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AZIMUTHAL QUANTIZATION OF ANGULAR MOMENTUM*

Jean-Marc Lévy-Leblond**

Instituto de Física, Universidad Nacional de México

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ABSTRACT: Angular momentum in quantum mechanics is usually studied by diagonalizing its zenithal projection J_0. An alternate scheme is developed here by defining an operator Υ corresponding to the orientation of the azimuthal plane around the polar axis. In the $(2j+1)$-dimensional eigenspaces of $\boldsymbol{J}^2$, the eigenvalues of the azimuthal direction are regularly distributed with a spacing $2\pi/(2j+1)$. The corresponding eigenstates of Υ, when expanded upon those of J_0, show a great analogy with the coherent states of the quantum harmonic oscillator.

I. INTRODUCTION

The angular momentum operator in quantum mechanics has noncommuting components. Its properties are usually investigated by diagonal-

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** Permanent address: Laboratoire de Physique Theorique
Université Paris VII, Place Jussieu
75005-Paris, France.

izing one component, J_0, along with the squared norm J^2. The well-known result that J^2 has eigenvalues $j(j+1)$ ($2j$ integer) and J_0 eigenvalues m ranging by integers from $-j$ to $+j$, is referred to as a "spatial quantization" of the angular momentum; this therminology, sometimes given a (dangerously misleading) pictorial illustration, corresponds to the idea of considering a polar parametrization of the angular momentum vector:

$$\left.\begin{aligned} J_0 &= \text{“}J\cos\Theta\text{”} \\ J_\pm &= \text{“}J\sin\Theta\exp(\pm i\Phi)\text{”}, \end{aligned}\right\} \tag{1}$$

where Θ and Φ would be the azimuthal and zenithal angle respectively. One then speaks of the "quantization of the zenithal angle", $\cos\Theta$ being supposed to take on the discrete values $m/\sqrt{j(j+1)}$. Of course, this is a very loose way of speaking, metaphorical at best, since neither Θ, nor Φ above, have been given a precise mathematical definition as operators – which in particular would certainly not commute. This is why (1) has been written with quotation marks. Now these considerations may suggest an alternative scheme to the ordinary one: instead of emphasizing the zenital properties of J, that is, diagonalizing J_0 or "$\cos\Theta$", why not consider its azimuthal properties, that is, the "Φ" of (1)? I will show that such a program is feasible by giving first a precise meaning to "$\exp(i\Phi)$" and then diagonalizing the corresponding operator. The rather elegant and natural results which emerge may find some useful applications.

II. DEFINITION AND PROPERTIES OF THE AZIMUTHAL OPERATOR

If an operator such as Φ in (1) exists as a *bona fide* hermitian operator, the corresponding exponential $\Upsilon = \exp(i\Phi)$ is a unitary operator. From now on, we will focus on this unitary operator Υ and refer to it as the "azimuthal operator". Insofar as one deals essentially with finite-dimensional vector spaces (eigenspaces of J^2 for instance), there is complete equivalence between the study of $\Upsilon = \exp(i\Phi)$ and $\Phi = -i\,\mathrm{Log}\,\Upsilon$ (defined by the converging series $-i\sum_{n=0}^{\infty}\frac{1}{n}(\Upsilon - I)^n$).

It is the, here appropriately called, "polar decomposition" of any operator as a product of a hermitian operator by a unitary one, which gives the key to the polar decomposition in the geometrical sense, needed here. We thus define

$$J_+ = J_T \Upsilon , \tag{2a}$$

with Υ unitary and J_T hermitian, where the notation is supposed to emphasize the meaning of J_T as a "transverse" part of J. Taking the adjoint of this equation, we obtain

$$J_- = \Upsilon^{-1} J_T . \tag{2b}$$

By multiplying (2a) and (2b), we see that the transverse operator J_T must fulfill the condition

$$J_T^2 = J_+ J_- . \tag{3}$$

Using standard properties of the operators $J_\pm$ and J_0 (derived directly from their commutation relations), this may be written

$$J_T^2 = \boldsymbol{J}^2 - J_0^2 + J_0 . \tag{4}$$

Since $J_T^2 + J_0^2 \neq \boldsymbol{J}^2$, we see how dangerous such expressions as J_T = "$J \sin \Theta$" and J_0 = "$J \cos \Theta$", based on classical analogy, can be.

To emphasize this point, clearly linked to the non-commutativity of Υ and J_T, we may define *another* transverse component of $\boldsymbol{J}$ according to

$$J_\perp = \Upsilon^{-1} J_T \Upsilon . \tag{5}$$

Corresponding to (2a,b), we now have

$$J_+ = \Upsilon J_\perp \tag{6a}$$

$$J_- = J_\perp \Upsilon^{-1} , \tag{6b}$$

so that

$$J_\perp^2 = J_- J_+ \tag{7}$$

or

$$J_\perp^2 = J^2 - J_0^2 - J_0 \ . \tag{8}$$

In the usual basis $\{|jm>\}$ where both J^2 and J_0 are diagonalized, J_T^2 and $J_\perp^2$ also are, with matrix elements $[j(j+1) - m^2 + m]$ and $[j(j+1) - m^2 - m]$ respectively. We will take the square roots with a conventional choice of phases such that the matrix elements of J_T and $J_\perp$ now read:

$$<jm|J_T|jn> = \delta_{mn}\sqrt{(j+m)(j-m+1)}\ , \tag{9}$$

and

$$<jm|J_\perp|jn> = \delta_{mn}\sqrt{(j-m)(j+m+1)}\ . \tag{10}$$

Observe that the two transverse operators have the same spectrum (just change m into $-m$), which is necessary after (5), but they are *not* equal, and do not commute with Υ.

Another interesting property comes from considering the commutation relation of J_0 and Υ. Starting from the standard commutation rule

$$[J_0, J_\pm] = \pm J_\pm\ , \tag{11}$$

using expressions (5) and (6), and remembering that, after (4) and (8) (or (9) and (10)),

$$[J_0, J_T] = [J_0, J_\perp] = 0\ , \tag{12}$$

we obtain the following relationships:

$$J_T([J_0, \Upsilon] - \Upsilon) = 0\ , \tag{13a}$$

$$([J_0, \Upsilon] - \Upsilon) J_\perp = 0\ . \tag{13b}$$

However, it is not true that

$$[J_0, \Upsilon] = \Upsilon \qquad \text{(false)} \qquad (14)$$

since this would lead to

$$\Upsilon^{-1} J_0 \, \Upsilon = J_0 + \Upsilon \,, \qquad \text{(false)} \qquad (15)$$

implying that the spectra of J_0 and of $\Upsilon^{-1} J_0 \, \Upsilon$, which must be the same, are shifted by one unit; this is impossible within a finite space (the eigenspaces of J^2). Yet, (14) would have been a desirable property, corresponding to a canonical commutation rule between J_0 (the generator of rotation around the axis) and Φ (the operator of angular localization)

$$[J_0, \Phi] = -iI \,. \qquad \text{(false)} \qquad (16)$$

The difficulty here is similar to the one of defining a "canonical" phase operator for the harmonic oscillator quantum problem, and is due to the spectrum boundedness (one-sided for the oscillator, two-sided here) of the relevant operators.

In fact J_T and $J_\perp^*$ are *singular* operators, without inverse, as (9) and (10) show clearly since they both have one null eigenvalue ($m = -j$ for J_T, $m = +j$ for $J_\perp$), so that one cannot go from (13) to (14). It is clear however that $\Gamma = [J_0, \Upsilon] - \Upsilon$, being annihilated from the left by J_T and from the right by $J_\perp$, is of rank one for a given j, with all matrix elements zero in the $\{|jm>\}$ basis except for the $<j\,-j\,|\Gamma|\,j\,+j>$ one. In a sense, then, (14) is "almost true".

III. MATRIX ELEMENTS AND DIAGONALIZATION OF THE AZIMUTHAL OPERATOR

We operate within a given eigenspace of J^2, characterized by the total angular momentum number j. In the standard basis of the eigenvectors $|jm>$ of J^2 and J_0, we know the matrix elements*

* For non-hermitian operators, the Dirac notation is ambiguous. The convention, which I prefer to make explicit, is that $<u|A|v>$ is the scalar product of $|u>$ with $A|v>$.

$$< jm \,|\, J_+ \,|\, jn > \; = \delta_{m,n+1} \sqrt{(j-n)(j+n+1)} \;, \tag{17a}$$

$$< jm \,|\, J_- \,|\, jn > \; = \delta_{m,n-1} \sqrt{(j+n)(j-n+1)} \;. \tag{17b}$$

Taking now the matrix element of both sides of the defining relation (2a) and using the expressions (9) and (17a), we obtain at once

$$\sqrt{(j+m)(j-m+1)} \; [< jm \,|\, \Upsilon \,|\, jn > - \, \delta_{m,n+1}] = 0 \;, \tag{18}$$

so that $< jm \,|\, \Upsilon \,|\, jn > \; = \delta_{m,n+1}$ *except for* $m = -j$. Further, the same calculation starting from (6a) gives

$$\sqrt{(j-n)(j+n+1)} \; [< jm \,|\, \Upsilon \,|\, jn > - \, \delta_{m,n+1}] = 0 \;. \tag{19}$$

Finally

$$< jm \,|\, \Upsilon \,|\, jn > \; = \delta_{m,n+1} \;, \qquad \textit{for } m \neq -j \textit{ and } n \neq +j \;. \tag{20}$$

The matrix representing Υ in the basis $\{ |\, jm > \}$ thus has only one unknown element. The condition that the matrix be unitary requires it to be of modulus unity. We finally have

$$^{J_0}\{\Upsilon\} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & & & & 1 \\ e^{i\varphi} & 0 & \cdots & \cdots & 0 \end{bmatrix} , \qquad ^{J_0}\{\Upsilon^{-1}\} = \begin{bmatrix} 0 & 0 & \cdots & 0 & e^{-i\varphi} \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{21}$$

where the notation $^{J_0}\{\Upsilon\}$ emphasizes that this is the matrix representative of Υ in the basis where J_0 is diagonal. The parameter φ is completely arbitrary as is easily seen from the fact that it enters through the dyadic $|j-j><j+j|$ which is annihilated from the left by J_T in (2a) (see (9) with $m=-j$) and from the right by $J_\perp$ in (6a) (see (10) with $m=+j$).

In particular we may verify easily that

$$\Gamma = [J_0, \Upsilon] - \Upsilon = -(2j+1)\, e^{i\varphi} |j-j><j+j| \tag{22}$$

as required by (13) and our subsequent discussion.

It is almost trivial to compute the characteristic polynomial of Υ:

$$\mathrm{Det}\,(\Upsilon - \eta I) = (-1)^{2j+1}(\eta^{2j+1} - e^{i\varphi}) \tag{23}$$

The eigenvalues of Υ thus, are the root of the unity of order $(2j+1)$, displaced by an angle $\varphi/(2j+1)$. Let us from now on take the arbitrary phase φ equal to zero, for the sake of simplicity; however, one should keep in mind its arbitrariness, which could provide a useful flexibility in possible applications. We may also mention here that a different choice of phases in the definition of J_T and $J_\perp$ (allowing for some minus signs in (9) and (10)) could only shift the eigenvalues of Υ by $\pi/2$. Defining

$$\omega = \exp\,[2\pi i/(2j+1)]\ , \tag{24}$$

we write the eigenvalues of Υ as

$$\eta_\kappa = \omega^\kappa\ , \quad (\kappa = +j, j-1, \ldots, -j)\ , \tag{25}$$

with a rather natural ordering. Correspondingly the eigenvalues of the azimuthal operator angle Φ may be written

$$\alpha_\kappa = 2\pi\kappa/(2j+1), \quad (\kappa = j, j-1, \ldots, -j)\ . \tag{26}$$

In classical geometrical (hence metaphorical) terms, this may be interpreted as a regular quantization in the orientation of the azimuthal plane of $\boldsymbol{J}$. The existence of Υ being given, such a result was foreseeable; there are not many ways to distribute $(2j+1)$ points on the unit circle in a rotation symmetrical way!

The eigenvectors of Υ are computed very easily and we choose their phase so that they write

$$|j\,\kappa> = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{+j} \omega^{\kappa m}\,|jm> . \tag{27}$$

The analogy is striking with the coherent states of the quantum harmonic oscillator problem. However, the finite-dimensionality of the eigenspaces of J^2 imply that these "angular coherent states" form an ordinary discrete (finite) complete basis, and not a continuous overcomplete one. Also these states are eigenstates of Υ, and not of the lowering or raising operators $J_{\pm}$, for neither one of these can be diagonalized, still due to the finite-dimensionality. It is easy to derive the transformation properties of these azimuthal states under a rotation by an angle θ around the polar axis. The corresponding unitary operator $\exp(iJ_0\theta)$ has the matrix elements:

$$<j\kappa|\exp(iJ_0\theta)|j\lambda> = \frac{1}{2j+1} \sum_{m=-j}^{+j} \omega^{(\lambda-\kappa)m}\, e^{im\theta} , \tag{28}$$

that is

$$<j\,\kappa|\exp(iJ_0\theta)|j\lambda> = \frac{1}{2j+1} \frac{\sin(2j+1)\left(\frac{\lambda-\kappa}{2j+1}\pi + \frac{1}{2}\theta\right)}{\sin\left(\frac{\lambda-\kappa}{2j+1}\pi + \frac{1}{2}\theta\right)} . \tag{29}$$

As could have been foreseen, a rotation by an angle $\mu\,\frac{2\pi}{2j+1}$ (μ integer), transforms the eigenvector $|j\,\kappa>$ of Υ in the eigenvector $|j\,\kappa+\mu>$. We may now compute the matrix representative of J_0 itself in our new basis. This can be done by direct calculation of the matrix elements

$$<j\,\kappa|J_0|j\,\lambda> = \frac{1}{2j+1} \sum_{m=-j}^{+j} m\,\omega^{(\lambda-\kappa)m} , \tag{30}$$

or by derivation with respect to θ from the matrix (29) of a finite rotation. The result is

$$\left.\begin{aligned} \langle j\,\kappa \,|\, J_0 \,|\, j\lambda\rangle &= \frac{i}{2}\,\frac{(-1)^{\kappa-\lambda}}{\sin\left(\frac{\kappa-\lambda}{2j+1}\pi\right)} \qquad (\kappa \neq \lambda) \\ \langle j\,\kappa \,|\, J_0 \,|\, j\,\kappa\rangle &= 0 \qquad \text{(diagonal elements)} \end{aligned}\right\} \qquad (31)$$

Further developments of this scheme, including a study of the "orbital" eigenstates of Υ, that is realizations as functions on the sphere (linear combinations (27) of spherical harmonics), as well as applications to specific physical problems, are in progress.

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RESUMEN

El momento angular en mecánica cuántica se estudia generalmente diagonalizando su proyección cenital J_0. Aquí se desarrolla una forma alternativa definiendo un operador Υ que corresponde a la orientación del plano azimutal alrededor del eje polar. En los eigenespacios de $(2j+1)$ dimensiones de $\boldsymbol{J}^2$, los eigenvalores de la dirección azimutal están distribuídos regularmente con espaciamientos de $2\pi/(2j+1)$. El eigenestado correspondiente de Υ, cuando se desarrolla en términos de los eigenestados de J_0, muestra una gran analogía con los estados coherentes del oscilador armónico cuántico.