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THE ANALYSIS OF TWO-DIMENSIONAL NEUTRON TRANSPORT PROBLEMS BY MEANS OF SINGULAR INTEGRAL EQUATIONS IN TWO COMPLEX VARIABLES

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ABSTRACT: The two-dimensional quarter-space problem of neutron transport theory is analyzed by means of the Fourier transformation. Two procedures for the approximate determination of the neutron density are developed and their convergence analyzed. Assuming that the two-dimensional dispersion function is factorized in a convenient manner, a closed-form solution is established.

INTRODUCTION

The analysis of two-dimensional transport problems can be attempted through either the integrodifferential equation or the integral equation for the neutron angular density. Recently, the application of the theory of two complex variables to problems in analytic function theory[1] has led us to re-examine the integral transport equations.

We have found that this method is very powerful and that we can indeed evaluate some previously insoluble problems. One of these problems is presented in this paper along with a complete description of the method of solution.

In Section 1, an appropriate integral equation for the angular density is developed. Section 2 contains transformed functional relations in two complex variables. In Section 3, two different iteration schemes are developed to solve, in an approximate manner, the transformed equations. In Section 4, an exact, closed form, solution of the transformed equation is given by assuming that an appropriate factorization of the dispersion function exists. Conclusions and suggestions for future work are given in Section 5.

1. THE FUNCTIONAL RELATION FOR THE TRANSFORM OF THE NEUTRON DENSITY.

The time-independent neutron angular density $\phi(\mathbf{r},\Omega)$ in a homogeneous, isotropic scattering medium for monoenergetic neutrons satisfies the equation

$$(1+\Omega\cdot\nabla)\,\phi(\mathbf{r},\Omega)=(c/4\pi)\,\rho(\mathbf{r})+q(\mathbf{r},\Omega)\ , \tag{1.1}$$

where

$$\rho(\mathbf{r})=\int_{\text{all }\Omega}\phi(\mathbf{r},\Omega)\,d\Omega\ ; \tag{1.2}$$

length is measured in units of mean free path, Ω is a unit vector in the direction in which neutrons are travelling, c is the average number of secondary neutrons per collision and q is a known source. The solution of equation (1.1) for a region V (bounded or unbounded) with convex surface S is completely determined (for $c < 1$) when the following boundary condition is imposed:

$$\phi(\mathbf{r}_S,\Omega)=\phi_S(\mathbf{r},\Omega),\quad \Omega \text{ inward to } V\ ,\quad \mathbf{r}_S\in S\ . \tag{1.3}$$

Case and Hazeltine[2] have obtained from equation (1.1) an integral equation for $\phi(\mathbf{r},\Omega)$ by means of the Green's Function technique, and after applying the Fourier transformation, they have translated the general problem to monoenergetic neutron transport theory, into the problem of the determination of the functions $R(\mathbf{k})$ and $R_V(\mathbf{k})$ which appear in the functional relation:

$$R(\boldsymbol{k}) = [1-\Lambda(\boldsymbol{k})]\, R_V(\boldsymbol{k}) + (c/4\pi)\, Q_0(\boldsymbol{k}) + B(\boldsymbol{k}) \;, \tag{1.4}$$

where $\boldsymbol{k}$ is a vector with *real* components k_1, k_2, k_3 in the transformed space, and

$$\Lambda(\boldsymbol{k}) = 1-(c/k)\tan^{-1} k \tag{1.5}$$

the principal branch being taken, and

$$R_V(\boldsymbol{k}) = \int_V \exp(i\boldsymbol{k}\cdot\boldsymbol{r})\,\rho(\boldsymbol{r})\,d\boldsymbol{r} \;, \tag{1.6}$$

$$R(\boldsymbol{k}) = \int_{\text{all}\,\Omega} \Phi(\boldsymbol{k},\boldsymbol{\Omega})\,d\boldsymbol{\Omega} \;, \tag{1.7}$$

$$\Phi(\boldsymbol{k},\boldsymbol{\Omega}) = \int_{\text{all}\,\boldsymbol{r}} \exp(i\boldsymbol{k}\cdot\boldsymbol{r})\,\phi(\boldsymbol{r},\boldsymbol{\Omega})\,d\boldsymbol{r} \;, \tag{1.8}$$

$$Q_0(\boldsymbol{k}) = \int_{\text{all}\,\Omega} [Q(\boldsymbol{k},\boldsymbol{\Omega})/(1-i\boldsymbol{k}\cdot\boldsymbol{\Omega})]\,d\boldsymbol{\Omega} \;, \tag{1.9}$$

$$Q(\boldsymbol{k},\boldsymbol{\Omega}) = (4\pi/c)\int_{\text{all}\,\boldsymbol{r}} \exp(i\boldsymbol{k}\cdot\boldsymbol{r})\,q(\boldsymbol{r},\boldsymbol{\Omega})\,d\boldsymbol{r} \;, \tag{1.10}$$

$$\begin{aligned} B(\boldsymbol{k}) = \int_S d\boldsymbol{r}_S \, \exp(i\boldsymbol{k}\cdot\boldsymbol{r}_S) \times \\ \times \int_{\text{all}\,\Omega} \big(\hat{\boldsymbol{n}}_i\cdot\boldsymbol{\Omega}/(1-i\boldsymbol{k}\cdot\boldsymbol{\Omega})\big)\phi_S(\boldsymbol{r},\boldsymbol{\Omega})\,d\boldsymbol{\Omega} \;, \end{aligned} \tag{1.11}$$

in this last equation $\hat{\boldsymbol{n}}_i$ is the inward unit normal to S.

We will describe our attempts for the determination of R_V in a particular instance.

2. THE QUARTER-SPACE PROBLEM

When the region V is one-quarter of the whole space, no neutrons enter the region and the internal source q is isotropic and independent of the coordinate z, equation (1.4) yields, after one integration over k_3, the equation:

$$R(k_1, k_2) = [1 - \Lambda(k_1, k_2)]\,[R_V(k_1, k_2) + Q(k_1, k_2)]\ , \tag{2.1}$$

where, in this instance,

$$Q(k_1, k_2) = (4\pi/c)\int q(x, y)\exp[i(k_1 x + k_2 y)]\,dxdy\ . \tag{2.2}$$

Also, in equation (2.1) we are using the notation

$$R(k_1, k_2) = \int \rho(x, y)\exp[i(k_1 x + k_2 y)]\,dxdy\ , \tag{2.3}$$

so that by introducing the following definition

$$R_n(k_1, k_2) = \int_{V_n} \rho(x, y)\exp[i(k_1 x + k_2 y)]\,dxdy\ ,\quad n = 1, \ldots, 4, \tag{2.4}$$

where

$$V_1 = \{(x, y) : x, y > 0\}\ , \tag{2.5}$$

$$V_2 = \{(x, y) : x < 0,\ y > 0\}\ , \tag{2.6}$$

$$V_3 = \{(x, y) : x, y < 0\}\ , \tag{2.7}$$

$$V_4 = \{(x, y) : x > 0,\ y < 0\}\ , \tag{2.8}$$

we can write from equation (2.1)

$$R(k_1, k_2) = \sum_{n=1}^{4} R_n(k_1, k_2) = [1 - \Lambda(k_1, k_2)]\,[R_1(k_1, k_2) + Q_1(k_1, k_2)]\ , \tag{2.9}$$

where $Q_1 = Q$ and

$$\Lambda(k_1, k_2) = 1 - c\,(k_1^2 + k_2^2)^{-\frac{1}{2}} \tan^{-1}(k_1^2 + k_2^2)^{\frac{1}{2}} \,. \tag{2.10}$$

3. INTEGRAL EQUATIONS FOR THE TRANSFORM OF THE NEUTRON DENSITY

A. The Hilbert Boundary Value Problem Approach.

If we rewrite equation (2.9) in the form

$$\Lambda(k_1, k_2)\, R_1(k_1, k_2) + \sum_{n=2}^{4} R_n(k_1, k_2) = [1 - \Lambda(k_1, k_2)]\, Q_1(k_1, k_2) \tag{3.1}$$

and recall that the k_j's are real, we can interpret equation (3.1) as a Hilbert boundary value problem, in two complex variables on the real axes, for properly behaved functions R_n and Q_1.

Let us consider the sectionally analytic function

$$R(z_1, z_2) = (2\pi i)^{-2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\phi(k_1, k_2)/(k_1 - z_1)(k_2 - z_2)]\, dk_1\, dk_2 \tag{3.2}$$

where the function $\phi(k_1, k_2)$ is to be determined in such a way that

$$R(z_1, z_2) \equiv R_1(z_1, z_2)\,, \quad \mathrm{Im}(z_j) > 0\,, \quad j = 1, 2.\,, \tag{3.3}$$

$$R(z_1, z_2) \equiv - R_2(z_1, z_2)\,, \quad \mathrm{Im}(z_1) < 0, \quad \mathrm{Im}(z_2) > 0, \tag{3.4}$$

$$R(z_1, z_2) \equiv R_3(z_1, z_2)\,, \quad \mathrm{Im}(z_j) < 0\,, \quad j = 1, 2.\,, \tag{3.5}$$

$$R(z_1, z_2) \equiv - R_4(z_1, z_2)\,, \quad \mathrm{Im}(z_1) > 0\,, \quad \mathrm{Im}(z_2) < 0\,, \tag{3.6}$$

and that when z_1, z_2 are real, equation (3.1) be satisfied.

Employing the Sokhotski formulae[3],

$$R_1(k_1,k_2) = \tfrac{1}{4}\,[\phi(k_1,k_2) + S_1\phi + S_2\phi + S_{12}\phi] \ , \tag{3.7}$$

$$\sum_{n=2}^{4} (-1)^{n+1} R_n(k_1,k_2) = \phi(k_1,k_2) - R_1(k_1,k_2) \ , \tag{3.8}$$

where

$$S_1\phi = (\pi i)^{-1}\int_{-\infty}^{\infty} [\phi(\tau_1,k_2)/(\tau_1 - k_1)]\, d\tau_1 \ , \tag{3.9}$$

$$S_2\phi = (\pi i)^{-1}\int_{-\infty}^{\infty} [\phi(k_1,\tau_2)/(\tau_2 - k_2)]\, d\tau_2 \ , \tag{3.10}$$

$$S_{12}\phi = -\pi^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\phi(\tau_1,\tau_2)/(\tau_1 - k_1)(\tau_2 - k_2)]\, d\tau_1 d\tau_2 \ , \tag{3.11}$$

and the integrals are taken in the sense of the principal value, enables us to write from equation (3.1) the following singular integral equation for ϕ:

$$\phi(k_1,k_2) = \big(4KQ_1/(4-K)\big)(k_1,k_2) + \big(K/(4-K)\big)S_t(\phi) \ , \tag{3.12}$$

where

$$S_t(\phi) = (S_1 + S_2 + S_{12})\,\phi \ , \tag{3.13}$$

$$K(k_1,k_2) = 1 - \Lambda(k_1,k_2) \ . \tag{3.14}$$

It is also convenient to define the functions

$$\gamma(k_1,k_2) = \big(4KQ_1/(4-K)\big)(k_1,k_2) \ , \tag{3.15}$$

$$\alpha(k_1,k_2) = \big(K/(4-K)\big)(k_1,k_2) \;, \tag{3.16}$$

and write equation (3.12) in terms of these:

$$\phi(k_1,k_2) = \gamma(k_1,k_2) + \alpha(k_1,k_2)\, S_t\phi \;, \tag{3.17}$$

from here, the following iterative procedure for the approximate determination of ϕ suggests itself:

$$\phi^{(n)}(k_1,k_2) = \gamma(k_1,k_2) + \alpha(k_1,k_2)\, S_t\phi^{(n-1)} \;,\; n = 1,2,\ldots \tag{3.18}$$

Proof of the convergence of this iterative procedure is given in Appendix A.

Taking

$$\phi^{(0)}(k_1,k_2) = 0 \;, \tag{3.19}$$

gives

$$\phi^{(1)}(k_1,k_2) = \gamma(k_1,k_2) \;, \tag{3.20}$$

$$\phi^{(2)}(k_1,k_2) = \gamma(k_1,k_2) + \alpha(k_1,k_2)\, S_t\gamma \;, \tag{3.21}$$

$$\phi^{(3)}(k_1,k_2) = \gamma(k_1,k_2) + \alpha(k_1,k_2)\, S_t\gamma + \alpha(k_1,k_2)\, S_t\alpha S_t\gamma \;, \tag{3.22}$$

and the general from of the nth iteration is evident.

Equation (3.7) gives us the nth iteration for the transform of the neutron density in the quarter space:

$$R_1^{(n)}(k_1,k_2) = \tfrac{1}{4}\,[\phi^{(n)}(k_1,k_2) + S_t\phi^{(n)}] \ , \tag{3.23}$$

in particular

$$R_1^{(1)}(k_1,\,k_2) = \tfrac{1}{4}\,\gamma(k_1,k_2) + \tfrac{1}{4}\,S_t\gamma \ , \tag{3.24}$$

$$R_1^{(2)}(k_1,k_2) = \tfrac{1}{4}\,\gamma(k_1,k_2) + (1/(4-K))S_t\gamma + \tfrac{1}{4}\,S_t\alpha S_t\gamma \ , \tag{3.25}$$

$$R_1^{(3)}(k_1,k_2) = \tfrac{1}{4}\,\gamma(k_1,k_2) + (1/(4-K))(1+S_t\alpha)\,S_t\gamma + \tfrac{1}{4}\,S_t\alpha S_t\alpha S_t\gamma \ . \tag{3.26}$$

B. Bochner's Decomposition

We now consider an alternate approach to the approximate solution of equation (2.9). To this end, we first determine the region of the z_1, z_2 complex planes where equation (2.9) is valid (we will reserve the symbols k_1, k_2 for the real parts of z_1, z_2 respectively). As discussed in reference 4, the asymptotic behavior of the neutron density is given by

$$\rho(x,y) = \mathcal{O}(\exp(-x/\nu_0)) \quad \text{as } x \to \infty$$

$$\rho(x,y) = \mathcal{O}(\exp x) \quad \text{as } x \to -\infty$$

$$\rho(x,y) = \mathcal{O}(\exp(-y/\nu_0)) \quad \text{as } y \to \infty$$

$$\rho(x,y) = \mathcal{O}(\exp y) \quad \text{as } y \to -\infty$$

where ν_0 satisfies the equation

$$1 = \tfrac{1}{2}\,c\nu_0 \,\mathrm{Ln}\,\frac{1 + 1/\nu_0}{1 - 1/\nu_0} \ , \tag{3.27}$$

and for $0 < c < 1$, ν_0 is a monotonically increasing function of c with values on the interval $(1,\infty)$, ($\nu_0 \to \infty$ as $c \to 1$). Hence,

$$0 < 1/\nu_0 < 1 \ . \tag{3.28}$$

The asymptotic behavior of the neutron density determines the region of analyticity of the R_n's and one obtains from equation (2.4) the following results:

$R_1(z_1, z_2)$ is analytic in the region

$$\mathrm{Im}(z_1) > -1/\nu_0\,, \quad \mathrm{Im}(z_2) > -1/\nu_0\ ,$$

$R_2(z_1, z_2)$ is analytic in the region

$$\mathrm{Im}(z_1) < 1\,, \quad \mathrm{Im}(z_2) > -1/\nu_0\ ,$$

$R_3(z_1, z_2)$ is analytic in the region

$$\mathrm{Im}(z_1) < 1\,, \quad \mathrm{Im}(z_2) < 1\ ,$$

and $R_4(z_1, z_2)$ is analytic in the region

$$\mathrm{Im}(z_1) > -1/\nu_0\,, \quad \mathrm{Im}(z_2) < 1\ .$$

These regions are shown in figures 1 to 4.

The function $\Lambda(z_1, z_2)$ is analytic in the region determined by the conditions

$$|\mathrm{Im}(z_j)| < 1/\sqrt{2}\,, \quad j = 1,2\ . \tag{3.29}$$

We see then that in the region

$$|\mathrm{Im}(z_j)| < \min(1/\sqrt{2}\,,\ 1/\nu_0)\,, \quad j = 1,2\ , \tag{3.30}$$

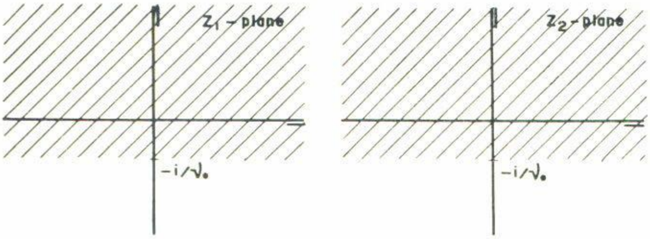

FIG. I REGION OF ANALITICITY OF R_1

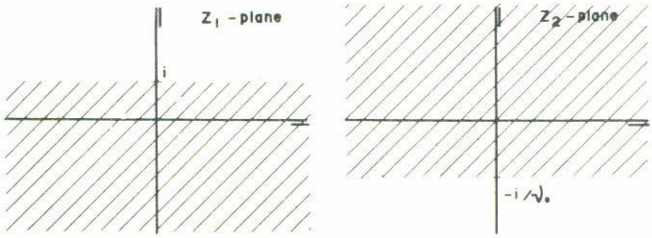

FIG. 2 REGION OF ANALITICITY OF R_2

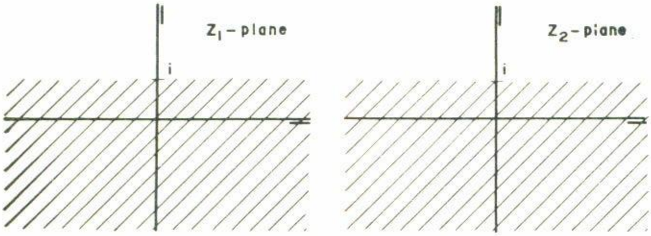

FIG. 3 REGION OF ANALITICITY OF R_3

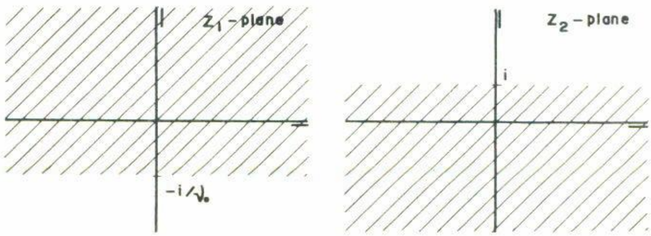

FIG. 4 REGION OF ANALITICITY OF R_4

shown in figure 5, the functions R_n and Λ are analytic. We can also assume, without loss of generality, that the function $Q_1(z_1, z_2)$ is analytic in this region. Hence the equation

$$\Lambda R_1(z_1, z_2) + \sum_{n=2}^{4} R_n(z_1, z_2) = KQ_1(z_1, z_2) \tag{3.31}$$

is valid in the region determined by the conditions given in equation (3.30).

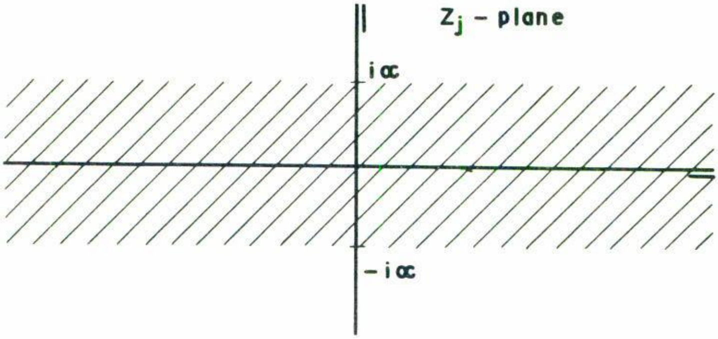

FIG. 5 COMMON REGION OF ANALITICITY

We now invoke a remark due to Bochner[5] and made more precise by Kraut[6] which for our purposes can be stated as follows: If a function $f(z_1, z_2)$ is analytic in a region of the form given in equation (3.30) and if the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(z_1, z_2)|^2 dk_1 dk_2 ,$$

converges in the region, then there exists in the region a decomposition $f = \sum_{j=1}^{4} f_j$, where each f_j is analytic and bounded in the Cartesian product of two-half-planes. When the f_j are obtained by means of Cauchy integrals, the decomposition is unique, up to additive constants.

We now illustrate the way in which the decomposition is accomplished: If one considers the contours L_1, L_2 shown in figure 6, Cauchy's formula in two complex variables is

$$f(z_1, z_2) = (2\pi i)^{-2} \int_{L_1}\int_{L_2} [f(\tau_1, \tau_2)/(\tau_1 - z_1)(\tau_2 - z_2)]\, d\tau_1 d\tau_2 \quad ,(3.32)$$

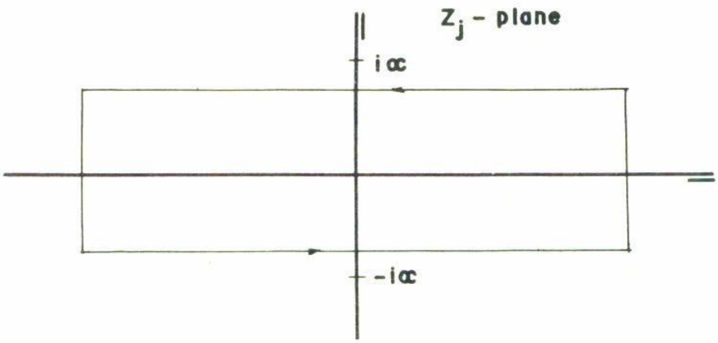

FIG. 6 CONTOURS L$_j$

where $z_j \in D_j^+$, $j = 1, 2$; D_j^+ is the interior of L_j, and f is continuous on $L_1 \times L_2$. Letting now $a_j \to \infty$ and under the condition

$$f(\tau_1, \tau_2)/\tau_j \to 0 \quad \text{as} \quad |\operatorname{Re}(\tau_j)| \to \infty \ , \tag{3.33}$$

we have

$$f(z_1, z_2) = (2\pi i)^{-2} \int_{\gamma_1} \int_{\gamma_2} [f(\tau_1, \tau_2)/(\tau_1 - z_1)(\tau_2 - z_2)] \, d\tau_1 \, d\tau_2 \ ,$$

where

$$\gamma_j = \gamma_j^+ \cup \gamma_j^- \ , \quad j = 1, 2 \ , \tag{3.34}$$

and the contours $\gamma_j^{\pm}$ are shown in figure 7. Now we can write

$$f(z_1, z_2) = (2\pi i)^{-2} \left[\int_{\gamma_I} + \int_{\gamma_{II}} + \int_{\gamma_{III}} + \int_{\gamma_{IV}} \right] [f(\tau_1, \tau_2)/(\tau_1 - z_1)(\tau_2 - z_2)] \, d\tau_1 d\tau_2 \ , \tag{3.35}$$

where

$$
\begin{aligned}
\gamma_{I} &= \gamma_1^{+} \times \gamma_2^{+} \, , \\
\gamma_{II} &= \gamma_1^{-} \times \gamma_2^{+} \, , \\
\gamma_{III} &= \gamma_1^{-} \times \gamma_2^{-} \, , \\
\gamma_{IV} &= \gamma_1^{+} \times \gamma_2^{-} \, ,
\end{aligned}
\tag{3.36}
$$

and

$$
\left| \operatorname{Im}(z_j) \right| < b_j \ , \quad j = 1, 2 \ . \tag{3.37}
$$

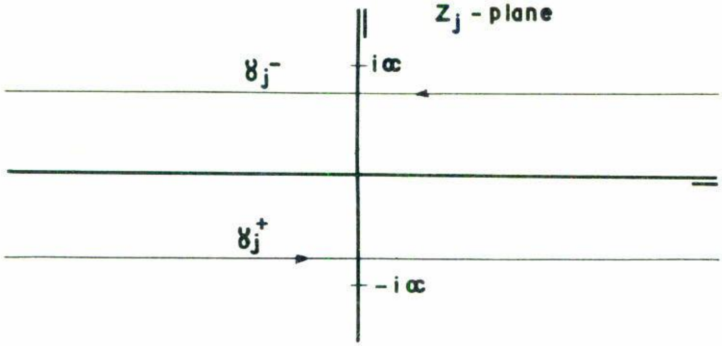

FIG. 7 CONTOURS γ_j

By defining the regions

$$
\begin{aligned}
(I) &= \{ (z_1, z_2) : \operatorname{Im}(z_j) > -b_j \ , \ j = 1, 2 \} \, , \\
(II) &= \{ (z_1, z_2) : \operatorname{Im}(z_1) < b_1 \ , \ \operatorname{Im}(z_2) > b_2 \} \, , \\
(III) &= \{ (z_1, z_2) : \operatorname{Im}(z_1) < b_1 \ , \operatorname{Im}(z_2) < b_2 \} \, , \\
(IV) &= \{ (z_1, z_2) : \operatorname{Im}(z_1) > -b_1 \ , \operatorname{Im}(z_2) < b_2 \}
\end{aligned}
\tag{3.38}
$$

we can say that the function given by

$$(2\pi i)^{-2}\int_{\gamma_\alpha} [f(\tau_1-\tau_2)/(\tau_1-z_1)(\tau_2-z_2)]\, d\tau_1 d\tau_2 \;,$$

where γ_α is any one of the contours given in equation (3.36), is analytic in the corresponding region (α) given in equation (3.38).

The application of these results to the functions $KQ_1(z_1, z_2)$, $\Lambda R_1(z_1, z_2)$, appearing in equation (3.31) yields after some labor the following integral equation

$$\iint_{-\infty}^{\infty} [\Lambda R_1(\tau_1, \tau_2)/(\tau_1-z_1)(\tau_2-z_2)]\, d\tau_1 d\tau_2 =$$

$$= \iint_{-\infty}^{\infty} [KQ_1(\tau_1, \tau_2)/(\tau_1-z_1)(\tau_2-z_2)]\, d\tau_1 d\tau_2 \;, \quad \mathrm{Im}(z_j) > 0, \; j = 1, 2 \tag{3.39}$$

or, equivalently,

$$R_1(z_1, z_2) = (2\pi i)^{-2}\iint_{-\infty}^{\infty} \{K[Q_1+R_1](\tau_1, \tau_2)/(\tau_1-z_1)(\tau_2-z_2)\}\, d\tau_1 d\tau_2 \;,$$

$$\mathrm{Im}(z_j) > 0, \; j = 1, 2 \tag{3.40}$$

it is proved in Appendix B that for $c < 1$, ΛR_1 is of bounded L_2 norm in the region given in equation (3.30) and hence that the decomposition is valid.

We now let $\mathrm{Im}(z_j) \to 0+$, $j = 1, 2$ and obtain, according to the Sokhotski formula:

$$R_1(k_1, k_2) = \tfrac{1}{4}[KQ_1(k_1, k_2) + S_t KQ_1] + \tfrac{1}{4}[KR_1(k_1, k_2) + S_t KR_1] \tag{3.41}$$

and, solving for $R_1(k_1, k_2)$:

$$R_1(k_1, k_2) = (1/(4-K))[KQ_1(k_1, k_2) + S_t KQ_1 + S_t KR_1] \;, \tag{3.42}$$

where the singular integral operator S_t is defined by equation (3.13).

Again we prescribe the iterative procedure

$$R_1^{(n)}(k_1, k_2) = (1/(4-K))[KQ_1(k_1, k_2) + S_t KQ_1 + S_t KR_1^{(n-1)}], \; n = 1, 2, .. \tag{3.43}$$

and set

$$R_1^{(0)}(k_1, k_2) = 0 \tag{3.44}$$

giving

$$R_1^{(1)} = (1/(4-K))[KQ_1 + S_t KQ_1] \tag{3.45}$$

$$R_1^{(2)} = (1/(4-K))[KQ_1 + S_t(4KQ_1/(4-K)) + S_t(K/(4-K)\; S_t KQ_1] \, , \tag{3.46}$$

or, in terms of the functions α, γ, given in equations (3.15) and (3.16):

$$R_1^{(2)} = \tfrac{1}{4}\gamma(k_1, k_2) + (1/(4-K))(S_t\gamma + S_t\alpha\, S_t KQ_1) \tag{3.46'}$$

also

$$R_1^{(3)} = \tfrac{1}{4}\gamma(k_1, k_2) + (1/(4-K))[(1 + S_t\alpha)\, S_t\gamma + S_t\alpha\, S_t\alpha\, S_t KQ_1] \, . \tag{3.47}$$

A comparison of equations (3.45), (3.46') and (3.47) with equations (3.24) and (3.26) of section 3A shows that, except for the last summand on the right hand side, both approaches yield the same results. Specifically, whereas in the iterative procedure of section 3A there appears the term

$$(1/(4-K))S_T\alpha \ldots S_T(4KQ_1/(4-K)),$$

in this section the corresponding term is

$$(1/(4-K))S_t \alpha \ldots S_T K Q_1 \quad .$$

4. CLOSED FORM SOLUTION[7]

We now return our attention to equation (3.1) and consider the following Hilbert boundary value problem on the real axes:

$$A\Phi_1(k_1,k_2) + B\Phi_2(k_1,k_2) + C\Phi_3(k_1,k_2) + D\Phi_4(k_1,k_2) = F(k_1,k_2) \; , \tag{4.1}$$

where the functions A, B, C, D do not vanish anywhere on the real axes.

We first set

$$A\Phi_1(k_1,k_2) + B\Phi_2(k_1,k_2) = f(k_1,k_2) \tag{4.2}$$

and make the assumptions

$$(B/A)(k_1,k_2) = -(\zeta_1/\zeta_2)(k_1,k_2) \; , \tag{4.3}$$

$$(D/C)(k_1,k_2) = -(\xi_3/\xi_4)(k_1,k_2) \; , \tag{4.4}$$

where:

$\zeta_1(z_1,z_2)$ is an analytic function in the region $\mathrm{Im}(z_j) > 0,\ j = 1,2$,

$\zeta_2(z_1,z_2)$ is an analytic function in the region $\mathrm{Im}(z_1) < 0,\ \mathrm{Im}(z_2) > 0$,

$\xi_3(z_1,z_2)$ is an analytic function in the region $\mathrm{Im}(z_1) < 0,\ \mathrm{Im}(z_2) < 0$,

$\xi_4(z_1,z_2)$ is an analytic function in the region $\mathrm{Im}(z_1) > 0,\ \mathrm{Im}(z_2) < 0$,

then we write equation (4.2) in the form

$$\Phi_1(k_1,k_2) - (\zeta_1/\zeta_2)\,\Phi_2(k_1,k_2) = (f/A)(k_1,k_2) \tag{4.5}$$

or

$$(\Phi_1/\zeta_1)(k_1,k_2) - (\Phi_2/\zeta_2)(k_1,k_2) = (f/A\zeta_1)(k_1,k_2)\ ; \tag{4.5'}$$

proceeding in a similar fashion one can write the equation

$$C\Phi_3(k_1,k_2) + D\Phi_4(k_1,k_2) = F(k_1,k_2) - f(k_1,k_2)\ , \tag{4.6}$$

in the form

$$(\Phi_3/\xi_3)(k_1,k_2) - (\Phi_4/\xi_4)(k_1,k_2) = ((F-f)/C\xi_3)(k_1,k_2)\ . \tag{4.7}$$

By introducing the sectionally analytic function

$$\psi(z_1,z_2) = (2\pi i)^{-2}\iint_{-\infty}^{\infty} [\phi(\tau_1,\tau_2)/(\tau_1-z_1)(\tau_2-z_2)\, d\tau_1 d\tau_2 \tag{4.8}$$

and applying the appropriate Sokhotski formulae, we have that

$$\psi_1(k_1,k_2) - \psi_2(k_1,k_2) = \tfrac{1}{2}\,[\phi(k_1,k_2) + S_2\phi]\ , \tag{4.9}$$

and observe that by making

$$(f/A\zeta_1)(k_1,k_2) = \tfrac{1}{2}\,[\phi(k_1,k_2) + S_2\phi]\ , \tag{4.10}$$

we will be able to identify the functions ψ_1 and ψ_2 with the functions Φ_1/ζ_1, Φ_2/ζ_2, respectively.

Using the expression

$$\psi_3(k_1,k_2) - \psi_4(k_1,k_2) = \tfrac{1}{2}\,[\phi(k_1,k_2) - S_2\phi] \tag{4.11}$$

which is also a consequence of the Sokhotski formulae, we see that we must set

$$\big((F-f)/C\xi_3\big)(k_1,k_2) = \tfrac{1}{2}\,[\phi(k_1,k_2) - S_2\phi] \tag{4.12}$$

in order to identify ψ_3 and ψ_4 with Φ_3/ξ_3, Φ_4/ξ_4, respectively.

Solving for f in equation (4.10) and substituting in equation (4.12) we obtain

$$[(F - \tfrac{1}{2}A\zeta_1(\phi + S_2\phi))/C\xi_3]\,(k_1,k_2) = \tfrac{1}{2}\,[\phi(k_1,k_2) - S_2\phi] \tag{4.13}$$

and from this equation we find the singular integral equation that ϕ must satisfy:

$$(C\xi_3 + A\zeta_1)\,\phi + (A\zeta_1 - C\xi_3)\,S_2\phi = 2F(k_1,k_2)\ . \tag{4.14}$$

This singular integral equation can be solved by the standard procedures given in reference 8 and in this manner, the special Hilbert boundary value problem is solved.

Applying the technique employed above to equation (3.1), the corresponding form of equation (4.14) for the quarter space problem is

$$[1 - \zeta_2(k_1,k_2)]\,\phi(k_1,k_2) - [1 + \zeta_2(k_1,k_2)]\,S_2\phi = 2K\underset{\sim}{Q}_1(k_1,k_2)\ , \tag{4.15}$$

where we have assumed that we can express Λ in the form

$$\Lambda = -\,\zeta_2(k_1,k_2)/\zeta_1(k_1,k_2)\ . \tag{4.16}$$

By letting

$$a(k_1, k_2) = 1 - \zeta_2(k_1, k_2) \tag{4.17}$$

$$b(k_1, k_2) = -1 - \zeta_2(k_1, k_2) \tag{4.18}$$

$$c(k_1, k_2) = 2KQ_1(k_1, k_2) \tag{4.19}$$

we can write equation (4.15) in the standard form

$$a\phi(k_1, k_2) + [b(k_1, k_2)/\pi i] \int_{-\infty}^{\infty} [\phi(k_1, \tau_2)/(\tau_2 - k_2)]\, d\tau_2 = c(k_1, k_2) \tag{4.20}$$

A physically acceptable solution of equation (4.20) (i.e., a solution without delta function singularities) is given by

$$\phi(k_1, k_2) = a^* c(k_1, k_2) - [b^* Z(k_1, k_2)/\pi i] \times$$

$$\times \int_{-\infty}^{\infty} [c(k_1, \tau_2)/Z(k_1, \tau_2)(\tau_2 - k_2)]\, d\tau_2 \ , \tag{4.21}$$

where

$$a^*(k_1, k_2) = \big(a/(a^2 - b^2)\big)(k_1, k_2) \ , \tag{4.22}$$

$$b^*(k_1, k_2) = \big(b/(a^2 - b^2)\big)(k_1, k_2) \ , \tag{4.23}$$

$$Z(k_1, k_2) = (a^2 - b^2)(k_1, k_2) \times$$

$$\times \exp\{-(2\pi i)^{-1} \int_{-\infty}^{\infty} [\mathrm{Ln}\, G(k_1, k_2)/(\tau_2 - k_2)]\, d\tau_2\} \ , \tag{4.24}$$

$$G(k_1, k_2) = \big((a - b)/(a + b)\big)(k_1, k_2) \ . \tag{4.25}$$

In terms of ζ_2 and KQ_1, a^*, b^*, G and Z are given by

$$a^*(k_1,k_2) = \tfrac{1}{4}\,[1 - \zeta_2^{-1}(k_1,k_2)]\,, \tag{4.26}$$

$$b^*(k_1,k_2) = \tfrac{1}{4}\,[1 + \zeta_2^{-1}(k_1,k_2)]\,, \tag{4.27}$$

$$G(k_1,k_2) = -\,\zeta_2^{-1}(k_1,k_2)\,, \tag{4.28}$$

$$Z(k_1,k_2) = 2i\,\zeta^{\frac{1}{2}}(k_1,k_2)\times$$
$$\times\exp\left\{-(2\pi i)^{-1}\int_{-\infty}^{\infty}[\mathrm{Ln}(-\zeta_2)(k_1,\tau_2)/(\tau_2-k_2)]\,d\tau_2\right\}. \tag{4.29}$$

We must recall that

$$R_1(z_1,z_2) = \zeta_1(z_1,z_2)\,\psi_1(z_1,z_2),\ \ \mathrm{Im}\,(z_j)\geqslant 0,\ j = 1,2\,. \tag{4.30}$$

where

$$\psi_1(z_1,z_2) = (2\pi i)^{-2}\iint_{-\infty}^{\infty}[\phi(\tau_1,\tau_2)/(\tau_1-z_1)(\tau_2-z_2)]\,d\tau_1\,d\tau_2,\ \mathrm{Im}\,(z_j)\geqslant 0, \tag{4.31}$$

so that

$$R_1(z_1,z_2) = \zeta_1(z_1,z_2)(2\pi i)^{-2}\iint_{-\infty}^{\infty}\tfrac{1}{2}\,[(1-\zeta_2^{-1})(\tau_1,\tau_2)/(\tau_1-z_1)]\times$$
$$\times\ [K\underset{\sim}{Q}_1(\tau_1,\tau_2)/(\tau_2-z_2)]d\tau_1\,d\tau_2\,+$$
$$-\,\zeta_1(z_1,z_2)(2\pi i)^{-2}\iint_{-\infty}^{\infty}\{[\zeta_2^{\frac{1}{2}}(1+\zeta_2^{-1})/(\tau_1-z_1)(\tau_2-z_2)]\,(2\pi i)^{-1}\times$$
$$\times\int_{-\infty}^{\infty}[KQ_1\,\zeta_2^{-\frac{1}{2}}(k_1,\tau_2')\exp(-\Gamma)/(\tau_2'-\tau_2)]\,d\tau_2'\}\exp\Gamma\,d\tau_1\,d\tau_2\,, \tag{4.32}$$

where $\mathrm{Im}(z_j) \geqslant 0$, $j = 1, 2$, and

$$\Gamma(k_1, k_2) = \exp\left\{-(2\pi i)^{-1}\int_{-\infty}^{\infty}\left[\mathrm{Ln}\left(-\zeta_2(k_1, \tau_2)\right)/(\tau_2 - k_2)\right] d\tau_2\right\}, \tag{4.33}$$

5. CONCLUSIONS AND EXTENSIONS

The solution to the behavior of neutrons in a quarter space has been resolved through the application of complex variable theory to the transformed integral transport equation. Two approximate techniques were developed and their convergence analysed, An exact solution was also obtained, based on the decomposition of one equation into two simpler equations, and an application of singular integral equation theory à là Muskhelishvili. The factorization of the dispersion function is suggested as an area of research.

The "solutions" described above were found in the transformed space and none were inverted back. We suggest this as another problem for future work. If appropriate solutions in the real space can be found these will perhaps lead to an elementary decomposition and completeness theorem. We expect this theorem will require mathematics similar to that described herein for its solution.

In principle at least, any two-dimensional problem can be cast into a Hilbert boundary problem in two complex variables (with or without terms containing integrals of the functions) and techniques resembling the approaches we have followed can be developed to obtain the transform of the neutron density.

The simplest logical extension to a three-dimensional problem would be the octant-space problem. One must recall, however, that in this case the Hilbert boundary value problem will consist in the determination of 2^3 unknown functions but, apart from this fact, the same approach used here can be successfully applied.

APPENDICES

A. Convergence of the Iterative Procedures.[9]

We will define the operator S as follows

$$Sf = \gamma(k_1, k_2) + \alpha(k_1, k_2)\, S_T f, \;\; f \in L_2 \;, \tag{A.1}$$

where, γ, α and S_T are defined in section 3A. Then we can write equation (3.17) in the form

$$\phi(k_1, k_2) = S\phi \tag{A.2}$$

hence if we show that S is a contraction mapping with respect to the L_2 norm, then equation (A.2) will have one and only one solution of integrable square in the real axes.

Taking the functions $\phi^{(n)}$, $\phi^{(m)}$, in L_2, one can write

$$\| S\phi^{(n)} - S\phi^{(m)} \|_2 = \| \alpha\, S_t(\phi^{(n)} - \phi^{(m)}) \|_2 \leqslant \max|\alpha|\; \| S_t(\phi^{(n)} - \phi^{(m)}) \|_2. \tag{A.3}$$

Using Minkowski's inequality and the properties

$$\| S_j f \|_2 = \| f \|_2 \;, \;\; j = 1, 2 \;, \tag{A.4}$$

$$\| S_{12} f \|_2 = \| f \|_2 \;, \tag{A.5}$$

given in reference 10, we obtain

$$\| S\phi^{(n)} - S\phi^{(m)} \|_2 \leqslant \max|\alpha|\, 3\, \| \phi^{(n)} - \phi^{(m)} \|_2 \;, \tag{A.6}$$

and since it can be shown that

$$3 \max |\alpha| \leqslant {}^3\!/_4\, c \ , \tag{A.7}$$

we see that for all values of c such that

$$c < {}^4\!/_3 \ , \tag{A.8}$$

S is a contraction mapping and by Banach's fixed point theorem,

$$\phi = \lim_{n\to\infty} S^n \phi_0 \ , \tag{A.9}$$

where $\phi_0 \in L_2$.

A similar proof can be given for the convergence of the iterative procedure of section 3B, the only difference is that instead of defining the operator S, one would define the operator

$$Mf = g(k_1, k_2) + \tfrac{1}{4}\,[Kf + S_t Kf] \ , \tag{A.10}$$

where

$$g(k_1, k_2) = \tfrac{1}{4}\,[KQ_1 + S_t K\underset{\sim}{Q}_1] \ , \tag{A.11}$$

the condition for convergence of the iterative procedure is in this case that $c < 1$.

B. Boundedness of the Norm of ΛR_1.

From equation (3.41), and Minkowski's inequality we write

$$\|R_1\|_2 \leqslant \tfrac{1}{4}\,[\|KR_1\|_2 + \|K\underset{\sim}{Q}_1\|_2 + \|S_1 KR_1\|_2 + \|S_1 K\underset{\sim}{Q}_1\|_2 + \\ + \|S_2 KR_1\|_2 + \|S_2 K\underset{\sim}{Q}_1\|_2 + \|S_{12} KR_1\|_2 + \|S_{12} K\underset{\sim}{Q}_1\|_2] \ , \tag{B.1}$$

and, using the properties given in equations (A.4) and (A.5),

$$\| R_1 \|_2 < \| KR_1 \|_2 + \| K\underset{\sim}{Q}_1 \|_2 \ , \tag{B.2}$$

also, since max $|K| < \infty$

$$\| KR_1 \|_2 \leqslant \max |K| \, \| R_1 \|_2 \ , \tag{B.3}$$

so that (B.2) is

$$(1 - \max |K|) \, \| R_1 \|_2 \leqslant \| K\underset{\sim}{Q}_1 \|_2 \ , \tag{B.4}$$

and by letting

$$H(k_1, k_2) = [\tan^{-1}(k_1^2 + k_2^2)^{\frac{1}{2}}](k_1^2 + k_2^2)^{-\frac{1}{2}} \tag{B.5}$$

we write

$$\| R_1 \|_2 \leqslant \big(c/(1 - c \max |H|)\big) \| H\underset{\sim}{Q}_1 \|_2 \ , \tag{B.6}$$

hence, if we choose c and Q_1, in such a way that

$$c \max |H| < 1 \ , \ \| H\underset{\sim}{Q}_1 \|_2 < \infty \ , \tag{B.7}$$

then

$$\| R_1 \|_2 < \infty \tag{B.8}$$

and $\| \Lambda R_1 \|_2$ is also finite. The conditions (B.7) together with the fact that max $|H| < 1$, imply that for $c < 1$, ΛR_1 has a bounded L_2 norm.

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RESUMEN

El problema bidimensional en un cuadrante, para la teoría de transporte de neutrones se analiza por medio de la transformación de Fourier. Se desarrollan dos procedimientos para la determinación aproximada de la densidad de neutrones y se analiza su convergencia. Se establece una solución de forma cerrada, suponiendo que la función de dispersión bidimensional se factoriza de forma conveniente.