AN APPLICATION OF LINEAR CANONICAL TRANSFORMATIONS: COHERENT STATES*

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ABSTRACT: The problem of the photon field radiated by a classical electric current distribution is equivalent to that of a forced harmonic oscillator. It is shown how certain aspects of the time evolution of this problem arise, in a natural way, by first considering the canonical transformation that describes the evolution of the corresponding classical problem. We then construct the quantum-mechanical evolution through the unitary representation of the classical canonical transformation.

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Let us consider the problem of the photon field radiated by an electric current distribution, which is of a classical nature and does not suffer any noticeable reaction from the process of radiation. We may then represent the radiating current by a prescribed vector function of space and time $\boldsymbol{J}(\boldsymbol{r}, t)$ which will then be treated as a c-number.

As usual[1], the vector potential can be expanded in the form

$$\mathbf{A}(\boldsymbol{r}, t) = c \sum_k (\hbar/2\omega_k)^{\frac{1}{2}} \left[a_k(t) \boldsymbol{u}_k(\boldsymbol{r}) + a_k^*(t) \boldsymbol{u}_k^*(\boldsymbol{r}) \right] , \tag{1}$$

where a_k is dimensionless, $\boldsymbol{u}_k$ is a plane wave and the polarization index has been suppressed. The field is quantized and the total hamiltonian H' can be written, in the Schrödinger picture, as

$$H' = \sum_k \left[\hbar\omega_k (a_k^+ a_k + \tfrac{1}{2}) + \beta_k'(t)\, a_k^+ + \beta_k'(t)^*\, a_k \right] , \tag{2}$$

where

$$\beta_k'(t) = -(\hbar/2\omega_k)^{\frac{1}{2}} \int \boldsymbol{J}(\boldsymbol{r}, t) \cdot \boldsymbol{u}_k^*(\boldsymbol{r})\, d^3\boldsymbol{r} \ . \tag{3}$$

Since the various modes are uncoupled, from now on we shall work with a single mode and drop the index k. For each mode, the hamiltonian will thus be written as

$$H' = \hbar\omega (a^+ a + \tfrac{1}{2}) + \beta'(t)\, a^+ + \beta'(t)^*\, a \ . \tag{4}$$

We shall now assume that at $t = 0$ there were no photons present and seek the solution for later times, the hamiltonian being the explicitly time-dependent expression (4). The solution to this problem is well known and has been discussed, for example, by Glauber[2], who employs field theoretical techniques, and by P. Carruthers and M.M. Nieto[3], who present a particularly simple approach to the problem. We want to show in this paper how that solution can be obtained in a simple way by first considering the canonical transformation that describes the evolution of the classical counterpart of

(4), which is a driven harmonic oscillator. We then construct the quantum-mechanical evolution through the unitary representation of the classical canonical transformation.

Introducing the dimensionless variables x, p through

$$a^{+} = (2)^{-\frac{1}{2}}(x - ip), \qquad a = (2)^{-\frac{1}{2}}(x + ip), \tag{5}$$

and measuring the energy in units of $\hbar\omega$, we have

$$H = \tfrac{1}{2}(p^2 + x^2) + \beta_1(t)\, x + \beta_2(t)\, p \ , \tag{6}$$

where $H = H'/(\hbar\omega)$, $\beta = \beta'/(\hbar\omega) = (\beta_1 + i\beta_2)/\sqrt{2}$.

Considering H as a classical hamiltonian, the time evolution of x and p, in terms of the values x_0, p_0 at time $t = 0$, is given by the canonical transformation

$$x = ax_0 + bp_0 + e$$

$$p = cx_0 + dp_0 + f \ , \tag{7}$$

where

$$a = \cos t\,,\ \ b = \sin t\,,\ \ e = \int_0^t [\cos(t-t')\beta_2(t') - \sin(t-t')\beta_1(t')]\, dt'\,,$$

$$(8)$$

$$c = -\sin t\,,\ \ d = \cos t\,,\ \ f = -\int_0^t [\sin(t-t')\beta_2(t') + \cos(t-t')\beta_1(t')]\, dt'\,.$$

It is shown in Appendix I that a *linear* canonical transformation[4,5] of the type (7) describes the time evolution of any problem, whose hamiltonian is at most quadratic in x and p with coefficients that can be arbitrary functions of time. In passing to the quantum mechanical problem, one can consider (7) as the relation between the position operators at time $t = 0$ and at time t, in the Heisenberg picture. Several conclusions can be drawn in a very elementary way, concerning the time evolution of the spread of a wave packet. This is presented in Appendix II.

One of the authors has found[4] the unitary representation U of an inhomogeneous canonical transformation of the type (7). U is defined by

$$x = U x_0 U^{-1} , \tag{9}$$

where x is the position operator at time t in the Heisenberg picture, and x_0 the corresponding operator at time $t = 0$. If we denote by $|\psi_0>$ a state vector in the Heisenberg representation, we have

$$<\psi_0|x|\psi_0> = <\psi_0|U x_0 U^{-1}|\psi_0> = <\psi|x_0|\psi> , \tag{10}$$

where

$$|\psi> = U^{-1}|\psi_0> . \tag{11}$$

Therefore U^{-1} describes the time evolution of the state, in the Schrödinger representation.

We shall write $U = U_{hom} U_{inh}$, where U_{hom} corresponds to the homogeneous part of the canonical transformation (7), characterized by the parameters a, b, c, d; U_{inh} corresponds to the inhomogeneous part and is characterized by the parameters e, f. In the particular case in which a, b, c, d are given by (8), the unitary representation U_{hom} takes on a very simple form in the basis $|n'>$ in which the hamiltonian of the harmonic oscillator is diagonal:

$$<n'|U_{hom}|n''> = \exp\left[i(n' + \tfrac{1}{2})t\right] \delta_{n'n''} . \tag{12}$$

That U_{hom} with a, b, c, d given by (8) is diagonal in the basis $|n>$ is clear from the fact that this is just the symmetry group of the one-dimensional oscillator. The particular form (12) is just the conjugate of the time evolution of the state $|n>$ of a free harmonic oscillator.

On the other hand, U_{inh} has a simple expression in the basis $|x'>$ in which the operator x_0 of Eq. (9) is diagonal:

$$< x' \,|\, U_{\text{inh}} \,|\, x'' > = \exp(-ifx'')\,\delta(x' + e - x'') \quad . \tag{13}$$

The displacement e in the coordinate gives rise to the δ-function, while the displacement f in the momentum gives the exponential as one can easily verify passing to momentum space.

One can then express U_{inh} in the basis $|n'>$, making use of the generating function of the Hermite polynomials[4]. Combining the resulting expression with (12), one obtains

$$\begin{aligned} < n \,|\, U^{-1} \,|\, n_0 > &= \exp\left[-i(n_0 + \tfrac{1}{2})t\right](f + ie)^{n_0}(f - ie)^{n} \times \\ &\times \exp\left[-\tfrac{1}{4}(e^2 - 2ief + f^2)\right](n_0!\,n!/2^{n_0+n})^{\frac{1}{2}}\, i^{n_0+n} \times \\ &\times \sum_r \left[(-2)^r (f^2 + e^2)^r / r!\,(n_0 - r)!\,(n - r)!\right] \quad , \end{aligned} \tag{14}$$

which gives, at time t, the overlap with $|n>$ of the state that was $|n_0>$ at time $t = 0$. If $n_0 = 0$, this result reduces to the simpler expression

$$< n \,|\, U^{-1} \,|\, 0 > = \exp\left[\chi(t)\right]\left[\lambda(t)\right]^n / (n!)^{\frac{1}{2}} \tag{15a}$$

$$\begin{aligned} \lambda(t) &= -i\int_0^t \exp\left[-i(t - t')\right]\beta(t')\,dt' \,, \\ \chi(t) &= -\tfrac{1}{2}it - \tfrac{1}{4}(e^2 + f^2 - 2ief) \,, \end{aligned} \tag{15b}$$

so that the state at time t is

$$|\psi(t)> = \exp\left[\chi(t)\right]\exp\left[\lambda(t)\, a^+\right]|\, 0 > \quad . \tag{16}$$

We see that

$$a\,|\psi(t)> = \lambda(t)\,|\psi(t)> \quad ; \tag{17}$$

i.e. $|\psi(t)>$ is an eigenfunction of the annihilation operator, and this is the definition of a coherent state[1]. It is convenient to write $|\psi(t)>$ as a unitary operator acting on $|0>$. Using the fact that $\exp(-\lambda^* a)|0> = |0>$ and making use of the identity[6]

$$\exp A \exp B = \exp\left(A + B + \tfrac{1}{2}[A,B]\right) , \tag{18}$$

valid if $\big[[A,B],A\big] = \big[[A,B],B\big] = 0$, the state $|\psi(t)>$ takes the form

$$|\psi(t)> = \exp[\chi(t) + |\lambda(t)|^2] \exp[\lambda(t)a^+ - \lambda^*(t)a]\,|0> . \tag{19}$$

Suppose that one proposes a wave function $|\psi>$ with the structure of Eq. (16)

$$|\psi> = \exp[\chi'(t)]\exp[\lambda'(t)a^+]\,|0> , \tag{20}$$

and asks that it satisfies Schrödinger's equation

$$i\partial|\psi>/\partial t = H|\psi> . \tag{21}$$

The function $\lambda'(t)$ that one obtains is identical with the $\lambda(t)$ given by (15b); however $\chi'(t)$ does not coincide with $\chi(t)$

$$\chi(t) - \chi'(t) = \tfrac{1}{2} i \int_0^t (f^2 - e^2 + 2\beta_2 f)\,dt . \tag{22}$$

The difference is pure imaginary and does not affect, of course, any transition probability. Notice that if $|\psi>$ satisfies Schrödinger's equation (21), then $\psi \exp[i\gamma(t)]$ satisfies the modified Schrödinger's equation

$$[(i\partial/\partial t) + (d\gamma(t)/\partial t)]\,[\exp(i\gamma(t))\,\psi] = H\,[\exp(i\gamma(t))\,\psi] . \tag{23}$$

This is similar to the freedom that one has[7] in defining the momentum operator as

$$p = (-i\partial/\partial x) + (d\gamma(x)/dx) \ , \qquad (24)$$

which reduces to the usual definition by an appropriate choice of the phase in the wave function. When one starts from the classical solution and derives from it the quantum-mechanical one, as we are doing here, one looses control on the quantity γ, because the various quantum solutions all map into the same classical one. However, it is interesting to note that while (12) and (13) give an extra phase in the problem of the forced harmonic oscillator, they do not in the case of the free oscillator, nor in the case of the free particle. At present it is not clear what is the condition that has to be imposed, in order to obtain a solution of the ordinary Schrödinger's equation, with the simple form $i\partial/\partial t$ for the energy operator.

APPENDIX I

HAMILTONIANS THAT GIVE RISE TO A LINEAR CANONICAL TRANSFORMATION.

Consider the classical mechanical problem of a one dimensional system satisfying Hamilton's equations of motion. If we specify the position and momentum x_0 and p_0 at time 0, their values x, p for future times will be connected to x_0, p_0 by a canonical transformation. In this section we shall find the most general hamiltonian for which this canonical transformation is linear in x_0, p_0.

Using Hamilton's equations of motion

$$\dot{x} = \partial H/\partial p \ , \quad \dot{p} = -\partial H/\partial x \ , \qquad (A1.1)$$

we can write the first two terms of a Taylor expansion of $x(t)$ and $p(t)$ near the original instant $t = 0$ as

$$x(t) \approx x(0) + \dot{x}(0)\,t = x_0 + [\,\partial H(x_0, p_0, 0)/\partial p_0\,]\,t \ , \qquad (A1.2a)$$

$$p(t) \approx p(0) + \dot{p}(0)\,t = p_0 - [\,\partial H(x_0, p_0, 0)/\partial x_0\,]\,t \ . \qquad (A1.2b)$$

If the canonical transformation has to be linear for every time, it must be linear for the infinitesimal transformation (A1.2). If we allow the potential to depend on x, p and t, we ask

$$\partial H/\partial p = f_1(t)\,x + f_2(t)\,p + f_3(t) \ , \tag{A1.3a}$$

$$\partial H/\partial x = f_4(t)\,x + f_5(t)\,p + f_6(t) \ , \tag{A.1.3b}$$

which, on integrating, lead to a hamiltonian that is the most general bilinear expression in x, p, with coefficients which can be arbitrary functions of time:

$$H(x,p,t) = \sum_{\substack{n,m \\ 0 \leqslant n+m \leqslant 2}} f_{nm}(t)\,x^n p^m \ . \tag{A1.4}$$

It has now to be shown that (A1.4) leads to a linear transformation for any finite time interval. To this end, we shall construct the time evolution of any function $f(x,p,t)$, knowing that

$$df(x,p,t)/dt = \hat{H}(x,p,t)\,f(x,p,t) + \partial f(x,p,t)/\partial t \ , \tag{A1.5}$$

where

$$\hat{H}(x,p,t) \equiv (\partial H/\partial p)(\partial/\partial x) - (\partial H/\partial x)(\partial/\partial p) \ . \tag{A1.6}$$

Notice that the first term on the right-hand side of (A1.5) is just the Poisson bracket of H and f. We are looking for an operator $\hat{U}(x_0, p_0, t)$ with the property that acting on any function $f(x_0, p_0, t')$, it takes x_0 to $x(x_0, p_0, t)$ and p_0 to $p(x_0, p_0, t)$

$$\hat{U}(x_0, p_0, t)\,f(x_0, p_0, t') = f\big(x(x_0, p_0, t),\, p(x_0, p_0, t),\, t'\big) \quad \forall f \tag{A1.7}$$

$x(x_0, p_0, t)$ and $p(x_0, p_0, t)$ are the values of the position and momentum at time t if they have evolved according to the equations of motion from x_0, p_0

at time zero. If H does not depend on t explicitly, it is clear that

$$f(x,p,t) = \exp\left[\hat{H}(x_0, p_0)\, t\right] f(x_0, p_0, t) \ . \tag{A1.8}$$

However, if H depends explicitly on time, one has to be more careful, because $\hat{H}(x_0, p_0, t_1)$ will not commute with $\hat{H}(x_0, p_0, t_2)$. Differentiating both sides of (A1.7) with respect to t, we obtain

$$\begin{aligned}
&[\partial \hat{U}(x_0, p_0, t)/\partial t]\, f(x_0, p_0, t') \\
&= [\partial f\big(x(x_0, p_0, t),\, p(x_0, p_0, t),\, t'\big)/\partial x]\,(dx/dt) + \\
&+ [\partial f\big(x(x_0, p_0, t),\, p(x_0, p_0, t),\, t'\big)/\partial p]\,(dp/dt) \\
&= \hat{H}(x,p,t)\, f(x,p,t')
\end{aligned} \tag{A1.9}$$

Using property (A1.7) we have

$$[\partial \hat{U}(x_0, p_0, t)/\partial t]\, f(x_0, p_0, t') = \hat{U}(x_0, p_0, t)\, \hat{H}(x_0, p_0, t)\, f(x_0, p_0, t') \ , \tag{A1.10}$$

and since this equation must hold for any function f, we have

$$[\partial \hat{U}(x_0, p_0, t)/\partial t] = \hat{U}(x_0, p_0, t)\, \hat{H}(x_0, p_0, t) \ . \tag{A1.11a}$$

The operator $\hat{U}$ is uniquely defined by this differential equation, together with the initial condition

$$\hat{U}(x_0, p_0, 0) = 1 \ . \tag{A1.11b}$$

One can check directly that the operator

$$\hat{U}(x_0, p_0, t) = 1 + \int_0^t dt_1\, \hat{H}(x_0, p_0, t_1) + \\ + \int_0^t dt_2 \int_0^{t_2} dt_1\, \hat{H}(x_0, p_0, t_1)\, \hat{H}(x_0, p_0, t_2) + \dots \tag{A1.12}$$

satisfies (A1.11a, b).

We shall now apply these considerations to the hamiltonian (A1.4). The operator $\hat{H}$ is, in this case

$$\hat{H}(x_0, p_0, t) = [f_{01}(t) + f_{11}(t)x_0 + 2f_{02}(t)p_0](\partial/\partial x_0) + \\ - [f_{10}(t) + f_{11}(t)p_0 + 2f_{20}(t)x_0](\partial/\partial p_0) . \tag{A1.13}$$

It is clear that the repeated application of (A1.13) to x_0 or p_0, in order to construct $x = \hat{U}x_0$, will always be linear in these quantities. We have thus proved our statement that (A1.4) is the most general hamiltonian for which x and p evolve in time according to a canonical transformation that is linear in the initial conditions x_0, p_0.

APPENDIX II

TIME EVOLUTION OF THE SPREAD OF A WAVE PACKET

We shall consider a system whose classical time evolution is given by the linear canonical transformation (7), without restricting the parameters to the specific values given by (8); they will only have to satisfy the relation

$$ad - bc = 1 , \tag{A2.1}$$

so that the transformation is canonical.

Notice that, having solved the classical problem, we have also solved the quantum-mechanical one in the Heisenberg representation. That is, if now x_0 and p_0 stand for the Heisenberg operators at time $t = 0$ and x and p at time t, they will be related precisely by Eq. (7).

Consider a wave packet and call

$$(\Delta x)^2 \equiv <x^2> - <x>^2, \quad (\Delta p)^2 \equiv <p^2> - <p>^2, \tag{A2.2}$$

the square of the spreads in x and p at time t. One can easily show, for any linear canonical transformation of the type (7), that

$$(\Delta x)^2 = a^2(\Delta x)^2_{t=0} + b^2(\Delta p)^2_{t=0} + ab\,[<x_0 p_0 + p_0 x_0> - 2<x_0><p_0>]$$

$$(\Delta p)^2 = c^2(\Delta x)^2_{t=0} + d^2(\Delta p)^2_{t=0} + cd\,[<x_0 p_0 + p_0 x_0> - 2<x_0><p_0>]\,. \tag{A2.3}$$

We thus obtain the result that the inhomogeneous part of a linear canonical transformation does not affect the time evolution of the spread of the wave packet. If the system is a driven harmonic oscillator, ***the driving force does not affect the spread as a function of time.***

It is interesting that if the wave function at $t = 0$ (in the Heisenberg picture it remains constant in time) is a gaussian of the type

$$\psi(x) = (2\pi\sigma^2)^{-\frac{1}{4}} \exp\left[-\left((x-a)^2/4\sigma^2\right) + ikx\right], \tag{A2.4}$$

the square bracket in (A2.3) vanishes and we have

$$\begin{aligned}(\Delta x)^2 &= a^2(\Delta x)^2_{t=0} + b^2(\Delta p)^2_{t=0}\\ (\Delta p)^2 &= c^2(\Delta x)^2_{t=0} + d^2(\Delta p)^2_{t=0}\,.\end{aligned} \tag{A2.5}$$

The wave packet (A2.4) is such that

$$\begin{aligned}(\Delta x)^2_{t=0} &= \sigma^2\\ (\Delta p)^2_{t=0} &= \hbar^2/4\sigma^2\,.\end{aligned} \tag{A2.6}$$

We shall consider some particular cases, assuming that (A2.5) are fulfilled.

1) *Free particle*. In this case the parameters of the canonical transformation are

$$a = 1, \quad b = t/m, \quad e = 0$$
$$c = 0, \quad d = 1, \quad f = 0 \tag{A2.7}$$

Therefore

$$(\Delta x)^2 = \sigma^2 [1 + (\hbar t/2m\sigma^2)^2]$$
$$(\Delta p)^2 = (\Delta p)^2_{t=0} = \hbar^2/4\sigma^2 , \tag{A2.8}$$

2) *Free fall*.

$$a = 1, \quad b = t/m, \quad e = \tfrac{1}{2} g t^2$$
$$c = 0, \quad d = 1, \quad f = gt . \tag{A2.9}$$

Δx and Δp behave exactly as for the free particle.

3) *Free harmonic oscillator*.

$$a = \cos \omega t , \quad b = (1/m\omega) \sin \omega t , \quad e = 0$$
$$c = - m\omega \sin \omega t , \quad d = \cos \omega t , \quad f = 0 \tag{A2.10}$$

$$(\Delta x)^2 = \sigma^2 \cos^2 \omega t + (\hbar/2m\omega\sigma)^2 \sin^2 \omega t$$
$$(\Delta p)^2 = (m\omega\sigma)^2 \sin^2 \omega t + (\hbar/2\sigma)^2 \cos^2 \omega t . \tag{A2.11}$$

The spread Δx does not increase indefinitely with time as in the case of the free particle, as is clear physically from the fact that the system is confined. Even more, both Δx and Δp are periodic in time.

4) *Forced harmonic oscillator*. Δx and Δp are again given by (A2.11). Notice that if we choose

$$\sigma = (\hbar/2m\omega)^{\frac{1}{2}} , \tag{A2.12}$$

then Δx and Δp become time independent: *the wave packet thus shows a coherent behaviour.*

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RESUMEN

El problema del campo de fotones radiado por una distribución clásica de corriente, es equivalente al problema de un oscilador armónico forzado. Se muestra cómo algunos aspectos de la evolución temporal de este problema se describen, de una manera natural, considerando primero la transformación canónica que describe la evolución del correspondiente problema clásico. Se construye después la evolución cuántica mediante la representación unitaria de la transformación canónica clásica.