractive (in particular orthogonal) matrix, then the matrix $I - EA$ is nonsingular (see (4), Lemma 5.1).

We then have the invariance principle [3, 4]:

When c and b tend to infinity under the condition $c^2/b \to 1$, the process *X converges in distribution (in the space of continuous functions) to a Gaussian random function with stationary and independent increments, mean zero, and covariance matrix function*

$$EX(t)\,X(s)' =$$

$$= \frac{\min(s,t)}{dET_1}\ \{\mathrm{Var}\ T_1 I + (ET_1)^2\ [(I - EA_1)^{-1} + (I - EA_1')^{-1} - I\,]\ \}$$

(the prime denotes transposition).

The covariance matrix is in general not diagonal with equal eigenvalues, and may have less than full rank (i.e. the limit process may be degenerate), but in the case of the isotropic scattering transport process we have $ET_1 = \mathrm{Var}\ T_1 = 1$ and $EA_1 = 0$, and therefore the covariance matrix becomes $EX(t)\,X(s)' = (2/d)\min(s,t)I$, which corresponds to the result of ref. (11).

Note. For the probabilistic concepts involved here see for example ref. (1) or ref. (2).

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RESUMEN

Se demuestra que la convergencia en distribución del proceso multidimensional de transporte por dispersión isotrópica al movimiento Browniano, es una consecuencia de un principio general de invariancia para una clase de movimientos aleatorios.