Revista Mexicana de Física 22 (1973) 273-280

SIMPLE MODEL FOR NUCLEAR DENSITY OSCILLATIONS

M. de Llano*

Instituto de Física, Universidad Nacional de México

and

J. Quintanilla**

Centro Nuclear, Instituto Nacional de Energía Nuclear,

Salazar, México

(Recibido: marzo 20, 1973)

ABSTRACT: Using rigid-wall boundary conditions, the probability density of an N-fermion system is calculated in one dimension as the sum of the N lowest-energy standing waves enclosed in a given length L. The result agrees qualitatively both with previous, more elaborate theories, and with experiments on nuclear density oscillations and surface thickness. A simple calculation of the surface energy is done and found to be of the right sign.

* Work supported by Instituto Nacional de Energía Nuclear, México

** Work supported by Instituto de Física, Universidad de México, México 20, D. F.

1. PROBABILITY DENSITY

Confine a system of N particles in a one-dimensional box of length L with rigid walls. The complete, orthonormal set of independent-particle states are then the standing waves

$$\varphi_n(x) = \sqrt{2/L}\ \sin(n\pi x/L)\ , \quad (n = 1, 2, \ldots, N, N+1, \ldots \infty)\ . \quad (1)$$

The probability density associated with the lowest-energy configuration will be

$$\rho(x) = \sum_{n=1}^{N} |\varphi_n(x)|^2 = (2/L) \sum_{n=1}^{N} \sin^2(n\pi x/L)$$

$$= (N/L)\ \{1 - [\cos(N+1)\pi x/L)\ \sin(N\pi x/L)]\ /N\ \sin(\pi x/L)\}\ , \quad (2)$$

the last step following from ref. 1. It is easy to verify that

$$\rho(0) = \rho(L) = 0\ , \quad (3)$$

as well as the limiting forms

$$\rho(x) \xrightarrow[x \to 0]{} (2\pi^2/3)(N/L)^3 x^2 - O(x^4) \quad (4)$$

and

$$\rho(x) \xrightarrow[\substack{N \gg 1 \\ x/L \ll 1}]{} (N/L)\,[1 - j_0(2\pi N x/L)]\ , \quad (5)$$

where $j_0(y) \equiv \sin y/y$ is the zero-order spherical Bessel function. The bracketed expression in (2) is plotted for $0 \lesssim x \lesssim L/2$ (the center of the "nucleus"), for $N = 2, 4, 10, 50$ and 250 particles in Figures 1 to 5 respectively, with $N = L$ (for scale convenience). There appear as many "humps" as particles in the length L; hence, for N not too small, the wavelength of oscillation $\lambda \simeq L/N = \pi/k_F$, where k_F is the Fermi "surface" of a one-dimensional gas of fermions in a length L, and is related to the density as $N/L = k_F/\pi$. The result $\lambda \simeq \pi/k_F$, which becomes a strict equality for $N \gg 1$, as can be seen from (5), agrees with the treatments of Swiatecki[2] and Thorpe and Thouless[3], and also with electron scattering experiments on ^{40}Ca (ref. 4) and ^{208}Pb (ref. 5). Figure 6 shows the bracketed expression of (5) and represents the probability density of a very large system very close to its surface. The "skin depth" s, here defined for $N \gg 1$ and $x/L \ll 1$ as the distance from the "rigid wall" (the point $x = 0$) to where the function $\rho(x)$ first takes on the equilibrium value N/L, is just (see Figure 6 and eq. (5)) $s = L/2N \simeq [2(0.17)^{1/3}]^{-1} \simeq 0.9$ fermi, where the empirical density of 0.17 nucleons per (fermi)^3 has been inserted. The discrepancy with the empirical value[6] of $s_{\exp} \simeq 2.4$ fermi can be roughly explained in terms of a "smearing" out of the surface in the real system, i.e., non-rigid walls.

2. SURFACE ENERGY

We now deduce expressions for the surface energy in terms of the interparticle force to first order, i.e., the difference in expectation values per particle of the system hamiltonian, taken between Slater determinants composed, on the one hand, of standing-wave (SW) orbitals (1) and, on the other hand, of plane-wave (PW) orbitals

$$\varphi_{k_n}(x) = L^{-1/2} \exp(ik_n x), \quad (n = \pm 1, \pm 2, \ldots, \pm N/2), \tag{6}$$

which are normalized to unity in the length L. Namely, we seek

$$\Delta E/N \equiv (E_{\mathrm{SW}} - E_{\mathrm{PW}})/N , \tag{7}$$

$$E_\alpha \equiv \langle \Phi_0^{(\alpha)} | H | \Phi_0^{(\alpha)} \rangle , \quad \alpha = \mathrm{SW} \text{ or } \mathrm{PW} , \tag{8}$$

$$\Phi_0^{(\alpha)}(x_1, x_2, \ldots, x_N) \equiv (N!)^{-1/2} \det | \varphi_n^{(\alpha)}(x_i) | , \tag{9}$$

$$H = -(\hbar^2/2m) \sum_{i=1}^{N} \nabla_i^2 + \sum_{i<j}^{N} v_{ij} \tag{10}$$

where v_{ij} is the two-body interaction. Eq. (8), with (9) and (10), can be expressed as

$$E_\alpha = -(\hbar^2/2m)\int dx\,\left[(d^2/dx^2)\rho_\alpha(x',x)\right]_{x'=x} +$$

$$+\tfrac{1}{2}\int dx_1\int dx_2\, v(x_1-x_2)\left\{\rho_\alpha(x_1)\rho_\alpha(x_2) - \left|\rho_\alpha(x_1,x_2)\right|^2\right\}\ , \qquad (11)$$

where we have defined

$$\rho_\alpha(x,x') \equiv \sum_{n=1}^{N} \varphi_n^{(\alpha)*}(x)\,\varphi_n^{(\alpha)}(x') \xrightarrow[x'\to x]{} \rho_\alpha(x)\ . \qquad (12)$$

Specifically, from eq. (6), one has

$$\rho_{PW}(x,x') = (1/L)\sum_{n=\pm 1}^{\pm N/2} \exp\left[-ik_n(x-x')\right]\ . \qquad (13)$$

For a large system

$$N = \sum_{n=\pm 1}^{\pm N/2} 1 \longrightarrow (L/2\pi)\int_{-k_F}^{+k_F} dk = Lk_F/\pi \qquad (14)$$

and, likewise, eq. (13) becomes

$$\rho_{PW}(x,x') \longrightarrow (1/L)(L/2\pi)\int_{-k_F}^{+k_F} dk\, \exp\left[-ik(x-x')\right]$$

$$= \sin\left[N\pi(x-x')/L\right]/\pi(x-x') \xrightarrow[x'\to x]{} N/L\ , \qquad (15)$$

giving the expected space-independent probability density of plane-wave states. Also, from eq. (1) follows

$$\rho_{SW}(x, x') = (2/L) \sum_{n=1}^{N} \sin(n\pi x/L) \sin(n\pi x'/L)$$

$$= (1/L) \sum_{n=1}^{N} \{\cos[n\pi(x - x')/L] - \cos[n\pi(x + x')/L]\}$$

$$= (1/L)\left\{\frac{\cos[(N+1)\pi(x-x')/2L]\sin[N\pi(x-x')/2L]}{\sin[\pi(x-x')/2L]} - \right.$$

$$\left. - \frac{\cos[(N+1)\pi(x+x')/2L]\sin[N\pi(x+x')/2L]}{\sin[\pi(x+x')/2L]}\right\} , \qquad (16)$$

the last equality coming from ref. 7. For a large system ($N >> 1$) with short-ranged forces ($L >> x_1 - x_2$) this simplifies to

$$\rho_{SW}(x, x') \xrightarrow[\substack{N>>1 \\ L>>x-x'}]{} \rho_{PW}(x, x') - \frac{\sin[N\pi(x+x')/L]/2L}{\sin[\pi(x+x')/2L]} \qquad (17)$$

which gives for (7), using (11),

$$\Delta E \xrightarrow[\substack{N \gg 1 \\ L \gg (x_1 - x_2)}]{} \Delta T + \Delta v ; \qquad (18)$$

$$\Delta T \equiv (\hbar^2/2m) \int_0^L dx_1 \left[\frac{d^2}{dx_2^2} \frac{\sin[N\pi(x_1+x_2)/L]}{2L \sin[\pi(x_1+x_2)/2L]}\right]_{x_2 = x_1} = 0 , \qquad (19)$$

$$\Delta v \equiv \frac{1}{2}\left(\frac{N}{L}\right)^2 \int_0^L dx_1 \int_0^L dx_2\, v(x_1 - x_2) \left\{ \frac{\sin(2N\pi x_1/L)}{2N\pi x_1/L}\, \frac{\sin(2N\pi x_2/L)}{2N\pi x_2/L} - \right.$$

$$\left. - \frac{\sin(2N\pi x_1/L}{2N\pi x_1/L} - \frac{\sin(2N\pi x_2/L)}{2N\pi x_2/L} \right\} - \frac{1}{2}\, \frac{1}{(2L)^2} \int_{-L}^{+L} dx\, v(x) \int_{x/2}^{L+x/2} dX\, g^2(X) +$$

$$+ \frac{1}{2}\, \frac{1}{(2L)} \int_{-L}^{+L} dx\, v(x)\, \frac{\sin(N\pi x/L)}{\pi x} \int_{x/2}^{L+x/2} dX g(X)\ ; \qquad (20)$$

$$X \equiv \tfrac{1}{2}(x_1 + x_2),\quad x \equiv x_1 - x_2,\quad g(X) \equiv \sin(2N\pi X/L)/\sin(\pi X/L)\ .$$

The vanishing of (19) readily follows from

$$\int_0^L dx_1 \left[\frac{d^2}{dx_2^2}\, \frac{\sin[N\pi(x_1 + x_2)/L]}{\sin[\pi(x_1 + x_2)/2L]} \right]_{x_2 = x_1} = \int_0^{2L} d\xi \left[\frac{d^2}{d\xi^2}\, \frac{\sin(N\pi\xi/L)}{\sin(\pi\xi/2L)} \right] =$$

$$= \left[\frac{d}{d\xi}\, \frac{\sin(N\pi\xi/L)}{\sin(\pi\xi/2L)} \right]_0^{2L} = 2 \left[\frac{d}{d\xi} \sum_{n=1}^{N} \cos\{2n-1)\pi\xi/2L\} \right]_0^{2L} = 0\ ,$$

the next to last equality being a well-known[8] identity $\sin y = 2 \sum_{n=1}^{N} \cos(2n-1)y$. Using this identity throughout, as well as $\sin(N\pi + y) = (-1)^N \sin y$, the integrals in (20) not involving $v(x)$ become

$$\int_{x/2}^{L+x/2} dX g(X) = (L/\pi) \int_{\pi x/2L}^{(\pi x/2L)+\pi} dy\, (\sin 2Ny/\sin y)$$

$$= -(4L/\pi) \sum_{n=1}^{N} \sin[(2n-1)\pi x/2L]/(2n-1)\ ;$$

$$\int_{x/2}^{L+x/2} dX g^2(X) = (4L/\pi) \sum_{n,m=1}^{N} \int_{\pi x/2L}^{(\pi x/2L)+\pi} dy\, \cos(2m-1)y\, \cos(2n-1)y = 2LN\ ,$$

(21)

the last integral being just $(\pi/2)\,\delta_{m,n}$. The remaining quantities in (20) can be evaluated for a definite interaction; we shall take only the simplest one leaving simple integrals, namely

$$v(x) = v_0\,\delta(x)\ . \qquad (22)$$

Then, (20) becomes

$$\Delta v = (v_0 N/4\pi L)\int_0^{2\pi N} dy\,\left[(\sin y/y)^2 - 2\sin y/y\right] - v_0 N/4L$$
$$\xrightarrow[N\gg 1]{} -(3Nv_0/8L) - O(1/N)\ . \qquad (23)$$

Therefore,

$$(\Delta E/N) = (E_{SW} - E_{PW})/N \xrightarrow[\substack{N\gg 1\\ L\gg (x_1-x_2)}]{} (\Delta T + \Delta v)/N$$
$$\xrightarrow[N\gg 1]{} -(3v_0/8)(N/L)(1/N) + O(1/N^2) \qquad (24)$$

so that for attractive forces ($v_0 < 0$) the effect of the surface is to *increase* the energy per particle, in accordance with the liquid-drop semi-empirical mass formula[6].

In conclusion, we see that wall boundary conditions, as opposed to periodic boundary conditions, suffice to explain qualitatively the density oscillations, the surface thickness and the positive surface energy of a nucleus, and semi-quantitatively the value of π/k_F of the oscillation wavelength.

REFERENCES

1. L. B. W. Jolley, *Summation of Series* (Dover, 1961) 2nd edition, p. 82, formula 437.
2. W. J. Swiatecki, Proc. Phys. Soc. (Lond.) 64 (1951) 227.
3. M. A. Thorpe & D. J. Thouless, Nuc. Phys. A156 (1970) 225.
4. J. B. Bellicard *et al*, Phys. Rev. Lett. 19 (1967) 527, (erratum in 20 (68) 977).
5. J. Heisenberg *et al*, Phys. Rev. Lett. 23 (1969) 1402.
6. R. R. Roy & B. P. Nigam, *Nuclear Physics* (Wiley, 1967).
7. L. B. W. Jolley, *loc. cit.*, p. 78, formula 418.
8. L. B. W. Jolley, *loc. cit.*, p. 78, formula 420.

RESUMEN

La densidad de probabilidad de un sistema de N fermiones en una dimensión, se calcula usando condiciones a la frontera de paredes rígidas, como la suma de las N ondas estacionarias de más baja energía que se encuentran de una longitud L. El resultado concuerda cualitativamente tanto con teorías anteriores más complejas, como con experimentos nucleares que revelan oscilaciones en la densidad y que determinan el espesor de la superficie. Se realiza un cálculo simple de la energía superficial y ésta resulta ser del signo correcto.