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SOME APPLICATIONS OF CANONICAL TRANSFORMATIONS IN BARGMANN HILBERT SPACE

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ABSTRACT: The relation between the Hilbert space H of quantum mechanics and the Hilbert space F of entire analytic functions introduced by Bargmann is used to derive unitary projective representations of the group $SL(2, R)$ of linear canonical transformations. These representations are applied to an analysis of various nuclear models.

1. INTRODUCTION

The first part of this paper deals with linear canonical transformations in a Hilbert space F_m of entire analytic functions introduced by Bargmann[1]. Properties of this space, its relation to the Hilbert space H_m of quantum mechanics and to coherent states are summarized in sections 2 and 3. Unitary projective representations of linear canonical transformations in H_m have been derived by Moshinsky and Quesne[2]. In section 4 we construct these representations for the group $SL(2, R)$ of linear canonical transformations

in the spaces F_1 and F_3. The results are used in section 5 to derive generating functions for these representations in an oscillator basis.

In the second part we sketch some models of nuclear structure with a view on applications of canonical transformations. In sections 6 and 7 we derive a unified viewpoint on the two-center and the oscillator cluster model and discuss the ground state of ^{6}Li along these lines. In section 8 we generalize these models and derive the corresponding exchange integrals in terms of complex canonical transformations. These considerations may be regarded as a first step towards an investigation of the cluster model in Bargmann Hilbert space.

2. BARGMANN HILBERT SPACE F_m

On the complex Euclidean space C_m with points $\mathbf{z} = (z_1 z_2 \dots z_m)$ and the scalar product

$$a \cdot b = \sum_{j=1}^{m} a_j b_j , \tag{2.1}$$

we consider entire analytic functions

$$f(\mathbf{z}) = f(z_1 z_2 \dots z_m) . \tag{2.2}$$

Bargmann[1] introduced the Hermitean scalar product of two such functions f_1, f_2 by

$$(f_1 | f_2) = \int \overline{f_1(\mathbf{z})}\, f_2(\mathbf{z})\, d\mu(\mathbf{z}) \tag{2.3}$$

where the measure is defined by

$$d\mu(\mathbf{z}) = d\mu(\mathbf{z}) = \pi^{-m} \exp\{-\bar{\mathbf{z}}\mathbf{z}\} \prod_{j=1}^{m} d\mathrm{Re}(z_j)\, d\mathrm{Im}(z_j) . \tag{2.4}$$

The integration extends over C_m and the bar always denotes complex conjugation. The Bargmann Hilbert space F_m consists of the entire analytic functions fulfilling $(f|f) < \infty$. The elements of the Hilbert space H_m of

quantum mechanics are the complex functions $\psi(\mathbf{x})$ defined on the real Euclidean space R_m with points $\mathbf{x} = (x_1\, x_2 \ldots x_m)$, and the Hermitean scalar product

$$<\psi_1|\psi_2> = \int \overline{\psi_1(\mathbf{x})}\, \psi_2(\mathbf{x})\, d\mathbf{x} \ , \tag{2.5}$$

$$d\mathbf{x} = \prod_{j=1}^{m} dx_j \ , \tag{2.6}$$

which fulfill $<\psi|\psi><\infty$. Bargmann[1] defined a mapping between the Hilbert spaces F_m and H_m by the integral transform

$$f(\mathbf{z}) = \int A(\mathbf{z}, \mathbf{x})\, \psi(\mathbf{x})\, d\mathbf{x} \ , \tag{2.7}$$

$$\psi(\mathbf{x}) = \int \overline{A(\mathbf{z}, \mathbf{x})}\, f(\mathbf{z})\, d\mu(\mathbf{z}) \ , \tag{2.8}$$

$$A(\mathbf{z}, \mathbf{x}) = A_m(\mathbf{z}, \mathbf{x}) = \pi^{-\frac{1}{4}m} \exp\{-\tfrac{1}{2}(\mathbf{xx}) - \tfrac{1}{2}(\mathbf{zz}) + \sqrt{2}(\mathbf{xz})\} \ . \tag{2.9}$$

The mapping is isometric,

$$<\psi_1|\psi_2> = (f_1|f_2) \ , \tag{2.10}$$

which may be summarized by the equations

$$\int A(\mathbf{z}', \mathbf{x})\, \overline{A(\mathbf{z}, \mathbf{x})}\, d\mathbf{x} = \exp\{\mathbf{z}' \cdot \overline{\mathbf{z}}\} = (\mathbf{z}'|\mathbf{z}) \ , \tag{2.11}$$

$$\int \overline{A(\mathbf{z}, \mathbf{x}')}\, A(\mathbf{z}, \mathbf{x})\, d\mu(\mathbf{z}) = \delta(\mathbf{x}' - \mathbf{x}) = \prod_{j=1}^{m} \delta(x_j' - x_j) \ . \tag{2.12}$$

Eq. (2.11) defines the reproducing kernel of F_m with the property

$$\int \exp\{\mathbf{a} \cdot \overline{\mathbf{z}}\}\, f(\mathbf{z})\, d\mu(\mathbf{z}) = f(\mathbf{a}) \ . \tag{2.13}$$

Eq. (2.12) is valid only on taking an appropriate limit[1].

3. HARMONIC OSCILLATOR AND COHERENT STATE

The kernel $\overline{A(z, x)}$ of the integral transform eqs. (2.7), (2.8) is an element of H_m because of eq. (2.11). Moreover it is an eigenstate of the annihilation operator

$$\xi_j = (\eta_j)^+ = \sqrt{\tfrac{1}{2}}(x_j + ip_j) \tag{3.1}$$

since

$$\xi_j \overline{A(z, x)} = \overline{A(z, x)}\, \bar{z}_j \tag{3.2}$$

and hence is a coherent state[3]. It is easy to expand the states $\overline{A(z,x)}$ in terms of oscillator states $|N>$. For $m = 1$ one obtains

$$\overline{A(z, x)} = \sum_{N=0}^{\infty} <x|N> \frac{\bar{z}^N}{\sqrt{N!}} = \sum_{N=0}^{\infty} \frac{(\bar{z}\eta)^N}{N!} |0>$$

$$= \exp\{\bar{z}\eta\}\,|0> . \tag{3.3}$$

For $m = 3$, we find in an angular momentum basis for the coefficients in the expansion

$$\overline{A(z, x)} = \sum_{NLM} <x|NLM> \overline{A(z, NLM)} , \tag{3.4}$$

the recursion relation

$$\sum_{N'L'M'} <NLM|\xi_j|N'L'M'> \overline{A(z, N'L'M')} = \bar{z}_j \overline{A(z, NLM)} . \tag{3.5}$$

On complex conjugation this becomes precisely the recursion relation for the homogeneous polynomials $P(\eta)$ which create the state $|NLM>$ from the ground state. The solution of this recursion relation as given by Brody and Moshinsky[4] yields

$$A(\mathbf{z}, NLM) = P^N_{LM}(\mathbf{z}) = A_{NL}(\mathbf{z}\cdot\bar{\mathbf{z}})^{\frac{1}{2}(N-L)} \psi_{LM}(\mathbf{z}) \quad , \tag{3.6}$$

$$A_{NL} = (-)^{\frac{1}{2}(N-L)} \left[\frac{4\pi}{(N+L+1)!!\,(N-L)!!}\right]^{\frac{1}{2}} . \tag{3.7}$$

That the result is correctly normalized may be checked by use of the relation

$$\sum_{LM} P^N_{LM}(\mathbf{z}')\,\overline{P^N_{LM}(\mathbf{z})} = \frac{(\mathbf{z}'\bar{\mathbf{z}})^N}{N!} \tag{3.8}$$

which can be proved for example by interpreting the polynomials P^N_{LM} as matrix elements of irreducible representations of the group $U(3)$. In Dirac notation, eqs. (3.3) and (3.6) may be interpreted as

$$f_N(z) = (z\,|N) = \frac{z^N}{N!} \quad , \tag{3.9}$$

$$f_{NLM}(\mathbf{z}) = (\mathbf{z}\,|NLM) = P^N_{LM}(\mathbf{z}) \quad . \tag{3.10}$$

4. LINEAR CANONICAL TRANSFORMATIONS IN F_m

Moshinsky and Quesne[2] have constructed the unitary operators which correspond to linear canonical transformations. We shall in particular consider the linear canonical transformations for $m = 1$ and $m = 3$ related to the group $SL(2, R)$. If g is an element of this group, the corresponding operators U in the x-representation are given by

$$g \to U(g) \tag{4.1}$$

$$g = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad da - bc = 1 \tag{4.2}$$

$$\langle x' \,|\, U(g) \,|\, x \rangle = C(g) \exp\{-(i/2b)(a(x'x') + d(xx) - 2(x'x))\} \tag{4.3}$$

and form a projective unitary representation of $SL(2, R)$, that is,

$$U(g_1)\,U(g_2) = F(g_1, g_2, g_1 g_2)\,U(g_1 g_2)\,. \tag{4.4}$$

We now construct these operators in F_m by defining

$$(z' \,|\, U(g) \,|\, z) = \iint A(z', x') \langle x' \,|\, U(g) \,|\, x \rangle \, \overline{A(z, x)}\, dx'dx \tag{4.5}$$

The result of the integration may be formally described as follows. If $A(z', x')$ and $\overline{A(z, x)}$ are regarded as complex canonical transformations, the corresponding matrix g_0 is given in both cases by

$$g_0 = \begin{pmatrix} \sqrt{(1/2)} & -i\sqrt{(1/2)} \\ -i\sqrt{(1/2)} & \sqrt{(1/2)} \end{pmatrix} \tag{4.6}$$

according to eqs. (4.2), (4.3) and (2.9). Applying formally eq. (4.4) to eq. (4.5) one finds

$$g_0 \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} g_0 = \begin{pmatrix} r & -is \\ -is^* & -r^* \end{pmatrix}$$

$$= \begin{pmatrix} \tfrac{1}{2}[(d-a) + i(b+c)] & -i\tfrac{1}{2}[(a+d) + i(c-b)] \\ -i\tfrac{1}{2}[(a+d) - i(c-b)] & -\tfrac{1}{2}[(d-a) - i(b+c)] \end{pmatrix}$$

$$ss^* - rr^* = 1\,. \tag{4.7}$$

The explicit integration yields in agreement with this formal manipulation

$$(\mathbf{z}'\,|\,U(g)\,|\,\mathbf{z}) = E(g)\exp\{\frac{1}{2s}\,(r^*(\mathbf{z}'\mathbf{z}') - r(\bar{\mathbf{z}}\,\bar{\mathbf{z}}) + 2(\mathbf{z}'\bar{\mathbf{z}}))\}\,, \quad (4.8)$$

a result that has been given already by Bargmann[1]. We calculate the product of two operators $U(g_1)$, $U(g_2)$ and find

$$\int(\mathbf{z}'\,|\,U(g_1)\,|\,\mathbf{z}'')(\mathbf{z}''\,|\,U(g_2)\,|\,\mathbf{z})\,d\mu(\mathbf{z}'')$$

$$= E(g_1)E(g_2)F'(g_1, g_2, g_1 g_2)E^{-1}(g_1 g_2)(\mathbf{z}'\,|\,U(g_1 g_2)\,|\,\mathbf{z}) \quad (4.9)$$

The quantities $rr^* ss^*$ characterizing $U(g_1 g_2)$ are obtained by writing

$$g_0(g_1 g_2)g_0 = (g_0 g_1 g_0)\,g_0^{-2}(g_0 g_2 g_0)\,,$$

$$\begin{pmatrix} r & -is \\ -is^* & -r^* \end{pmatrix} = \begin{pmatrix} r_1 & -is_1 \\ -is_1^* & -r_1^* \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} r_2 & -is_2 \\ -is_2^* & -r_2^* \end{pmatrix}$$

$$= \begin{pmatrix} r_1 s_2^* + s_1 r_2 & -i(r_1 r_2^* + s_1 s_2) \\ -i(s_1^* s_2^* + r_1^* r_2) & -(s_1^* r_2^* + r_1^* s_2) \end{pmatrix} \quad (4.10)$$

and the factor F' has the form

$$F'(g_1, g_2, g_1 g_2) = \left[\frac{|s_1|\,|s_2|}{|s|}\right]^{m/2}\exp\{-i\,\frac{m}{2}\,\varphi\left(\frac{s}{s_1 s_2}\right)\} \quad (4.11)$$

where $\varphi(t)$ is the phase of the complex number t and $-\pi < \varphi(t) \leqslant \pi$. Eq. (4.9) is valid if

$$\mathrm{Re}\left(\frac{s}{s_1 s_2}\right) = \mathrm{Re}\left(\frac{r_1 r_2^* + s_1 s_2}{s_1 s_2}\right) \geqslant 0 \quad (4.12)$$

The factor eq. (4.11) suggests the choice

$$E(g) = |s|^{-m/2} \exp\{-i \frac{m}{2} \varphi(s)\} \tag{4.13}$$

which leads to the relations

$$(z' | U(e) | z) = (z' | z) = \exp\{z' \bar{z}\} \quad \text{(identity element)} \tag{4.14}$$

$$\overline{(z' | U(g) | z)} = (z | U(g^{-1}) | z') \quad \text{(unitarity)} \tag{4.15}$$

$$U(g_1)U(g_2) = \exp\{i \frac{m}{2} (\varphi(s) - \varphi(s_1) - \varphi(s_2) + \varphi(\frac{s}{s_1 s_2})\} U(g_1 g_2)$$

$$= \pm U(g_1 g_2) \quad \text{(real phase factor)} \tag{4.16}$$

These relations are valid for real numbers $abcd$ and hence for $s^* = \bar{s}$, $r^* = \bar{r}$. In this case the inequality (4.12) is always fulfilled. For complex numbers $abcd$ with $da - bc = 1$, that is to say for complex canonical transformations corresponding to elements of $SL(2, C)$, the operator eq. (4.8) with the numbers $ss^* rr^*$ defined by eq. (4.7) *may* still be meaningful in F_m. The unitarity eq. (4.15) is of course no longer valid and the multiplication of two such operators is possible only if eq. (4.12) is fulfilled.

5. GENERATING FUNCTIONS FOR MATRIX ELEMENTS

We shall demonstrate some uses of the results of section 4 by deriving the matrix elements for $U(g)$ between oscillator states. For $m = 1$ we write

$$(v | U(g) | w) = \sum_{N'N} (v | N') (N' | U(g) | N) \overline{(w | N)}$$

$$= \sum_{N'N} \frac{v^{N'}}{\sqrt{N'!}} (N' | U(g) | N) \frac{\bar{w}^N}{\sqrt{N!}}. \tag{5.1}$$

This is a generating function for the elements $(N' \,|U(g)|\, N)$. By expanding the exponential of eq. (4.8) in the form

$$\exp\left\{\frac{1}{2s}\,(r^*(vv) - r(\bar{w}\bar{w}) + 2(v\bar{w}))\right\}$$

$$= \sum_{q_1 q_2 q} \frac{1}{q_1!\, q_2!\, q!} \left[\left(\frac{r^*}{2s}\right)^{q_1} (vv)^{q_1} \left(-\frac{r}{2s}\right)^{q_2} (\bar{w}\bar{w})^{q_2} \left(\frac{1}{s}\right)^{q} (v\bar{w})^{q}\right] \tag{5.2}$$

and noting that $N' - q = 2q_1$ even, $N - q = 2q_2$ even, we obtain the only non-vanishing contributions for $N' + N$ even,

$$(N' \,|U(g)|\, N)$$

$$= |s|^{-\frac{1}{2}} \exp\{-i\tfrac{1}{2}\varphi(s)\}\, [N'!N!]^{\frac{1}{2}} \left(\frac{1}{s}\right)^{\frac{1}{2}(N'+N)} \times$$

$$\times \sum_{\substack{q \geqslant 0 \\ N-q \text{ even}}} [(\tfrac{1}{2}(N'-q))!(\tfrac{1}{2}(N-q))!\, q!]^{-1} \left(\frac{r^*}{2}\right)^{\frac{1}{2}(N'-q)} \left(-\frac{r}{2}\right)^{\frac{1}{2}(N-q)} \tag{5.3}$$

in complete agreement with Moshinsky and Quesne[2]. For $m = 3$ we find in the same way the generating function

$$(\mathbf{v} \,|U(g)|\, \mathbf{w})$$

$$= \sum_{N'L'M'} \sum_{NLM} P^{N'}_{L'M'}(\mathbf{v})(N'L'M' \,|U(g)|\, NLM)\, \overline{P^{N}_{LM}(\mathbf{w})}. \tag{5.4}$$

To evaluate the matrix element we use eq. (5.2) for $m = 3$, eq. (3.8) in the form

$$\frac{1}{q!}(\mathbf{v}\bar{\mathbf{w}})^q = \sum_{LM} P^q_{LM}(\mathbf{v})\, \overline{P^q_{LM}(\mathbf{w})} \tag{5.5}$$

and relations of the type

$$(\nu\nu)^{q_1} P^{q}_{LM}(\nu) = A^{-1}_{2q_1+qL} A_{qL} P^{2q_1+q}_{LM}(\nu) \tag{5.6}$$

to obtain the restriction $N'+N$ even and the final result

$$\begin{aligned}
&(N'L'M' \,|\, U(g) \,|\, NLM) \\
&= \delta_{L'L}\delta_{M'M} |s|^{-3/2} \exp\{-i\tfrac{3}{2}\varphi(s)\} A^{-1}_{N'L} A^{-1}_{NL} \left(\frac{1}{s}\right)^{\frac{1}{2}(N'+N)} \\
&\sum_{\substack{q \geq 0 \\ N'-q \text{ even}}} [(\tfrac{1}{2}(N'-q))!\,(\tfrac{1}{2}(N-q))!\,]^{-1} A^2_{qL} \left(\frac{r^*}{2}\right)^{\frac{1}{2}(N'-q)} \left(-\frac{r}{2}\right)^{\frac{1}{2}(N-q)}
\end{aligned} \tag{5.7}$$

We believe that the use of F_m may lead to other useful generating functions for linear canonical transformations.

6. TWO-CENTER AND CLUSTER MODEL STATES

We shall show in the next three sections that linear canonical transformations provide a very helpful tool for the discussion and analysis of nuclear models. A simple two-center ansatz ψ_A for states of a nucleus like ^{6}Li is obtained by using 1s-oscillator orbits φ_a and φ_b centered at a given distance. The state ψ_A may be written as

$$\psi_A = \prod_{j=1}^{n_a} \varphi_a(j) \prod_{p=n_a+1}^{n_a+n_b} \varphi_b(p) \ , \tag{6.1}$$

$$\varphi_a(j) = \pi^{-\frac{3}{4}} \exp\{-\tfrac{1}{2}(x^j - [\frac{n_b}{nn_a}]^{\frac{1}{2}} q)^2\} \ , \tag{6.2}$$

$$\varphi_b(p) = \pi^{-\frac{3}{4}} \exp\{-\tfrac{1}{2}(x^p + [\frac{n_a}{nn_b}]^{\frac{1}{2}} q)^2\} \ . \tag{6.3}$$

The distance vector between the two centers is

$$d = \left[\frac{n}{n_a n_b}\right]^{1/2} q \tag{6.4}$$

and all distances are measured in units of the oscillator parameter

$$b = \left[\frac{\hbar}{m\omega}\right]^{1/2} . \tag{6.5}$$

Alternatively we consider the cluster model ansatz

$$\psi_B = \psi_a \psi_b \chi_{ab} \quad \exp\{-\tfrac{1}{2} R^2\} \tag{6.6}$$

In eq. (6.6) the internal states ψ_a and ψ_b of the two clusters a, b may be assumed in the simplest form as

$$\psi_a = \pi^{-\frac{3}{4}(n_a - 1)} \exp\{-\tfrac{1}{2} \sum_{j=1}^{n_a} (x^j - \frac{1}{n_a} R^a)^2\} \tag{6.7}$$

$$\psi_b = \pi^{-\frac{3}{4}(n_b - 1)} \exp\{-\tfrac{1}{2} \sum_{p=n_a+1}^{n_a+n_b} (x^p - \frac{1}{n_b} R^b)^2\} \tag{6.8}$$

while χ_{ab} describes the relative motion. The vectors R^a, R^b, x, R are defined by

$$R^a = \sum_{j=1}^{n_a} x^j , \quad R^b = \sum_{p=n_a+1}^{n_a+n_b} x^p , \tag{6.9}$$

$$x = \left[\frac{n_a n_b}{n}\right]^{1/2} (\frac{1}{n_a} R^a - \frac{1}{n_b} R^b) , \tag{6.10}$$

$$R = \left[\frac{1}{n}\right]^{1/2} (R^a + R^b) . \tag{6.11}$$

In the cluster model ansatz one would set up a variational principle for χ_{ab}. If χ_{ab} is expanded in terms of harmonic oscillator states we obtain the oscillator cluster model studied elsewhere[5]. We shall make here a special choice of χ_{ab} and show that it leads to a simple relation of the models A and B. Consider the state

$$\chi_{ab}(x) \to \chi_{\underline{z}}(x) = \overline{A_3(z,x)} = \pi^{-\frac{3}{4}} \exp\{-\tfrac{1}{2}x^2 - \tfrac{1}{2}\bar{z}^2 + \sqrt{2}\,x\bar{z}\}$$

$$= \pi^{-\frac{3}{4}} \exp\{-\tfrac{1}{2}(x-\sqrt{2}\bar{z})^2 + \tfrac{1}{2}\bar{z}^2\}\,. \tag{6.12}$$

Upon inserting $\chi_{\underline{z}}(x)$ into eq. (6.6) for ψ_b one may rearrange the quadratic expressions according to

$$\sum_{j=1}^{n_a}\left(x^j - \frac{1}{n_a}R^a\right)^2 + \sum_{p=n_a+1}^{n_a+n_b}\left(x^p - \frac{1}{n_b}R^b\right)^2 + R^2 + (x-\sqrt{2}\bar{z})^2 + \bar{z}^2$$

$$= \sum_{j=1}^{n_a}\left(x^j - \left[\frac{2n_b}{nn_a}\right]^{\frac{1}{2}}\bar{z}\right)^2 + \sum_{p=n_a+1}^{n_a+n_b}\left(x^p + \left[\frac{2n_a}{nn_b}\right]^{\frac{1}{2}}\bar{z}\right)^2 + \bar{z}^2\,. \tag{6.13}$$

We conclude that with the replacement of eq. (6.12) the state ψ_B reduces to

$$\psi_B \to \psi_A \exp\{\tfrac{1}{2}\bar{z}^2\}\,, \tag{6.14}$$

where $\sqrt{2}\bar{z} = q$ is now a complex vector. Its interpretation is obtained from the expectation values[3]

$$\langle \chi_{\underline{z}} | x_j | \chi_{\underline{z}} \rangle = \sqrt{\tfrac{1}{2}}(\bar{z}_j + z_j)\langle \chi_{\underline{z}} | \chi_{\underline{z}} \rangle\,, \tag{6.15}$$

$$\langle \chi_{\underline{z}} | p_j | \chi_{\underline{z}} \rangle = \sqrt{\tfrac{1}{2}}\,\frac{1}{i}(\bar{z}_j - z_j)\langle \chi_{\underline{z}} | \chi_{\underline{z}} \rangle\,, \tag{6.16}$$

$$\langle \chi_{\underline{z}} | (xx) | \chi_{\underline{z}} \rangle = [\tfrac{1}{2}(\bar{z}+z)(\bar{z}+z) + \tfrac{3}{2}]\langle \chi_{\underline{z}} | \chi_{\underline{z}} \rangle\,, \tag{6.17}$$

$$\langle \chi_{\bar{z}} | (pp) | \chi_{\bar{z}} \rangle = [-\tfrac{1}{2}(\bar{z} - z)(\bar{z} - z) + \tfrac{3}{2}] \langle \chi_{\bar{z}} | \chi_{\bar{z}} \rangle . \qquad (6.18)$$

In physical terms $\chi_{\bar{z}}$ describes a minimal Gaussian wave packet with the average position and momentum given by the real and imaginary part of $\bar{z}$ respectively. The choice of a real vector $\bar{z}$ implies zero average momentum and minimal kinetic energy.

The coherent state eq. (6.12) and hence the two-center state ψ_A contain states of different parity and angular momentum. We make use of the results given in section 3 to develop it in terms of oscillator states,

$$\chi_{\bar{z}}(x) = \overline{A_3(z, x)} = \sum_{NLM} \langle x | NLM \rangle \overline{P^N_{LM}(z)} . \qquad (6.19)$$

It is now very easy to project from the two-center model states of angular momentum L for the relative motion. The application of a corresponding Young operator gives

$$c^L_{MK} \chi_{\bar{z}}(x) = \sum_{N \geqslant L} \langle x | NLM \rangle \overline{P^N_{LK}(z)} . \qquad (6.20)$$

7. ENERGY CALCULATIONS AND COMPARISON OF MODELS

In this section we shall discuss the ground state of ^{6}Li as a very simple example for the comparison of the models mentioned before. In all cases we shall choose the orbital partition $f = \{42\}$ and an interaction due to Brink and Boeker[6]. Except in case (A) we choose $n_a = 4$, $n_b = 2$.
(A) Two-center model: For ^{6}Li this model has been used by Deenen[7]. He calculated the energy $E_A(z)$ with a superposition of two states of the type given in eqs. (6.1-6.3) as a function of the distance $d = |\boldsymbol{d}|$. His results are shown in Fig. 1 and may be interpreted as follows: The normalized version of eq. (6.12) is

$$\chi_{\bar{z}}(x) \exp\{-\tfrac{1}{2}\bar{z}z\} = \sum_{NLM} \langle x | NLM \rangle \overline{P^N_{LM}(z)} \exp\{-\tfrac{1}{2}\bar{z}z\} . \qquad (7.1)$$

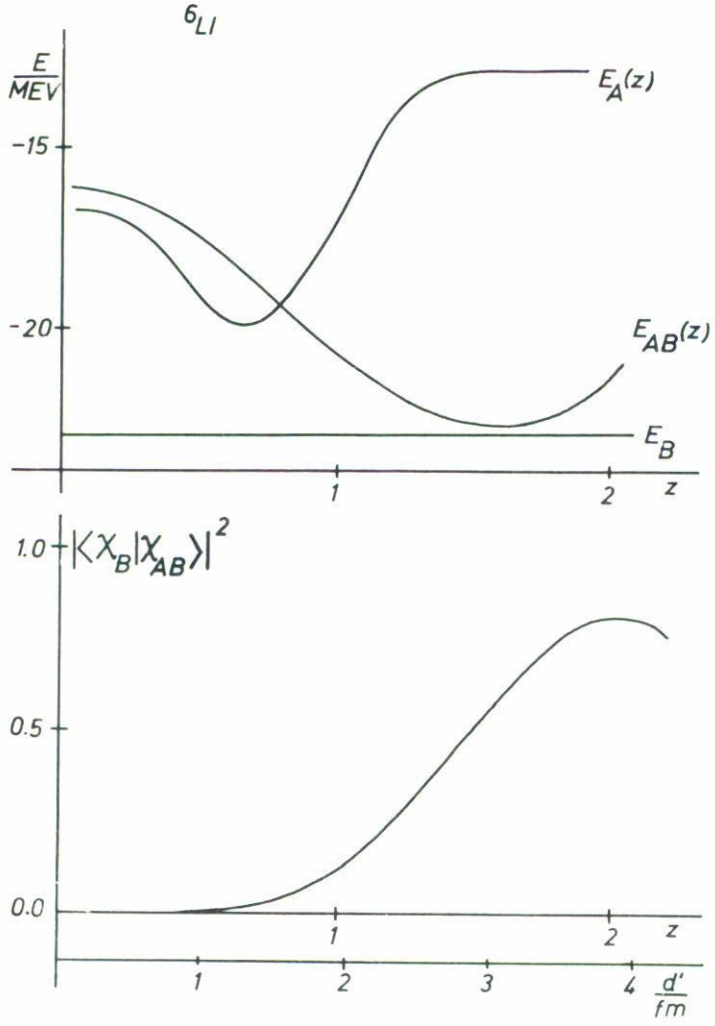

Fig. 1. Comparison of two-center and cluster model for the ground state of ^{6}Li with partition $f = \{42\}$ and the interaction of Brink and Boeker[6]. E_A, E_B and E_{AB} are energies for the unprojected two-center model according to Deenen[7], the oscillator cluster model and the two-center model with angular momentum $L = 0$ respectively. E_A and E_{AB} are given as functions of $z = |\mathbf{z}|$ which is related to the distance d' of the centers by $d' = \sqrt{3/2}\, zb$, $b = 1.6$ fm is the oscillator parameter. Included is the squared overlap of the states χ_{AB} and χ_B.

For $|z| << 1$ only the lowest values of N give a significant contribution. In the limit $|z| \to 0$ we therefore approach the lowest oscillator shell model state with $f = \{42\}$, that is, the s^4p^2 configuration,

$$|z| \to 0 \qquad E_A(z) \to E_{\text{shell model}} \,. \tag{7.2}$$

For large values of $|z|$ and a real vector $\bar{z} = z$ the states ψ_a and ψ_b contribute the internal energies E_a and E_b while the state $\chi_{\bar{z}}(x)$ gives only kinetic energy. From eq. (6.18) we obtain

$$|z| >> 1 \qquad \bar{z} = z \qquad E_A(z) \to E_a + E_b + \tfrac{3}{4}\hbar\omega \,. \tag{7.3}$$

The function $E_A(z)$ in Fig. 1 shows a behaviour in agreement with eqs. (7.2) and (7.3). A minimum is reached at $|z| = 0.72$.

(B) Oscillator cluster model: This model has been discussed elsewhere[5] in great detail. The ansatz for ^{6}Li with $L = 0$,

$$\chi_B(x) = \sum_N^{N_0} \langle x \,|\, N00 \rangle \, c_N \tag{7.4}$$

yields for $N_0 = 6$ the value E_B in Fig. 1 independent of $|z|$.

(AB) Two-center model with angular momentum and cut-off: If the two-center model is projected with respect to $L = 0$ we obtain

$$c_{00}^0 \, \chi_{\bar{z}}(x) = \sum_N \langle x \,|\, N00 \rangle \, \overline{P_{00}^N(z)}$$

$$= \sum_N \langle x \,|\, N00 \rangle \, [(N+1)!\,]^{-\frac{1}{2}} \, (\bar{z}\,\bar{z})^{\frac{1}{2}N} \,. \tag{7.5}$$

The corresponding normalized state is for $\bar{z} = z$

$$\chi_{AB} = c_{00}^{0} \chi_{\frac{-}{z}}(x) \left[\frac{|z|^2}{\sinh |z|^2} \right]^{\frac{1}{2}}$$

$$= \sum_{N} \langle x | N00 \rangle [(N+1)!]^{-\frac{1}{2}} |z|^{N} \left[\frac{|z|^2}{\sinh |z|^2} \right]^{\frac{1}{2}} . \qquad (7.6)$$

If the sum over N is cut off at $N_0 = 6$ we obtain the energy $E_{AB}(z)$ in Fig. 1. Clearly the two-center model is improved by this procedure and we approach the value E_B at a distance corresponding to $|z| = 1.6$. In Fig. 1 we have also shown the square of the overlap between the normalized states χ_B, χ_{AB} of the models B and AB. At $|z| = 2.0$ this squared overlap has the value 0.75. We have discussed a very simple system of six nucleons, but it is clear that all the concepts used have a generalization to more complex systems like more-center systems. Besides we have shown that there is a simple relation between the oscillator shell model, the oscillator cluster model and the two-center model, which allows us to obtain the projection on states of fixed angular momentum and parity.

8. THE CLUSTER MODEL IN BARGMANN HILBERT SPACE

A straight-forward generalization of the ansatz eq. (6.12) is the superposition

$$\chi_{ab}(x) = \int \overline{A_3(z, x)}\, f(z)\, d\mu(z) . \qquad (8.1)$$

From section 3 we know that this equation implies going from H_3 to F_3 with respect to the state $\chi_{ab}(x)$. The ansatz eq. (6.12) in connection with the states ψ_A of eq. (6.1) has been studied in detail by Brink and Weiguny[8]. In this section we shall not develop the dynamical equations for $f(z)$ in F_3 in greater detail but concentrate on the adaptation to orbital symmetry. It has been shown elsewhere[9] that this adaptation presupposes the calculation of basic exchange integrals, that is, of the matrix elements for the double coset generators Z_K. For two different two-center configurations with occupation numbers n_a, n_b, $n_a n_b$, the permutations Z_K are characterized

by the DC-symbol

$$DC(Z_K) = \begin{Bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{Bmatrix} \tag{8.2}$$

where the integers d_{ij} denote the number of nucleons exchanged and are subject to the restrictions

$$\begin{aligned} n_{a'} &= d_{11} + d_{12} , \qquad n_a = d_{11} + d_{21} , \\ n_{b'} &= d_{21} + d_{22} , \qquad n_b = d_{12} + d_{22} . \end{aligned} \tag{8.3}$$

The main point in calculating the matrix elements of Z_K between the states eq. (6.1) is the fact that these states are products of nonorthogonal single-particle states. Therefore it suffices to calculate the single-particle overlaps

$$\begin{aligned} \epsilon_{a'a} &= \int \bar{\varphi}_{a'} \varphi_a \, dx = \exp\left\{ -\frac{1}{2n}\left(\left[\frac{n_{b'}}{n_{a'}}\right]^{1/2} \bar{z}' - \left[\frac{n_b}{n_a}\right]^{1/2} z \right)^2 \right\} , \\ \epsilon_{a'b} &= \int \bar{\varphi}_{a'} \varphi_b \, dx = \exp\left\{ -\frac{1}{2n}\left(\left[\frac{n_{b'}}{n_{a'}}\right]^{1/2} \bar{z}' + \left[\frac{n_a}{n_b}\right]^{1/2} z \right)^2 \right\} , \\ \epsilon_{b'a} &= \int \bar{\varphi}_{b'} \varphi_a \, dx = \exp\left\{ -\frac{1}{2n}\left(\left[\frac{n_{a'}}{n_{b'}}\right]^{1/2} \bar{z}' + \left[\frac{n_b}{n_a}\right]^{1/2} z \right)^2 \right\} , \\ \epsilon_{b'b} &= \int \bar{\varphi}_{b'} \varphi_b \, dx = \exp\left\{ -\frac{1}{2n}\left(\left[\frac{n_{a'}}{n_{b'}}\right]^{1/2} \bar{z}' - \left[\frac{n_a}{n_b}\right]^{1/2} z \right)^2 \right\} . \end{aligned} \tag{8.4}$$

It is easy to prove and has been shown by Seligman[10] that the exchange integral corresponding to Z_K between states $\psi_{A'}$ and ψ_A is given by

$$\langle \psi_{A'} \,|\, Z_K \,|\, \psi_A \rangle$$

$$= (\epsilon_{a'a})^{d_{11}} (\epsilon_{a'b})^{d_{12}} (\epsilon_{b'a})^{d_{21}} (\epsilon_{b'b})^{d_{22}} \quad . \tag{8.5}$$

If we now use eq. (6.14) and the explicit expressions eq. (8.4) we find

$$\langle \psi_{a'} \psi_{b'} \chi_{\underline{z}'} \,|\, Z_K \,|\, \psi_a \psi_b \chi_{\underline{z}} \rangle$$

$$= \langle \psi_{A'} \,|\, Z_K \,|\, \psi_A \rangle \exp \{ \tfrac{1}{2} \mathbf{z}'^2 + \tfrac{1}{2} \bar{\mathbf{z}}^2 \}$$

$$= \exp \{ \frac{1}{s(K)} (\mathbf{z}' \bar{\mathbf{z}}) \} \tag{8.6}$$

with

$$\frac{1}{s(K)} = \frac{1}{n} \left\{ \left[\frac{n_{b'} n_b}{n_{a'} n_a} \right]^{\frac{1}{2}} d_{11} - \left[\frac{n_{b'} n_a}{n_{a'} n_b} \right]^{\frac{1}{2}} d_{12} - \left[\frac{n_{a'} n_b}{n_{b'} n_a} \right]^{\frac{1}{2}} d_{21} + \left[\frac{n_{a'} n_a}{n_{b'} n_b} \right]^{\frac{1}{2}} d_{22} \right\} \quad . \tag{8.7}$$

Comparing eq. (8.6) with eq. (4.8) we see that the exchange integral corresponds to a complex canonical transformation with

$$r = r^* = 0, \qquad s = (s^*)^{-1} = s(K) \quad . \tag{8.9}$$

This complex canonical transformation does not change the degree of a homogeneous polynomial in F_3 in agreement with the fact that a permutation cannot change the overall oscillator excitation of the system. From eq. (5.7) we obtain for the oscillator cluster model

$$(N'LM \,|\, Z_K \,|\, NLM) = \delta_{N'N} \left(\frac{1}{s(K)} \right)^N \quad .$$

Comparing with the entirely different derivation given in ref.(5) we conclude that $s^{-1}(K)$ is a special $9f$-symbol,

$$s^{-1}(K) = \begin{bmatrix} \{n-11\} & \{n_a\} & \{n_b\} \\ \{n_a'\} & d_{11} & d_{12} \\ \{n_b'\} & d_{21} & d_{22} \end{bmatrix}$$

We have given a simple example for the use of complex canonical transformations in nuclear physics. These considerations could be extended to the interaction kernels of the cluster model ansatz. Recent work by Sünkel and Wildermuth[11] using integral transforms similar to eq. (8.1) shows that reactions involving many nucleons may be calculated by an extension of the methods discussed in the last sections.

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RESUMEN

La relación entre el espacio de Hilbert H de la mecánica cuántica y el espacio de Hilbert F de las funciones analíticas enteras propuesta por Bargmann, se usa para obtener representaciones proyectivas unitarias del grupo $SL(2, R)$ de las transformaciones canónicas lineales. Estas representaciones se aplican a un análisis de varios modelos nucleares.