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RELATIONS BETWEEN THE HYPERSPHERICAL HARMONIC AND THE HARMONIC OSCILLATOR METHODS FOR THE THREE BODY PROBLEM

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ABSTRACT: We give a systematic and explicit procedure for deriving hyperspherical harmonics for the three body problem with given permutational symmetry. The matrix elements of a two body interaction with respect to these hyperspherical harmonics are determined in terms of the corresponding ones for harmonic oscillator states. This allows us to reduce the three body problem to a system of coupled ordinary differential equations for the hyperradial functions.

1. INTRODUCTION

In the last few years there have been many papers dealing with the hyperspherical harmonic approach to the few nucleon problem with particular emphasis on the three body case[1]. In the present work we describe a tech-

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nique that allows us to obtain a system of *orthonormal* spherical harmonics with well defined permutational symmetry for the three body problem. We also discuss a new method for the evaluation of matrix elements of two body interaction potentials with respect to the three body hyperspherical harmonics.

With the help of these matrix elements one can, as is well known[1], reduce the three body problem to a system of coupled ordinary differential equations. For the sake of simplicity we consider only the case of central interactions. Then the wave function can be written as

$$\Psi = \sum_{K\alpha} X_{K\alpha}(\rho)\, Y_{K\alpha}(\Omega) \ , \tag{1.1}$$

where ρ is the hyperradius given in terms of the relative coordinates and Ω is the set of angles on which the hyperspherical harmonic $Y_{K\alpha}$ depends. This last function is characterized by the integer K which gives the irreducible representation (IR) of the $O(6)$ group and a set of "inner" quantum numbers α which include the IR of the $S(3)$ group of permutation and completely define the state.

The equation we want to satisfy is then

$$H_{\text{int}}\, \Psi = E\Psi \ , \tag{1.2}$$

where H_{int} is the intrinsic hamiltonian in which we eliminated the center of mass motion. Substituting (1.1) in (1.2) one can immediately obtain[1] for the three body problem the system of coupled differential equations

$$[\rho^{-5}(d/d\rho)(\rho^5 d/d\rho) - \rho^{-2} K(K+4) - E]\, X_{K\alpha}(\rho)$$

$$= 6 \sum_{K'\alpha'} X_{K'\alpha'}(\rho)\, F^{K'K}_{\alpha'\alpha}(\rho) \ , \tag{1.3}$$

where, denoting by $\mathbf{r}'_s$, $s = 1, 2, 3$ the coordinates of the particles, we have

$$F^{K'K}_{\alpha'\alpha}(\rho) = \int Y^*_{K'\alpha'}(\Omega)\, V(|\mathbf{r}'_1 - \mathbf{r}'_2|)\, Y_{K\alpha}(\Omega)\, d\Omega \tag{1.4}$$

as the matrix element of the two body central interaction V with respect to hyperspherical harmonic functions. We can of course write $r_1' - r_2'$ in terms of ρ and Ω and thus this last matrix element is only a function of ρ.

As we see from Eq. (1.3), the basic ingredient for the determination of the hyperradial functions $X(\rho)$, and thus the wave function Ψ of Eq. (1.1), is the matrix element $F^{K'K}_{\alpha'\alpha}(\rho)$. In section 5 of the present paper we shall use, for the determination of these matrix elements, a method introduced earlier by two of us[2] for the determination of many body matrix elements in the Hartree-Fock approximation. In this approach we can thus make use of powerful techniques developed formerly in connection with harmonic oscillator wave functions[3].

In section 2 we give a derivation of a system of hyperspherical polynomials adequate for the description of the intrinsic motion of the three body system; we use a classification scheme involving the groups

$$O(6) \supset O^{(1)}(3) \oplus O^{(2)}(3)$$

which uniquely defines the functions.

Section 3 contains the derivation of harmonic oscillator states in relative coordinates, with a classification scheme similar to that of section 2, namely, a classification according to the group chain

$$U(6) \supset O(6) \supset O^{(1)}(3) \oplus O^{(2)}(3) \ .$$

We obtain also the transformation coefficient between these oscillator states and the more familiar ones[3] classified by the group chain

$$U(6) \supset U^{(1)}(3) \oplus U^{(2)}(3) \ .$$

In section 4 we describe a method by which, starting from the oscillator functions of section 3, we obtain linear combinations of them having a definite permutational symmetry. The basic step here is the diagonalization of the square of an operator $\mathfrak{M}$ which is a generator of a group $O(2)$, contained in $O(6)$ and which in turn contains a representation of the group $S(3)$ of permutations of the three body problem.

2. HYPERSPHERICAL POLYNOMIALS FOR THE THREE BODY PROBLEM

As usual in the three body problem we introduce the Jacobi coordinates

$$
\begin{aligned}
\mathbf{r}_1 &= 2^{-\frac{1}{2}}(\mathbf{r}_1' - \mathbf{r}_2') \\
\mathbf{r}_2 &= 6^{-\frac{1}{2}}(\mathbf{r}_1' + \mathbf{r}_2' - 2\mathbf{r}_3') \\
\mathbf{r}_3 &= 3^{-\frac{1}{2}}(\mathbf{r}_1' + \mathbf{r}_2' + \mathbf{r}_3') \ ,
\end{aligned}
\tag{2.1}
$$

where $\mathbf{r}_s'$, $s = 1, 2, 3$ are the original coordinates of the particles. We shall only be interested in functions of the relative coordinates $\mathbf{r}_1$, $\mathbf{r}_2$ in terms of which we can define the intrinsic wave function of the three body problem.

As is well known[1], the hyperspherical polynomials in the vectors $\mathbf{r}_1$, $\mathbf{r}_2$ will be homogeneous polynomials of degree K in the components of these vectors which satisfy the Laplace equation in this six dimensional configuration space. Besides, we can require that they are eigenfunctions of the angular momentum in each coordinate, i.e. of the operators $\mathbf{L}^{(1)} \cdot \mathbf{L}^{(1)}$, $\mathbf{L}^{(2)} \cdot \mathbf{L}^{(2)}$ where

$$\mathbf{L}^{(1)} = \mathbf{r}_1 \times \mathbf{p}_1 \ , \quad \mathbf{L}^{(2)} = \mathbf{r}_2 \times \mathbf{p}_2 \ . \tag{2.2}$$

We ask also that the polynomials be eigenfunctions of the total angular momentum L^2 and its 3rd component L_z, where

$$\mathbf{L} = \mathbf{L}^{(1)} + \mathbf{L}^{(2)} . \tag{2.3}$$

We denote then these polynomials by

$$P_{Kl_1l_2LM}(\mathbf{r}_1, \mathbf{r}_2) \ . \tag{2.4}$$

We determine them explicitly in algebraic fashion, in contrast with the analytic technique of Morse and Feshbach and Fabre de la Ripelle [4,5]. We consider first the case when

$$L = M = l_1 + l_2 \ . \tag{2.5}$$

We have then that

$$P_{Kl_1l_2} \equiv P_{K,l_1,l_2,l_1+l_2,l_1+l_2} \tag{2.6}$$

satisfies the following set of equations

$$(\mathbf{r}_1 \cdot \nabla_1 + \mathbf{r}_2 \cdot \nabla_2)\, P_{Kl_1l_2} = K P_{Kl_1l_2} \tag{2.7a}$$

$$(\nabla_1^2 + \nabla_2^2)\, P_{Kl_1l_2} = 0 \tag{2.7b}$$

$$L_+^{(1)} P_{Kl_1l_2} = 0 \ , \qquad L_+^{(2)} P_{Kl_1l_2} = 0 \tag{2.7c, d}$$

$$L_z^{(1)} P_{Kl_1l_2} = l_1 P_{Kl_1l_2} \ , \qquad L_z^{(2)} P_{Kl_1l_2} = l_2 P_{Kl_1l_2} \ , \tag{2.7e, f}$$

where ∇_s, $s = 1, 2$ is the gradient vector with components $(\partial/\partial x_s, \partial/\partial y_s, \partial/\partial z_s)$ in terms of the cartesian components of the Jacobi vectors $\mathbf{r}_s$.

The analysis of Eqs. (2.7) is much simplified if we introduce spherical components for the Jacobi vectors, namely x_{ms}, $m = 1, 0, -1$. In this notation the operators $L_+^{(s)}$ and $L_z^{(s)}$, $s = 1, 2$ become

$$\begin{aligned} L_+^{(s)} &= -\left(x_{1s}(\partial/\partial x_{0s}) + x_{0s}(\partial/\partial x_{-1s})\right), \\ L_z^{(s)} &= \left(x_{1s}(\partial/\partial x_{1s}) - x_{-1s}(\partial/\partial x_{-1s})\right) \ . \end{aligned} \tag{2.8}$$

The polynomial (2.4) can now be written as

$$P_{Kl_1l_2}(r_1,r_2) = x_{12}^K P_{Kl_1l_2}\left(x_{11}/x_{12},\, x_{01}/x_{12},\, x_{-11}/x_{12},\, x_{02}/x_{12},\, x_{-12}/x_{12}\right) . \tag{2.9}$$

We note that

$$r_s^2 = \sum_m (-1)^m x_{ms}x_{-ms} = -2x_{1s}x_{-1s} + x_{0s}^2 \tag{2.10}$$

and thus

$$\begin{aligned} x_{-12}/x_{12} &= \tfrac{1}{2}\left[(-r_2^2/x_{12}^2) + (x_{02}^2/x_{12}^2)\right]; \\ x_{-11}/x_{12} &= \tfrac{1}{2}\left[(-\rho^2/x_{12}^2) + (r_2^2/x_{12}^2) + (x_{01}^2/x_{12}^2)\right] , \end{aligned} \tag{2.11}$$

where

$$\rho^2 = r_1^2 + r_2^2 . \tag{2.12}$$

We see then that x_{-12}/x_{12} could be replaced by r_2^2/x_{12}^2, and x_{-11}/x_{12} by ρ^2/x_{12}^2, and we would still have a polynomial function which we could call P':

$$P_{Kl_1l_2}(r_1,r_2) = x_{12}^K P'\left(x_{11}/x_{12},\, x_{01}/x_{12},\, x_{02}/x_{12},\, r_2^2/x_{12}^2,\, \rho^2/x_{12}^2\right) . \tag{2.13}$$

Applying $L_+^{(1)}$ and $L_+^{(2)}$ to the polynomial (2.13) we immediately see that P' can not be a function of either x_{01}/x_{12} nor x_{02}/x_{12}. Writing it then explicitly and applying $L_z^{(1)}$, $L_z^{(2)}$ we have that

$$\begin{aligned} P_{Kl_1l_2}(r_1,r_2) &= x_{12}^K (x_{11}/x_{12})^{l_1} \sum_\nu C_\nu (\rho^2/x_{12}^2)^{\frac{1}{2}(K-l_1-l_2)-\nu} (r_2^2/x_{12}^2)^\nu \\ &= \eta_{Kl_1l_2} x_{11}^{l_1} x_{12}^{l_2} \sum_\nu C_\nu \rho^{K-l_1-l_2-2\nu} r_2^{2\nu} (l_1!\,l_2!)^{-\frac{1}{2}} \end{aligned} \tag{2.14}$$

Finally, the remaining equation (2.7b) gives a two-term recursion formula for C_ν, whose solution, as shown in Appendix A gives

$$C_\nu = (-1)^\nu \left(\tfrac{1}{2}(K + l_1 + l_2 + 2) + \nu\right)!/\nu!\left(\tfrac{1}{2}(K - l_1 - l_2) - \nu\right)!\,\Gamma(l_2 + 3/2 + \nu) \; . \tag{2.15}$$

The summation in Eq. (2.14) can now be identified with a Jacobi polynomial[4] if desired; the constant $n_{Kl_1l_2}$ plays the role of a normalization factor.

We have thus obtained the harmonic polynomial $P_{Kl_1l_2LM}$ of (2.4) with $L = M = l_1 + l_2$. In order to obtain the polynomial with arbitrary values of L, M, we notice that, without disturbing the part depending on ρ^2 and r_2^2, by application of $L_-^{(s)}$ on Eq. (2.14) we can transform $(x_{1s})^{l_s}(l_s!)^{-\frac{1}{2}}$ into solid spherical harmonics $[4\pi/(2l_s + 1)!!]^{\frac{1}{2}}\, \mathcal{Y}_{l_sm_s}(\mathbf{r}_s)$, $s = 1,2$, which then we can vector-couple to definite values of L and M.

Therefore the general harmonic polynomial will be

$$P_{Kl_1l_2LM}(\mathbf{r}_1, \mathbf{r}_2) = n_{Kl_1l_2}[(2l_1 + 1)!!(2l_2 + 1)!!]^{-\frac{1}{2}} \left[\mathcal{Y}_{l_1}(\mathbf{r}_1)\,\mathcal{Y}_{l_2}(\mathbf{r}_2)\right]_{LM} \times$$

$$\times\; 4\pi \sum_\nu C_\nu\, \rho^{-2\nu + K - l_1 - l_2}\, r_2^{2\nu} \; . \tag{2.16}$$

From the foregoing analysis it is apparent that these harmonic polynomials have a group theoretical classification according to the chain of groups

$$O(6) \supset \begin{pmatrix} O^{(1)}(3) & 0 \\ 0 & O^{(2)}(3) \end{pmatrix} \supset O(3) \supset O(2) \; . \tag{2.17}$$

In the next section we discuss harmonic oscillator functions with a similar classification scheme.

3. THE HARMONIC OSCILLATOR FUNCTIONS OF THE THREE BODY PROBLEM

The intrinsic motion of a system of three identical particles in harmonic oscillator potential can be described by wave functions of the two Jacobi relative vectors $\mathbf{r}_1, \mathbf{r}_2$ of Eq. (2.1). However, when dealing with oscillator systems it is often very convenient to express the wave functions as polynomials in creation operators acting on a ground state[3]. We shall follow the second alternative, and accordingly, let us introduce "relative" creation (η) and annihilation (ξ) operators, defined in terms of the relative coordinates and momenta, as

$$\eta_s = (2)^{-\frac{1}{2}} (\mathbf{r}_s - i\mathbf{p}_s) , \quad \xi_s = (2)^{-\frac{1}{2}} (\mathbf{r}_s + i\mathbf{p}_s) ; \quad s = 1, 2 . \quad (3.1)$$

(We shall use throughout this section a system of units in which $m = \hbar = \omega = 1$). We introduce also a normalized ground state $|0>$, characterized by the properties

$$<0|0> = 1, \quad \xi_{is}|0> = 0 ; \quad i = x, y, z ; \quad s = 1, 2 . \quad (3.2)$$

The operators (3.1) obey the commutation rules

$$[\xi_{is}, \eta_{i's'}] = \delta_{ii'}\delta_{ss'} , \quad [\xi_{is}, \xi_{i's'}] = [\eta_{is}, \eta_{i's'}] = 0 \quad (3.3)$$

and from (3.2) and (3.3) we immediately see that if $P(\eta)$ is a polynomial in creation operators, then

$$\xi_{is} P(\eta)|0> = (\partial P(\eta)/\partial \eta_{is})|0> \text{ for all } i, s . \quad (3.4)$$

The wave function of an oscillator in the relative coordinate $\mathbf{r}_1$ has the well known expression[3]

$$|n_1 l_1 m_1> = N_{l_1 m_1} (\eta_1 \cdot \eta_1)^{n_1} \mathcal{Y}_{l_1 m_1}(\eta_1)|0> \quad (3.5)$$

with

$$N_{l_1 n_1} = (-1)^{n_1} \left[4\pi / (2n_1)!!(2n_1 + 2l_1 + 1)!! \right]^{\frac{1}{2}} . \qquad (3.6)$$

These functions describe an oscillator with a number $2n_1 + l_1$ of quanta of excitation energy, and an angular momentum $l_1(l_1 + 1)$ with a component m_1 along the Z axis. By vector-coupling the state (3.5) with a similar state in the relative vector r_2, we obtain two-oscillator states

$$|n_1 l_1 n_2 l_2 LM> = N_{l_1 n_1} N_{l_2 n_2} (\eta_i \cdot \eta_1)^{n_1} (\eta_2 \cdot \eta_2)^{n_2} [\mathcal{Y}_{l_1}(\eta_1) \mathcal{Y}_{l_2}(\eta_2)]_{LM} |0> \qquad (3.7)$$

which, from the nature of the operators that they diagonalize, are seen to possess a classification according to the chain of groups

$$U(6) \supset \begin{pmatrix} U^{(1)}(3) & 0 \\ 0 & U^{(2)}(3) \end{pmatrix} \supset \begin{pmatrix} O^{(1)}(3) & 0 \\ 0 & O^{(2)}(3) \end{pmatrix} \supset O(3) \supset O(2) . \qquad (3.8)$$

The generators of some of the groups in this chain, are

$$U^{(s)}(3): \quad \eta_{is} \xi_{js} \; ; \quad i,j = x,y,z \; ; \quad s = 1,2 \qquad (3.9a)$$

$$O^{(s)}(3): \quad L_j^{(s)} = -i(\eta_s \times \xi_s)_j \; ; \quad j = x,y,z \; ; \quad s = 1,2 \qquad (3.9b)$$

$$O(3): \quad L_j = L_j^{(1)} + L_j^{(2)} \; ; \quad j = x,y,z \qquad (3.9c)$$

and therefore, the six operators which are diagonal with respect to the states (3.7) are

$$\hat{N}_s = \eta_s \cdot \xi_s \; ; \quad s = 1,2 \qquad (3.10)$$

$$(\mathbf{L}^{(1)}\cdot\mathbf{L}^{(1)})\ ,\ (\mathbf{L}^{(2)}\cdot\mathbf{L}^{(2)})\ ,\ (\mathbf{L}\cdot\mathbf{L})\ ,\ L_z\ . \tag{3.11}$$

Our goal in this paper is the discussion of hyperspherical functions which, as we saw in Section 2, are associated with a group $O(6)$. With this goal in mind we shall study now two-oscillator states with a classification scheme in which we introduce a group $O(6)$ instead of a group $U^{(1)}(3)\oplus U^{(2)}(3)$ of the chain of groups (3.8).

The group $O(6)$ has as generators the operators

$$\Lambda_{js,j's'} = -i\,(\eta_{js}\xi_{j's'} - \eta_{j's'}\xi_{js})\ ;\ \ j,j' = x,y,z\ ;\ \ s,s' = 1,2 \tag{3.12}$$

and its quadratic Casimir operator Λ^2, defined by

$$\Lambda^2 = \tfrac{1}{2}\sum_{jj'}\sum_{ss'}\Lambda_{js,j's'}\Lambda_{j's',js}\ , \tag{3.13}$$

has eigenvalues $K(K+4)$, with K being a non-negative integer. Using the commutation rules of ξ and η, we can rewrite Λ^2 in an equivalent form, which will turn out to be useful later, namely

$$\Lambda^2 = \hat{N}\,(\hat{N}+4) - (\eta_1\cdot\eta_1 + \eta_2\cdot\eta_2)(\xi_1\cdot\xi_1 + \xi_2\cdot\xi_2) \tag{3.14}$$

where $\hat{N}$ is the number operator

$$\hat{N} = \eta_1\cdot\xi_1 + \eta_2\cdot\xi_2\ \ . \tag{3.15}$$

Notice that the eigenvalues N of the operator $\hat{N}$ give the number of quanta of excitation energy of the two-oscillator system.

The new two-oscillator states classified by $O(6)$ will be denoted by $|NKl_1l_2LM>$ and they are eigenfunctions of the operators $\hat{N}$, Λ^2 and the four operators (3.11). In order to find the explicit expression for these states, we shall start by obtaining first the particular state

$$|NKl_1l_2, l_1+l_2, l_1+l_2> \equiv \mathbf{P}(\eta_1,\eta_2)|0>\ . \tag{3.16}$$

This state has highest weight in the groups $O^{(1)}(3)$ and $O^{(2)}(3)$, therefore $\mathsf{P}|0>$ satisfies the equations

$$\hat{N}\mathsf{P}|0> = N\mathsf{P}|0>,$$

$$(\eta_1\cdot\eta_1+\eta_2\cdot\eta_2)(\xi_1\cdot\xi_1+\xi_2\cdot\xi_2)\mathsf{P}|0> = (N-K)(N+K+4)\mathsf{P}|0> \tag{3.17a, b}$$

$$L_+^{(1)}\mathsf{P}|0> = 0,\; L_+^{(2)}\mathsf{P}|0> = 0,\; L_z^{(1)}\mathsf{P}|0> = l_1\mathsf{P}|0>,\; L_z^{(2)}\mathsf{P}|0> = l_2\mathsf{P}|0>. \tag{3.17c,d,e,f}$$

But if we write P as

$$\mathsf{P}(\eta_1,\eta_2) = (\eta_1\cdot\eta_1+\eta_2\cdot\eta_2)^{\frac{1}{2}(N-K)}\,P_{Kl_1l_2}(\eta_1,\eta_2), \tag{3.18}$$

it is easily verified that $P_{Kl_1l_2}|0>$ satisfies the equations

$$\hat{N}P_{Kl_1l_2}|0> = KP_{Kl_1l_2}|0>,\quad (\xi_1\cdot\xi_1+\xi_2\cdot\xi_2)P_{Kl_1l_2}|0> = 0 \tag{3.19a, b}$$

$$L_+^{(1)}P_{Kl_1l_2}|0> = 0,\quad L_+^{(2)}P_{Kl_1l_2}|0> = 0 \tag{3.19c, d}$$

$$L_z^{(1)}P_{Kl_1l_2}|0> = l_1P_{Kl_1l_2}|0>,\quad L_z^{(2)}P_{Kl_1l_2}|0> = l_2P_{Kl_1l_2}|0>. \tag{3.19e, f}$$

If we remember the correspondence $\xi_{is}\to\partial/\partial\eta_{is}$ of Eq. (3.4), we realize that the set of equations (3.19) is identical to the set of equations (2.7) of the last section which determine the harmonic polynomial $P_{Kl_1l_2}(r_1,r_2)$ of that section. Therefore, from Eqs. (3.18), (2.14) and (2.15) we have

$$|NKl_1l_2,l_1+l_2,l_1+l_2> = A^{NK}_{l_1l_2}[l_1!l_2!]^{-\frac{1}{2}}\eta_{11}^{l_1}\eta_{12}^{l_2}\times$$
$$\times\sum_\nu\frac{(-1)^\nu\left(\frac{1}{2}(K+l_1+l_2+2)+\nu\right)!}{\nu!\left(\frac{1}{2}(K-l_1-l_2)-\nu\right)!\,\Gamma(l_2+3/2+\nu)}(\eta_1\cdot\eta_1+\eta_2\cdot\eta_2)^{\frac{1}{2}(N-l_1-l_2)-\nu}(\eta_2\cdot\eta_2)^\nu|0>. \tag{3.20}$$

The normalization coefficient $A^{NK}_{l_1 l_2}$ is found by taking the scalar product of the state on the right hand side of (3.20) with itself; after a somewhat lengthy calculation and making a suitable phase choice, we obtain

$$A^{NK}_{l_1 l_2} = (-1)^{\frac{1}{2}(N-K)} \Gamma\left(\tfrac{1}{2}(K - l_1 + l_2 + 3)\right) \times$$
$$\times \left[\frac{(K+2)\left(\tfrac{1}{2}(K-l_1-l_2)\right)!(2l_2+1)!\left(\tfrac{1}{2}(K-l_1+l_2)\right)!}{2^{N-K}\left(\tfrac{1}{2}(N-K)\right)!\left(\tfrac{1}{2}(N+K+4)\right)!\,l_2!\left(\tfrac{1}{2}(K+l_1+l_2+2)\right)!(K-l_1+l_2+1)!} \times \right.$$
$$\left. \times \frac{\Gamma(l_1 + 3/2)}{\Gamma\left(\tfrac{1}{2}(K+l_1-l_2+3)\right)} \right]^{\frac{1}{2}} \qquad (3.21)$$

As is shown in section 2, the oscillator state with arbitrary values of L, M, i.e., $|NKl_1 l_2 LM>$ is obtained from (3.20) by replacing $[l_1!\, l_2!]^{-\frac{1}{2}} \eta_{11}^{l_1} \eta_{12}^{l_2}$ with the vector-coupled product of solid spherical harmonics

$$4\pi\, [(2l_1+1)!!\ 2l_2+1)!!]^{-\frac{1}{2}}\, [\mathcal{Y}_{l_1}(\eta_1)\, \mathcal{Y}_{l_2}(\eta_2)]_{LM}\,. \qquad (3.22)$$

Since we shall not need this general state, we do not discuss it further.

The determination of the matrix elements of interaction potentials with respect to harmonic oscillator states has been systematized for the case[3] when the states are expressed in the form of Eq. (3.7), i.e., the states $|n_1 l_1 n_2 l_2 LM>$. Since we want to use oscillator states of the type $|NKl_1 l_2 LM>$, it would be desirable to express the latter states in terms of the former. This can be done, provided we have an explicit algebraic formula for the scalar product

$$< n_1' l_1' n_2' l_2' L' M' \,|\, NK l_1 l_2 LM > \qquad (3.23)$$

From general group theoretical properties it is known[6] that this scalar product is diagonal in $l_1 l_2 LM$, and independent of the values LM. Therefore, for its explicit evaluation we can take $L = M = l_1 + l_2$, and thus we need only the states given in Eqs. (3.7) and (3.20). In Appendix B we give the details of the explicit calculation of the scalar product, which leads (denoting n'_s by n_s, $s = 1, 2$) to this result

$$\langle n_1 l_1 n_2 l_2 LM \mid NK l_1 l_2 LM \rangle \equiv \langle n_1 n_2 \mid NK \rangle_{l_1 l_2}$$

$$= (-1)^{n_1} \delta_{N, 2n_1 + l_1 + 2n_2 + l_2} \, 2^{K - n_1 - l_1 - n_2 - l_2} \times$$

$$\times \left[\frac{\left(\tfrac{1}{2}(N-K)\right)! \, n_1! \, n_2! \, (2n_1 + 2l_1 + 1)! \, (2n_2 + 2l_2 + 1)! \left(\tfrac{1}{2}(K - l_1 - l_2)\right)!}{\left(\tfrac{1}{2}(N+K+4)\right)! \, (n_1 + l_1)! \, (n_2 + l_2)! \, (K - l_1 + l_2 + 1)! \, (K + l_1 - l_2 + 1)!} \times \right.$$

$$\left. \times (K+2) \left(\tfrac{1}{2}(K + l_1 + l_2 + 2)\right)! \left(\tfrac{1}{2}(K - l_1 + l_2)\right)! \left(\tfrac{1}{2}(K + l_1 - l_2)\right)! \right]^{\frac{1}{2}} \times$$

$$\times \sum_{s=0} \frac{(-1)^s}{s! \left(\tfrac{1}{2}(N-K) - s\right)! \, (n_1 - s)! \left(\tfrac{1}{2}(K - 2n_1 - l_1 - l_2) + s\right)!}$$

$$\times \frac{\Gamma\left(\tfrac{1}{2}(K - l_1 + l_2 + 3)\right) \Gamma\left(\tfrac{1}{2}(K + l_1 - l_2 + 3)\right)}{\Gamma\left(\tfrac{1}{2}(K - 2n_1 - l_1 + l_2 + 3) + s\right) \Gamma\left(n_1 + l_1 + \tfrac{3}{2} - s\right)} . \qquad (3.24)$$

The Kronecker delta, obviously is the expression of the conservation of energy.

In the particular case when $K = N$, (which is in fact the only case we shall need), the transformation coefficient given above reduces identically to a coefficient formerly determined by Raynal and Revai[7], which as is easily seen, contains no summations:

$$\langle n_1 n_2 \mid KK \rangle_{l_1 l_2} = (-1)^{n_1} \begin{pmatrix} \tfrac{1}{2}(K - l_1 + l_2 + 1) \\ n_1 \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} \tfrac{1}{2}(K + l_1 - l_2 + 1) \\ n_2 \end{pmatrix}^{\frac{1}{2}} \times$$

$$\times \begin{pmatrix} K + 1 \\ \tfrac{1}{2}(K - l_1 - l_2) \end{pmatrix}^{-\frac{1}{2}} \delta_{K, 2n_1 + l_1 + 2n_2 + l_2} \qquad (3.25)$$

4. THREE BODY HYPERSPHERICAL HARMONICS WITH DEFINITE PERMUTATIONAL SYMMETRY

As shown first by Dragt in Ref. 1, (cf. also Ref. 11), for the purpose of analyzing the permutational symmetry of the 6-dimensional spherical harmonics of the three body problem, the most convenient classification scheme is one involving a group chain $\mathbf{O}(2) \supset S(3)$. Dragt introduced a classification according to the chain

$$O(6) \supset \mathbf{O}(2) \times SU(3) \tag{4.1}$$

the groups $\mathbf{O}(2)$ and $SU(3)$ being "complementary" within the IR (KOO) of $O(6)$, in the sense that the IR (μ) of $\mathbf{O}(2)$, $\mu = K, K-2, \ldots, 1$ or 0 determines two conjugate IR of $SU(3)$, namely $[K, \frac{1}{2}(K \pm \mu)]$. (We label an IR of $SU(3)$ by a partition $[p, q]$, $p \geqslant q \geqslant 0$). Rotational symmetry then drive us to introduce the rotation groups $SO(3) \supset SO(2)$ as subgroups of $SU(3)$ in the chain of groups (4.1). But at this point a trouble appears, consisting in the fact that the chain of groups

$$SU(3) \supset SO(3) \supset SO(2) \tag{4.2}$$

does not uniquely define the states of an IR of $SU(3)$. We shall mention two possible ways out of this difficulty. One is to work with a complete but *non-orthogonal* set of basis states which, when necessary, are distinguished among themselves by means of an arbitrary index q. From this non-orthonormal basis we can pass to an orthonormal one by using the standard Schmidt procedure as proposed by Efros[8]. The second alternative is to diagonalize an additional operator, let us call it Ω, independent of, and commuting with the Casimir operators of the groups in the chain (4.2); this alternative leads to orthonormal basis states, though it implies in general the numerical diagonalization of matrices. The two alternatives have been discussed in detail in Ref. 14. Other methods leading to non-orthogonal sets of permutationally adapted $O(6)$ spherical harmonics have been proposed in Ref. 8.

In the present paper we have preferred to introduce the group $\mathbf{O}(2)$ in our basis by numerically diagonalizing its Casimir operator $\mathfrak{M}^2$, whose matrix is constructed with respect to an orthonormal set of $O(6)$ spherical harmonics with good angular momentum, namely the states of Eq. (2.16). Faced with the unavoidable* fact of numerical diagonalization of matrices in order to ob-

* At any rate, it seems so at the present time.

tain an orthonormal basis, we think is far more convenient to diagonalize $\mathfrak{M}^2$ rather than Ω. For convenience we shall do our analysis in terms of harmonic oscillator states and creation operators and then translate the results to hyperspherical harmonics.

The oscillator states we need were obtained in the last section; they are denoted $|KKl_1 l_2 LM>$ and given by Eq. (3.20) with the substitution indicated before Eq. (3.22).

Let us introduce at this point an operator $\mathfrak{M}$ defined as

$$\mathfrak{M} = -i(\eta_1 \cdot \xi_2 - \eta_2 \cdot \xi_1) \tag{4.3}$$

$\mathfrak{M}$ is a generator of $O(6)$ and, being a scalar, commutes with the total orbital angular momentum $\mathbf{L}$. From the theory of angular momentum[9] the matrix elements of $\mathfrak{M}$ with respect to the states $|KKl_1 l_2 LM>$ are given by

$$< KKl_1'l_2'L'M' | \mathfrak{M} | KKl_1 l_2 LM > =$$

$$\delta_{LL'}\,\delta_{MM'}(-1)^{L+l_1+l_2'} \begin{Bmatrix} l_1' & l_1 & 1 \\ l_2' & l_2 & L \end{Bmatrix} < Kl_1'l_2' \| \mathfrak{M} \| Kl_1 l_2 > . \tag{4.4}$$

The last term is essentially a reduced matrix element and can be determined by evaluating directly the left hand side of the previous formula for $L = L' = l_1 + l_2$ and using Eq. (3.20) as well as hermitian conjugation. We find in Appendix B that there are four non-vanishing reduced matrix elements whose values are

$$< K, l_1 - 1, l_2 + 1 \| \mathfrak{M} \| Kl_1 l_2 > = i\,[l_1(l_2+1)(K+1+l_1-l_2)(K+3-l_1+l_2)]^{\frac{1}{2}} , \tag{4.5a}$$

$$< K, l_1 + 1, l_2 - 1 \| \mathfrak{M} \| Kl_1 l_2 > = -i\,[(l_1+1)\,l_2(K+3+l_1-l_2)(K+1-l_1+l_2)]^{\frac{1}{2}} , \tag{4.5b}$$

$$< K, l_1 - 1, l_2 - 1 \| \mathfrak{M} \| Kl_1 l_2 > = -i\,[l_1 l_2(K+2+l_1+l_2)(K+2-l_1-l_2)]^{\frac{1}{2}} , \tag{4.5c}$$

$$< K, l_1+1, l_2+1 \| \mathfrak{m} \| K l_1 l_2 > = i\,[(l_1+1)(l_2+1)(K+4+l_1+l_2)(K-l_1-l_2)]^{\frac{1}{2}} \,. \tag{4.5d}$$

These results agree with formulas obtained in references 8 and 10.

What is the operator $\mathfrak{m}$ useful for? It was shown in reference 11 that $\mathfrak{m}^2$ is the Casimir operator of a group $\mathbf{O}(2)$ which contains as a subgroup the symmetric group of permutations of 3 objects, $S(3)$, in the form of its two-dimensional IR $D^{\{21\}}$. Let us denote by $\mu = 0, 1, 2 \ldots$ the IR label of $\mathbf{O}(2)$, i.e. we are denoting with μ^2 the eigenvalues of $\mathfrak{m}^2$. The standard technique of characters[6] says to us that

$$\begin{aligned} &\text{For } \mu \equiv 1,2 \bmod 3: \ \text{IR}\mu \text{ of } \mathbf{O}(2) \supset \text{Rep. } \{21\} \text{ of } S(3) \\ &\text{For } \mu \equiv 0 \bmod 3: \ \text{IR}\mu \text{ of } \mathbf{O}(2) \supset \text{Rep. } \{3\} \oplus \{1^3\} \text{ of } S(3) \end{aligned} \tag{4.6}$$

When $\mu = 0$ only one symmetric *or* one antisymmetric state occurs; the way to tell which of these two symmetries a state has when $\mu \equiv 0 \bmod 3$ will be explained below, cf. Eqs. (4.11), (4.13).

The method we want to propose for obtaining hyperspherical harmonics with good permutational symmetry, consists in the computer diagonalization of the matrix of the operator $\mathfrak{m}^2$ in the basis of the oscillator states $|KKl_1 l_2 LM>$, i.e. the matrix

$$\| < KKl_1' l_2' LM | \mathfrak{m}^2 | KKl_1 l_2 LM > \| \ , \tag{4.7}$$

where KLM are fixed and l_1, l_2 are restricted by $|l_1 - l_2| \leqslant L \leqslant l_1 + l_2 \leqslant K$ with $K - l_1 - l_2 =$ even. It is seen from Eqs. (4.5) that $\mathfrak{m}$ does not connect values of l_1 with the same parity; thus by an adequate ordering of the rows and columns the matrix of $\mathfrak{m}$ takes a shape like

$$\| \mathfrak{m} \| = \begin{array}{c} \overbrace{}^{l_1 \text{ even}} \ \ \overbrace{}^{l_1 \text{ odd}} \\ \begin{pmatrix} 0 & -i\mathbf{M} \\ i\mathbf{M}^T & 0 \end{pmatrix} \end{array} \begin{array}{l} \} \ l_1 \text{ even} \\ \} \ l_1 \text{ odd} \end{array} \ , \tag{4.8}$$

where $\mathbf{M}$ is a real, in general rectangular, matrix and $\mathbf{M}^T$ is the transpose of $\mathbf{M}$. The matrix of $\mathfrak{M}^2$ is thus seen to have the form

$$\|\mathfrak{M}^2\| = \begin{pmatrix} \overbrace{\mathbf{M}\mathbf{M}^T}^{l_1 \text{ even}} & \overbrace{0}^{l_1 \text{ odd}} \\ 0 & \mathbf{M}^T\mathbf{M} \end{pmatrix} \begin{matrix} \} \; l_1 \text{ even} \\ \} \; l_1 \text{ odd} \end{matrix} \quad , \tag{4.9}$$

where $\mathbf{M}\mathbf{M}^T$ and $\mathbf{M}^T\mathbf{M}$ are square real symmetric matrices.

Diagonalizing the submatrix $\mathbf{M}\mathbf{M}^T$ in (4.9) we obtain a set of orthonormal eigenstates of $\mathfrak{M}^2$, which we shall denote as

$$|KK\mu wLM>_+ = \sum_{l_1 l_2} B^{l_1 l_2}_{\mu w +}(KL)\,|KKl_1 l_2 LM> \quad , \tag{4.10}$$

where w is an arbitrary index to distinguish among a set eigenfunctions having the same quantum numbers K, μ, L, M ; and the index + makes reference to the fact that in the sum on the right hand side only even values of l_1 occur. For fixed K, L, μ, w, the set of coefficients $B^{l_1 l_2}_{\mu w +}(KL)$ for the compatible, l_1, l_2 forms an eigenvector of $\mathbf{M}\mathbf{M}^T$ characterized by the eigenvalue μ^2 and the index w.

The transposition (1, 2) acting on a state $|KKl_1 l_2 LM>$ multiplies it by $(-1)^{l_1}$; then since l_1 is even in Eq. (4.10) we deduce that

$$(1,2)\,|KK\mu wLM>_+ = |KK\mu wLM>_+ \quad , \tag{4.11}$$

i.e. the functions (4.10) are symmetric in the first two particles.

By a reasoning similar to that of the last paragraph, if we diagonalize the submatrix $\mathbf{M}^T\mathbf{M}$ in (4.9) we obtain eigenfunctions

$$|KK\mu wLM>_- = \sum_{l_1 l_2} B^{l_1 l_2}_{\mu w -}(KL)\,|KKl_1 l_2 LM> \;;\; l_1 \text{ odd} \tag{4.12}$$

which are antisymmetric in the first two particles:

$$(1,2)\,|KK\mu wLM>_- = -\,|KK\mu wLM>_- \quad . \tag{4.13}$$

The set of coefficients $B^{l_1 l_2}_{\mu w -}(KL)$ has the same meaning as $B^{l_1 l_2}_{\mu w +}(KL)$ but now associated with $\mathbf{M}^T\mathbf{M}$ rather than $\mathbf{M}\mathbf{M}^T$. We have a computer program for the calculation of the coefficients $B^{l_1 l_2}_{\mu w \pm}(KL)$, with which we have made tables of these coefficients up to $K = 12$.

Using the results given in Eqs. (4.9), (4.11) and (4.13), we can now see that when $\mu \equiv 0$ mod 3 the states $|KK\mu wLM>_+$ have permutational symmetry $\{3\}$, (111), and the states $|KK\mu wLM>_-$ have symmetry $\{1^3\}$, (321), where $(s_3\, s_2\, s_1)$ denotes the Yamanouchi symbol[11]. On the other hand, when $\mu \equiv 1, 2$ mod 3, each state $|KK\mu wLM>_+$ can be considered as belonging to the row (211) of the IR $\{21\}$ of $S(3)$, and its corresponding partner function in the IR is[11]

$$|K\mu wLM(121)> = \sqrt{4/3}\,[(23) + \tfrac{1}{2}]\,|KK\mu wLM>_+ \tag{4.14}$$

But $(23) = (12)(123)$, and by the usual conventions[6] we must apply the operation $(321)(12)$ on the vectors η_1, η_2. Since the state $|KK\mu wLM>_+$ is symmetric in the particles 1 and 2, this amounts to the application on the state of

$$\exp(i\tfrac{2}{3}\pi\mathfrak{M}) = \cos\tfrac{2}{3}\pi\mathfrak{M} + i\,\mathfrak{M}\,(\sin\tfrac{2}{3}\pi\mathfrak{M}/\mathfrak{M}) \tag{4.15}$$

and as $\cos\tfrac{2}{3}\pi\mathfrak{M}$ and $\sin\tfrac{2}{3}\pi\mathfrak{M}/\mathfrak{M}$ are functions of $\mathfrak{M}^2$, they can be replaced by $\cos\tfrac{2}{3}\pi\mu$ and $\sin\tfrac{2}{3}\pi\mu/\mu$, respectively. Therefore, since we are in the case of $\mu \equiv 1, 2$ mod 3, we have $\cos\tfrac{2}{3}\pi\mu + \tfrac{1}{2} = 0$, $\sin\tfrac{2}{3}\pi\mu = \pm\sqrt{\tfrac{3}{4}}$, and the final result is

$$|K\mu wLM(121)> \equiv \pm(i/\mu)\,\mathfrak{M}\,|KK\mu wLM>_+$$

$$= \pm(i/\mu)\sum_{l_1' l_2' l_1 l_2} |KKl_1'l_2'LM><KKl_1'l_2'LM|\,\mathfrak{M}\,|KKl_1\,l_2LM>\,B^{l_1 l_2}_{\mu w +}(KL)\ , \tag{4.16}$$

where the matrix element is given by (4.4), (4.5).

In conclusion we have obtained harmonic oscillator states characterized by the chain of groups

$$U(6) \supset O(6) \supset \begin{pmatrix} O^{(1)}(3) & 0 \\ & \\ 0 & O^{(2)}(3) \end{pmatrix} \supset O(3) \supset O(2) \tag{4.17}$$

$$N \qquad K \qquad l_1 \qquad l_2 \qquad L \qquad M$$

where the integers below each group characterizes its irreducible representation (IR) in the state (3.16). Furthermore in the present section we have determined linear combinations of these states that are characterized by IR of $S(3)$.

We now turn to the problem of obtaining hyperspherical harmonics of definite permutational symmetry. For this purpose we first note that $|KKl_1 l_2 LM>$ can be written as

$$|KKl_1 l_2 LM> = \sqrt{2/(K+2)!}\ \rho^K Y_{Kl_1 l_2 LM}(\Omega) \exp(-\tfrac{1}{2}\rho^2) . \tag{4.18}$$

This can be seen from the fact that the harmonic oscillator hamiltonian when expressed in terms of hyperspherical variables becomes

$$H_{osc} = \tfrac{1}{2}[-\rho^{-5}(\partial/\partial\rho)(\rho^5\partial/\partial\rho) + \rho^{-2}\Lambda^2 + \rho^2] , \tag{4.19}$$

where Λ^2 is the Casimir operator of $O(6)$ given by (3.13). The state (4.18) corresponds to the eigenvalue $K+3$ of H_{osc} and thus if we apply (4.19) to (4.18) we immediately obtain that

$$\Lambda^2 Y_{Kl_1 l_2 LM}(\Omega) = K(K+4)\, Y_{Kl_1 l_2 LM}(\Omega) , \tag{4.20}$$

which implies that it is an hyperspherical harmonic. Note that the operators Λ^2, $\mathbf{L}^{(1)} \cdot \mathbf{L}^{(1)}$, $\mathbf{L}^{(2)} \cdot \mathbf{L}^{(2)}$, L^2, L_z of section 3 are all given in terms of $\Lambda_{js,j's'}$ of (3.12) and that the latter, from the definition (3.1) of creation and annihilation operators, can also be written as

$$\Lambda_{js,j's'} = -i\,(r_{js}\,\partial/\partial r_{j's'} - r_{j's'}\,\partial/\partial r_{js})\,;\; j,j' = x,y,z;\; s=1,2\,, \tag{4.21}$$

where r_{js} are the components of the two Jacobi vectors. Clearly the generators $\Lambda_{js,j's'}$ of $O(6)$ are only functions of the angles in Ω and their derivatives and thus the hyperspherical harmonics are also eigenfunction of $\boldsymbol{L}^{(1)}\cdot\boldsymbol{L}^{(1)}$, $\boldsymbol{L}^{(2)}\cdot\boldsymbol{L}^{(2)}$, L^2, L_z, which is the reason for the indices that characterize them. The factor $[2/(K+2)!]^{1/2}$ is put in so as to guarantee the normalization of the $Y_{Kl_1l_2LM}(\Omega)$.

From the developments (4.10), (4.13) we now see that the hyperspherical harmonic with definite permutational symmetry is given by

$$\overset{\pm}{Y}_{K\mu wLM}(\Omega) \equiv \sum_{l_1l_2} B^{l_1l_2}_{\mu w\pm}(KL)\, Y_{Kl_1l_2LM}(\Omega)\;. \tag{4.22}$$

This was the state that we designated as $Y_{K\alpha}(\Omega)$ in the introduction in which α stands now for $\mu, w, \pm, L, M$.

In the next section we shall discuss the matrix element of the two body interaction with respect to the hyperspherical harmonics (4.22)

5. MATRIX ELEMENTS OF A TWO BODY INTERACTION POTENTIAL WITH RESPECT TO THE HYPERSPHERICAL HARMONICS OF THE THREE BODY PROBLEM

We are interested in the matrix elements of the two body interaction

$$V(|r_1' - r_2'|) = V(\sqrt{2}\,r_1) = \mho(\sqrt{2}\rho, \Omega) \tag{5.1}$$

with respect to the hyperspherical harmonics (4.22). For simplicity we shall only consider central forces as the extension to other types is trivial. In (5.1) we designate by r_1', r_2' the coordinates of the first two particles, by r_1, the first Jacobi coordinate (2.1) and by script $\mho$ the central potential in terms of hyperspherical coordinates. From (4.22) it is clear that our matrix element will be a linear combination of expressions of the form

$$F^{K'K}_{l'_1 l'_2 L, l_1 l_2 L}(\rho) = \int Y^{*}_{K' l'_1 l'_2 LM}(\Omega)\, \mho(\sqrt{2}\rho, \Omega)\, Y_{K l_1 l_2 LM}(\Omega)\, d\Omega \ , \tag{5.2}$$

where, because of the central nature of the forces, the matrix element does not depend on M and is diagonal in L.

We shall now proceed to evaluate (5.2) with the help of harmonic oscillator functions in the chain (4.17). For this purpose we take $\hbar = m = 1$ as previously, but leave the frequency ω in the wave function. In that case the harmonic oscillator state we will be interested in, is

$$|KKl_1 l_2 LM>_{\omega} = \sqrt{2/(K+2)!}\ \omega^{\frac{1}{2}(K+3)} \rho^{K}\, Y_{Kl_1 l_2 LM}(\Omega) \exp(-\tfrac{1}{2}\omega\rho^2) \ , \tag{5.3}$$

where ρ, Ω have the same meaning as before, and we put the frequency ω in the ket as an index. The expression (5.3) follows immediately from (4.18).

We shall now consider the following matrix element

$$f^{K'K}_{l'_1 l'_2 L, l_1 l_2 L}(\omega) \equiv {}_{\omega}<K'K'l'_1 l'_2 LM \,|\, V(\sqrt{2}\, r_1) \,|\, KKl_1 l_2 LM>_{\omega} =$$

$$\int_0^{\infty} d\rho^2 \exp(-\omega\rho^2) \left[[(K+2)!(K'+2)!]^{-\frac{1}{2}} \omega^{3+\frac{1}{2}(K+K')} \rho^{4+K+K'} F^{K'K}_{l'_1 l'_2 L, l_1 l_2 L}(\rho) \right] \tag{5.4}$$

which clearly is the Laplace transform with respect to the variable ρ^2 of the expression in the square bracket. Using then the inverse of this transform we obtain

$$F^{K'K}_{l'_1 l'_2 L, l_1 l_2 L}(\rho) = (1/2\pi i) \int_{c-i\infty}^{c+i\infty} [(K+2)!(K'+2)!]^{\frac{1}{2}}\, \omega^{-3-\frac{1}{2}(K+K')}\, \rho^{-(K+K'+4)} \times$$

$$\times f^{K'K}_{l'_1 l'_2 L, l_1 l_2 L}(\omega) \exp(\omega\rho^2)\, d\omega \ . \tag{5.5}$$

We have thus derived the matrix elements with respect to hyperspherical harmonics of a two body interaction as an inverse Laplace transform of corresponding matrix elements associated with the harmonic oscillator states in the chain (4.17). Using the coefficients (3.25) of section 3 in the case $N = K$, which were originally obtained by Raynal and Revai[7], we can write

$$f^{K'K}_{l_1'l_2'L,\, l_1 l_2 L}(\omega) =$$

$$= \Big[\sum_{n_1 n_2 n_1'} \langle K'K' | n_1' n_2 \rangle_{l_1 l_2} \; {}_{\omega}\langle n_1' l_1 \| V(\sqrt{2} r_1) \| n_1 l_1 \rangle_{\omega} \langle n_1 n_2 | KK \rangle_{l_1 l_2} \Big] \delta_{l_1 l_1'} \delta_{l_2 l_2'} \tag{5.6}$$

where the matrix element of $V(\sqrt{2} r_1)$ with respect to the states in the

$$U(6) \supset \begin{pmatrix} U^{(1)}(3) & 0 \\ 0 & U^{(2)}(3) \end{pmatrix} \supset \begin{pmatrix} O^{(1)}(3) & 0 \\ 0 & O^{(2)}(3) \end{pmatrix} \supset O(3) \supset O(2) \tag{5.7}$$

chain reduces to the one body matrix element

$${}_{\omega}\langle n_1' l_1 \| V(\sqrt{2} r_1) \| n_1 l_1 \rangle_{\omega} = \int_0^\infty R_{n_1' l_1}(\omega, r_1)\, V(\sqrt{2} r_1)\, R_{n_1 l_1}(\omega, r_1)\, r_1^2 dr_1$$

$$= \sum_p B(n_1' l_1, n_1 l_1, p)\, [2\omega^{p+3/2} / \Gamma(p + 3/2)] \int_0^\infty r^{2p+2}\, V(\sqrt{2} r) \exp(-\omega r^2)\, dr\,, \tag{5.8}$$

where $B(n_1' l_1, n_1 l_1, p)$ are coefficients tabulated by Brody and Moshinsky[12].

Introducing (5.8) into (5.6), and the latter into (5.5), and interchanging the order of integration one obtains

$$F^{K'K}_{l_1'l_2'L,\,l_1l_2L}(\rho) = 2\sum_{n_1n_1'n_2}\left[(K+2)!(K'+2)!\right]^{\frac{1}{2}} < KK'\,|\,n_1'n_2>_{l_1l_2} < n_1n_2\,|\,KK>_{l_1l_2} \times$$

$$\times\ \rho^{-(K+K'+4)}\sum_p \frac{B(n_1'l_1, n_1l_1, p)}{\Gamma(p+3/2)\Gamma\left(\frac{1}{2}(K+K'+3)-p\right)} \times$$

$$\times\int_0^\rho r^{2p+2}(\rho^2-r^2)^{\frac{1}{2}(K+K'+1)-p}\, V(\sqrt{2}r)\, dr\ . \qquad (5.9)$$

It is obvious but nevertheless important to note that the polynomial $\rho^K Y_{Kl_1l_2LM}(\Omega)$ is homogeneous of degree K, and thus under reflection

$$\mathbf{r}_1 \to -\mathbf{r}_1\ ,\quad \mathbf{r}_2 \to -\mathbf{r}_2 \qquad (5.10)$$

the polynomial suffers a change of sign $(-1)^K$. Thus the parity of the hyperspherical harmonic $Y_{Kl_1l_2LM}(\Omega)$ is $(-1)^K$. As the central potential is invariant under reflection we conclude that the matrix element (5.15) will vanish unless $K+K'$ is even.

6. CONCLUSION

We have presented a systematic and explicit procedure for deriving the matrix elements of a two body potential with respect to the hyperspherical harmonics of the three body problem with given permutational symmetry. Thus now we can write out explicitly the system of coupled differential equation (1.3) in which α is replaced by $\mu, w, \pm, L, M$ as indicated in section 4.

These sets of equations could be solved both in relation to the bound state of a three body problem such as tritium, as well as for a scattering state that would appear for example in the collisions of neutrons and deuterons. Calculations of these types have been done by several authors[1] and we plan to carry them out also with the procedure outlined in this paper.

APPENDIX A

In section 2 we saw that the polynomial (2.14), namely

$$P_{Kl_1 l_2}(r_1, r_2) = x_{11}^{l_1} x_{12}^{l_2} \sum_\nu C_\nu \rho^{K-l_1-l_2-2\nu} r_2^{2\nu} \qquad (A.1)$$

satisfies all equations (2.7) with the single exception of the Eq.(2.7b); i.e. $P_{Kl_1 l_2}$ given above is not as yet a solution of

$$(\nabla_1^2 + \nabla_2^2)\, P_{Kl_1 l_2} = 0 \ . \qquad (A.2)$$

We shall prove in this appendix that enforcing condition (A.2) on the polynomial (A.1) gives a recursion formula for C_ν, whose solution we shall obtain.

By straightforward application of the operator

$$(\nabla_1^2 + \nabla_2^2) \equiv \sum_{m=1}^{-1} (-1)^m \big((\partial^2/\partial x_{m1}\, \partial x_{-m1}) + (\partial^2/\partial x_{m2}\, \partial x_{-m2}) \big) \qquad (A.3)$$

on the polynomial (A.1) we obtain

$$(\nabla_1^2 + \nabla_2^2)\, P_{Kl_1 l_2}$$

$$= - x_{11}^{l_1} x_{12}^{l_2} \ \{ \sum_\nu C_\nu (K - l_1 - l_2 - 2\nu)(K + l_1 + l_2 + 4 + 2\nu)\, \rho^{K-l_1-l_2-2\nu-2}\, r_2^{2\nu} \ +$$

$$+ \sum_{\nu'} C_{\nu'} (2\nu')(2l_2 + 1 + 2\nu')\, \rho^{K-l_1-l_2-2\nu'}\, r_2^{2\nu'-2} \} \ . \qquad (A.4)$$

If in the second sum we introduce as new dummy index $\nu \equiv \nu' - 1$, the curly bracket in (A.4) becomes

$$\sum_{\nu}[C_{\nu}(K-l_1-l_2-2\nu)(K+l_1+l_2+4+2\nu)+C_{\nu+1}(2\nu+2)(2l_2+3+2\nu)]\times$$

$$\times\rho^{K-l_1-l_2-2\nu-2}r_2^{2\nu} \tag{A.5}$$

and the condition that this be equal to zero, leads to the recursion formula

$$C_{\nu+1}=-C_{\nu}[\tfrac{1}{2}(K-l_1-l_2)-\nu][\tfrac{1}{2}(K+l_1+l_2+4)+\nu]/(\nu+1)(l_2+\tfrac{3}{2}+\nu) \tag{A.6}$$

for the coefficients C_{ν}. The solution of this formula, found by induction, is given in Eq. (2.15) of section 2.

APPENDIX B

The purpose of this appendix is to obtain the expansion of the harmonic oscillator state (3.20), which is classified by $O(6)$, in terms of oscillator states of the type (3.5) which are classified by $U^{(1)}(3)\oplus U^{(2)}(3)$.

Let us write again the $O(6)$ state of Eq. (3.20):

$$|NKl_1l_2,l_1+l_2,l_1+l_2> = A^{NK}_{l_1l_2}[l_1!\,l_2!]^{-\frac{1}{2}}\eta_{11}^{l_1}\eta_{12}^{l_2}\times$$

$$\times\sum_{\nu}C_{\nu}(\eta_1\cdot\eta_1+\eta_2\cdot\eta_2)^{\frac{1}{2}(N-l_1-l_2)-\nu}(\eta_2\cdot\eta_2)^{\nu}|0>, \tag{B.1}$$

where $A^{NK}_{l_1l_2}$ is given in Eq. (3.21) and C_{ν} in Eq. (2.15).

Expanding the binomial in (B.1) and grouping terms we obtain

$$|NKl_1l_2,l_1+l_2,l_1+l_2> = A^{NK}_{l_1l_2}[l_1!\,l_2!]^{-\frac{1}{2}}\sum_{\nu\lambda}C_{\nu}\begin{pmatrix}\frac{1}{2}(N-l_1-l_2)-\nu\\ \lambda\end{pmatrix}\times$$

$$\times\eta_{11}^{l_1}(\eta_1\cdot\eta_1)^{\frac{1}{2}(N-l_1-l_2)-\nu-\lambda}\eta_{12}^{l_2}(\eta_2\cdot\eta_2)^{\nu+\lambda}|0>. \tag{B.2}$$

The product of creation operators on $|0>$ which appears in Eq. (B.2) is, up to a normalization factor, the oscillator state

$$|\tfrac{1}{2}(N-l_1-l_2)-\nu-\lambda, l_1, \nu+\lambda, l_2, l_1+l_2, l_1+l_2> \tag{B.3}$$

of Eq. (3.5). Supplying the appropriate normalization factor, we have from (B.2),

$$|NKl_1 l_2, l_1+l_2, l_1+l_2>$$

$$= A^{NK}_{l_1 l_2}[l_1!\,l_2!]^{-\frac{1}{2}}(-1)^{\frac{1}{2}(N-l_1-l_2)}(4\pi)^{-1}\sum_{\nu\lambda}C_\nu \begin{pmatrix}\frac{1}{2}(N-l_1-l_2)-\nu\\ \lambda\end{pmatrix} \times$$

$$\times[(N-l_1-l_2-2\nu-2\lambda)!!(2\nu+2\lambda)!!(N+l_1-l_2+1-2\nu-2\lambda)!!(2l_2+1+2\nu+2\lambda)!!]^{\frac{1}{2}}\times$$

$$\times|\tfrac{1}{2}(N-l_1-l_2)-\nu-\lambda, l_1, \nu+\lambda, l_2, l_1+l_2, l_1+l_2> . \tag{B.4}$$

Introducing the explicit value of C_ν, given in Eq. (2.15), we obtain from (B.4)

$$<n_1 l_1 n_2 l_2, l_1+l_2, l_1+l_2|NKl_1 l_2, l_1+l_2, l_1+l_2> \equiv <n_1 n_2|NK>_{l_1 l_2}$$

$$=\delta_{2n_1+l_1+2n_2+l_2,N}\, A^{NK}_{l_1 l_2}(-1)^{\frac{1}{2}(N-l_1-l_2)}[l_1!\,l_2!]^{-\frac{1}{2}}[4\pi(\tfrac{1}{2}(N-l_1-l_2)-n_2)!]^{-1}\times$$

$$\times\;[(2n_1)!!(2n_2)!!(2n_1+2l_1+1)!!(2n_2+2l_2+1)!!]^{\frac{1}{2}}\,S\;, \tag{B.5}$$

where

$$S=\sum_\nu\frac{(-1)^\nu(\frac{1}{2}(K+l_1+l_2+2)+\nu)!\,(\frac{1}{2}(N-l_1-l_2)-\nu)!}{\nu!\,(\frac{1}{2}(K-l_1-l_2)-\nu)!\,(n_2-\nu)!\,\Gamma(l+3/2+\nu)} . \tag{B.6}$$

This formula for S has an unsymmetrical form. By following steps analogous to those described by Racah[13] in his symmetrization of Wigner coefficients of $SU(2)$, we can get at an alternative symmetric expression for S, namely

$$S = (-1)^{\frac{1}{2}(K-l_1-l_2)-n_1} \left(\tfrac{1}{2}(N-K)\right)!\ n_1!\left(\tfrac{1}{2}(K+l_1+l_2+2)\right)!\,\Gamma\left(\tfrac{1}{2}(K+l_1-l_2+3)\right)\times$$

$$\times \sum_s (-1)^s \left[s!\left(\tfrac{1}{2}(N-K)-s\right)!(n_1-s)!\left(\tfrac{1}{2}(K-l_1-l_2)-n_1+s\right)! \times\right.$$

$$\left.\times\,\Gamma\left(\tfrac{1}{2}(K-l_1+l_2+3)-n_1+s\right)\Gamma(n_1+l_1+3/2-s)\right]^{-1} . \qquad \text{(B.7)}$$

Introducing this S on (B.5) and placing the explicit value of $A^{NK}_{l_1 l_2}$ we obtain the expression given in section 3, Eq. (3.24). We want to mention that from the explicit formula (3.24) it is easy to verify that the coefficient has the symmetry property

$$\langle n_1 n_2 | NK \rangle_{l_1 l_2} = (-1)^{\frac{1}{2}(K-l_1-l_2)} \langle n_2 n_1 | NK \rangle_{l_2 l_1} . \qquad \text{(B.8)}$$

APPENDIX C

In this appendix we shall describe the basic steps that lead to the determination of the matrix elements of the operator

$$\mathfrak{M} = -i\,(\eta_1 \cdot \xi_2 - \eta_2 \cdot \xi_1) \qquad \text{(C.1)}$$

with respect to the harmonic oscillator states $|KKl_1 l_2 LM\rangle$ classified by $O(6)$. According to the formula (4.4) we only have to determine the reduced matrix elements, which are independent of LM and therefore, some of them at least, could be determined if we were able to evaluate the matrix element

$$\langle KKl_1' l_2', l_1+l_2, l_1+l_2 | \mathfrak{M} | KKl_1 l_2, l_1+l_2, l_1+l_2 \rangle . \qquad \text{(C.2)}$$

For the calculation of (C.2) we make use of the correspondence $\xi_{is} \to \partial/\partial\eta_{is}$ of Eq. (3.4) and apply $\mathfrak{M}$ on the ket as given in Eq. (3.20). Using recoupling techniques, as well as some simple properties of solid spherical harmonics, we find after an elementary but lengthy computation, that

$$\mathfrak{M} \,|\, KKl_1 l_2, l_1 + l_2, l_1 + l_2 >$$

$$= i \left[\frac{(l_1 + l_2 + 1)(2l_1 + 2l_2 + 3)(K - l_1 - l_2)(K + 4 + l_1 + l_2)}{(2l_1 + 1)(2l_1 + 3)(2l_2 + 1)(2l_2 + 3)} \right]^{\frac{1}{2}} \times$$

$$\times \,|\, KK, l_1 + 1, l_2 + 1, l_1 + l_2, l_1 + l_2 > +$$

$$+ i \,[\, l_1 (l_2 + 1)(K - l_1 + l_2 + 3)(K + l_1 - l_2 + 1)/(2l_1 + 1)(2l_2 + 3)]^{\frac{1}{2}} \times$$

$$\times \,|\, KK, l_1 - 1, l_2 + 1, l_1 + l_2, l_1 + l_2 > -$$

$$- i \,[\, l_2 (l_1 + 1)(K - l_1 + l_2 + 1)(K + l_1 - l_2 + 3)/(2l_2 + 1)(2l_1 + 3)]^{\frac{1}{2}} \times$$

$$\times \,|\, KK, l_1 + 1, l_2 - 1, l_1 + l_2, l_1 + l_2 > \qquad \text{(C.3)}$$

This formula, in combination with Eq. (4.4), permits to us the determination of all reduced matrix elements, except for the three cases

$$< K, l_1 - 1, l_2 - 1 \,\|\, \mathfrak{M} \,\|\, Kl_1 l_2 >, \; < K, l_1 - 1, l_2 \,\|\, \mathfrak{M} \,\|\, Kl_1 l_2 >, \; < K, l_1, l_2 - 1 \,\|\, \mathfrak{M} \,\|\, Kl_1 l_2 >.$$

But then, these three cases can be deduced from the previously known cases by hermitian conjugation. Therefore we have the possibility of evaluating all the reduced matrix elements. The calculation, once done, gives the four non-vanishing reduced matrix elements of Eq. (4.5a, b, c, d).

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RESUMEN

Se da un procedimiento sistemático y explícito para obtener armónicos hiperesféricos para el problema de tres cuerpos con una simetría permutacional dada. Los elementos de matriz de una interacción de dos cuerpos con respecto a estos armónicos hiperesféricos, se determinan en términos de los elementos de matriz correspondientes para estados del oscilador armónico. Esto nos permite reducir el problema de tres cuerpos a un sistema de ecuaciones diferenciales ordinarias acopladas para las funciones hiperradiales.