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POINT TRANSFORMATIONS IN QUANTUM MECHANICS

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ABSTRACT: We present first a review of the definition and properties of classical and quantum canonical transformations, from the point of view of a current program on the role of canonical transformations in Quantum Mechanics. The groups and the corresponding infinitesimal algebras are explored. The subgroup of point transformations (i. e. canonical transformations between pairs of conjugate observables, where one of a pair is function only of one of the other pair) is of special interest since it is a group of transformations for which classical and quantum mechanics follow each other. Only the group of inhomogeneous symplectic transformations has similar characteristics. Point transformations are treated in detail and a principal series of unitary representations is constructed.

1. INTRODUCTION

The role of canonical transformations in Quantum Mechanics has recently been a subject of active research at this university[1-6]. There is a two-fold purpose in this program: on one hand one would like to develop techniques to implement mappings of arbitrary physical problems onto simple and well known systems, such as the harmonic oscillator, the hydrogen atom, the point rotor or the free particle, and use the knowledge one has about the spectra and wave-functions, the symmetry and dynamical algebras and groups, transition operators and selection rules of the latter, and translate them to the former[2,3,7]. On the other –and farther– hand, one would like to have a clear and unambiguous answer on a number of fundamental questions which can be posed on the connections between Classical and Quantum Mechanics: What is a quantization scheme? When is it unique? What is the relation between Poisson brackets and commutators? What is the relation between canonical transformations in Classical and Quantum Mechanics? When do they follow each other? If they don't, what is the origin, nature and consequences of the disagreement?

For the time being, one can give answers to a fair part of these questions on the level of Schrödinger Quantum Mechanics (i. e. no spin, no relativity and not too much else), through the rather elegant method of embedding the problem in a group-theoretical context by introducing the Weyl ring of all quantum-mechanical operators built from the universal enveloping algebra of the fundamental Heisenberg-Weyl algebra[4,5]. Out of this work has come a statement on which we intend to elaborate, namely, that among all canonical transformations, Classical and Quantum Mechanics follow each other under inhomogeneous symplectic and point transformations.

Homogeneous symplectic transformations have been studied in Ref. 1, and the purpose of this article is to present some results on point transformations, i. e. canonical transformations between pairs of conjugate observables, where one of a pair is function only of one of the other pair. Point transformations form a function group, where the number of parameters is infinite and one has to deal with a non-locally-compact group. Although these groups are not quite Lie groups[8], one can in this case define infinitesimal generators, there being an infinity of them spanning a function algebra, and find a class of representations with the characteristics of a principal series. A supplementary series-like class of representations is being investigated[9] in connection with the program set out in Ref. 6.

Classically, the problem –if there ever was one– seems to be essentially solved. Quantum-mechanically, however, several restrictions must be made at this stage on the type of systems we want to work with: First, the domain of the position and momentum operators will be the space of infinitely-differentiable functions of compact support on the *full* line (extendable to the space of continuous linear functionals on the full line). The important part of this statement is to ask the full real line to be at our disposal. Only in this way will the Schrödinger representation of the quantum operators of position and momentum be well-defined, and close onto a Heisenberg-Weyl algebra. Problems related to the phase and time "operators"[10] –essentially Quantum Mechanics on a compact space– are thus avoided by, for the moment, not looking at them. Likewise, the point transformations considered will be those which map the full line on the full line. Second, we want to consider invertible one-to-one point transformations. This has to be so, if we want to find the unitary representation of the group which will leave the spectrum of a given operator invariant. This is a rather serious limitation on the general freedom we would like to enjoy, since n-to-1 canonical mappings of phase space have been successfully considered[3] for the anisotropic and sector harmonic oscillators. The fine print shows, however, that the canonical commutation relations preserved there have been preserved *weakly*, i.e. only when taking subsets of the matrix elements of the operators involved, in some particular basis, and extracting n nonequivalent operators and eigenfunctions sets. Classically, this corresponds to viewing phase space through a set of projection operators which act as a grid[11]. Third, we will not consider mappings which exhibit singular points such as those encountered by balls bouncing off walls or similar interface phenomena. Sometimes one can circumvent this restriction by choosing the domains of the operators to be restricted to a class of antisymmetric, periodic or periodic-antisymmetric functions. These domains, however, are not always left invariant by the operators of an algebra.

In conclusion, we ask our point transformations to be one-to-one, invertible, infinitely differentiable mappings of the whole of phase space onto itself.

In this approach we are emphasizing the importance of the Heisenberg-Weyl algebra as the basic building unit for Quantum Mechanics, since we generate all operators out of its universal enveloping algebra. There is an "inverse" approach which starts with the symmetry or dynamical algebra[12] and then looks for operators within the enveloping algebra which form canonically conjugate pairs and which transform under the generated group in the proper way to qualify them as position and momentum operators[13]. These two approaches are certainly not equivalent, since not all representations of the higher groups can be realized on a homogeneous space restricted by the dimensionality of physical space, although all can be realized on a generalized space-restricted to spheres and the like- of sufficiently high dimension and both local and non-local measures, while in some representations of the higher algebras, position and momentum operators are not to be found[13]. The first approach has a classical limit built in, but poses the problem of quantization, while the latter has no quantization problems, but a classical limit is not always defined. This is no problem for quantum characteristics with no classical analog (i.e. spin), in fact, it may seem as a welcome feature of the theory, but when dealing with canonical transformations of the system, it may not be clear how to implement them without recourse to the classical correspondence.

In Sections 2 and 3 we shall define the concepts which were handled informally in this Introduction, namely, the definitions of canonical and point transformations in Classical and Quantum Mechanics. In Section 4 we build a principal series of unitary representations of the group of quantum point transformations.

2. THE GROUP OF CLASSICAL POINT TRANSFORMATIONS

2.1 Classical Canonical Transformations

In Classical Mechanics, let q and p be a pair of canonically conjugate observables (i.e. such that their Poisson bracket $\{q, p\} = 1$), and consider a mapping

$$\phi: \quad q \to q' = \varphi(q,p) \,, \tag{2.1a}$$

$$p \to p' = \psi(q,p) \,, \tag{2.1b}$$

such that the functions $\varphi(q,p)$ and $\psi(q,p)$ are differentiable everywhere and the Poisson bracket is preserved,

$$\{\varphi(q,p),\ \psi(q,p)\} = 1 \ . \tag{2.1c}$$

The transformation (2.1) is then said to be canonical[14]. Equivalent definitions can be given in terms of Lagrange brackets, Pfaffians and conservation of measure in phase space[8, 15, 16]. It is proven that if $f'(q',p') = f(q,p)$ and $g'(q',p') = g(q,p)$ are elements of $\mathcal{S}$, the space of infinitely differentiable functions on phase space,

$$\{f',g'\}(q',p') = \{f',g'\}(q,p) = \{f,g\}(q,p) \ . \tag{2.2}$$

Clearly, the set K of invertible classical canonical transformations (2.1) forms a group, the unit element being the identity trasnformation, and the space $\mathcal{S}$ itself can be taken as a homogeneous space for K

2.2 The Classical Group and its Generators

One can define a correspondence between $\mathcal{S}$ and K, in the manner of Lie, introducing[17] for every $z(q,p) \in \mathcal{S}$, a first-degree operator

$$z_{\text{op}} = \frac{\partial z}{\partial q}\frac{\partial}{\partial p} - \frac{\partial z}{\partial p}\frac{\partial}{\partial q} \tag{2.3a}$$

which has the property, that for every $f(q,p) \in \mathcal{S}$

$$z_{\text{op}} f = \{z, f\} \ . \tag{2.3b}$$

The functions $z \in \mathcal{S}$ can then be used to define a one-parameter group of transformations of $\mathcal{S}$ on itself as

$$f' = \exp(\tau z_{\text{op}})\, f = f + \tau\,\{z, f\} + (\tau^2/2!)\,\{z\,\{z, f\}\} + \ldots \tag{2.4}$$

The linearity of the operator (2.3) allows us to write (2.4) as

$$f'(q,p) = \exp(\tau z_{\text{op}}) f(q,p) = f(q', p')$$

$$= f(\exp(\tau z_{\text{op}})\, q,\ \exp(\tau z_{\text{op}})\, p) \ . \tag{2.5}$$

In this way, every (up to an additive constant) $z \in \mathcal{S}$ generates a one-parameter subgroup, the set of which is to be identified with K. We can thus build the (pseudo)-Lie algebra of infinitesimal operators z_{op} themselves with the Lie bracket

$$[z_{1\text{op}}, z_{2\text{op}}] = \{z_1, z_2\}_{\text{op}} \ . \tag{2.6}$$

We call it "pseudo-Lie" algebra because it has an infinite number of elements, in fact, those of $\mathcal{S}/c$, i. e., functions which differ by an additive constant are identified.

One can define another correspondence between $\mathcal{S}/c$ and the elements of K by the use of the generating function[14, 15]. This "parametrization" of K is not convenient for our purposes, however, since we have no true way of building one-parameter subgroups. The correspondence between both approaches has been studied but is not, in general, simple[18].

2.3 The Inhomogeneous Symplectic Subgroup

One subgroup of K is the set of inhomogeneous symplectic transformations

$$q \to q' = aq + bp + e \quad , \tag{2.7a}$$

$$p \to p' = cq + dp + f \quad , \tag{2.7b}$$

$$a, b, \ldots, f \in \mathbf{R}, \text{ the real field,}$$

which is canonical if $ad - bc = 1$, and can be identified with a group $ISL(2, R) \cong ISp(2, R)$. The subset of $\mathcal{S}/c$ generating $ISp(2, R)$ can be seen to be all up-to-second order polynomials in q and p, of which we can choose the linearly independent set

$$j_1 = \tfrac{1}{4}(p^2 - q^2), \quad j_2 = \tfrac{1}{2}pq, \quad j_3 = \tfrac{1}{4}(p^2 + q^2), \quad q, p \ . \tag{2.8}$$

2.4 Classical Point Transformations and their Generators

The subgroup of K transformations of the form

$$q' = \varphi(q) \tag{2.9a}$$

$$p' = \psi(q, p) \tag{2.9b}$$

is the group P of point transformations.

We can generate a one-dimensional subgroup ϕ_τ of K depending on the parameter τ and entirely in P if we propose a generator function

$$z(q,p) = pf(q) + g(q) \ , \tag{2.10}$$

so that, from (2.4)

$$\varphi_\tau(q) = q - \tau f(q) + \frac{\tau^2}{2!} f(q) \frac{d}{dq} f(q) - \frac{\tau^3}{3!} f(q) \frac{d}{dq} f(q) \frac{d}{dq} f(q) + \ldots \tag{2.11}$$

which satisfies

$$\frac{\partial \varphi_\tau(q)}{\partial \tau} = z_{op}\, \varphi_\tau(q) = -f(q) \frac{\partial \varphi_\tau(q)}{\partial q} \tag{2.12a}$$

which, given $f(q)$, can be used to determine $\varphi_\tau(q)$ under the change of variable

$$r = \chi(q) \equiv \int [f(q)]^{-1} dq \ , \qquad q = \chi^{-1}(r) \ , \tag{2.12b}$$

as

$$\varphi_\tau(q) = \exp\,[\tau (pf + g)_{op}]\, q = \chi^{-1}(\chi(q) - \tau) \ , \tag{2.12c}$$

which can be checked to satisfy (2.12a) with the correct initial condition $\varphi_0(q) = q$.

Similarly, a $\psi_\tau(q,p)$ in (2.9b) will satisfy

$$\frac{\partial \psi_\tau(q,p)}{\partial \tau} = z_{op}\, \psi_\tau(q,p) = [(p\dot{f} + \dot{g}) \frac{\partial}{\partial p} - f \frac{\partial}{\partial q}]\, \psi_\tau(q,p) \ , \tag{2.13}$$

where the dot means differentiation with respect to the argument q. It can be seen that, in order to conserve the canonical Poison bracket relation, we must write

$$\psi_\tau(q,p) = p\,[\dot{\varphi}_\tau(q)]^{-1} + \gamma_\tau(q) \ . \tag{2.14}$$

Now, in order to determine $\gamma_\tau(q)$ we replace (2.14) in (2.13) and using (2.12) we get

$$\frac{\partial\gamma_\tau(q)}{\partial\tau} = [\dot{\varphi}_\tau(q)]^{-1}\,\dot{g}(q) - f(q)\,\frac{\partial\gamma_\tau(q)}{\partial q}\ . \tag{2.15}$$

This equation looks very much like (2.12) but is inhomogeneous with respect to it in the sense that the solution can be written as $\gamma_\tau(q) = c_1\,\varphi_\partial(q) + \varphi_\tau(q) + c_2$ where c_1 and c_2 are constants and $\delta_\tau(q)$ satisfies again (2.15). The initial condition $\gamma_0(q) = 0$, however, requires that $c_1 = 0 = c_2$. For the trivial case $g \equiv 0$ we have $\gamma_\tau \equiv 0$. The more general case will be solved below (eqs. (2.19) and (2.20)).

2.5 Ray Transformations

In order to solve (2.15) for $\gamma_\tau(q)$, it can be remarked that a general point transformation (2.12)-(2.14) can be made in a two-step process: one where the generator function (2.10) is $z(q,p) = pf(q)$ so that $\gamma_\tau \equiv 0$ in (2.14), and a second, which will be called *ray* transformation (because it will give rise to ray representations of the Heisenberg-Weyl group), which is generated by functions of the kind $z(q,p) = pq + g(q)$ i.e. $f(q) = q$ hence $\chi = \ln$ and $\chi^{-1} = \exp$, so that (2.12) and (2.14) yield

$$q' = e^{-\tau} q\ , \tag{2.16a}$$

$$p' = e^{\tau} p + \gamma_\tau(q)\ . \tag{2.16b}$$

The set of these transformations forms a subgroup R of P. We can now find a series expansion for $\gamma_\tau(q)$ in powers of τ by using (2.4) on p, i.e.

$$\gamma_\tau(q) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} z_{\mathrm{op}}^n\, p - e^{\tau} p = \sum_{n=0}^{\infty} \frac{\tau^n}{n!}\,\gamma_n(q)\ , \tag{2.17a}$$

where $\gamma_0(q) \equiv 0$, $\gamma_1(q) = \dot{g}(q)$ and

$$\gamma_n(q) = \dot{g}(q) - q\,\dot{\gamma}_{n-1}(q)\ . \tag{2.17b}$$

It is a simple matter to check that the series (2.17) satisfies the differential equation (2.15) for the ray transformation (2.16). We can determine directly that for $g(q) = q^m$, we obtain $\gamma_n(q) = [1-(1-m)^n]\, q^{m-1}$ and thence $\gamma_\tau(q) = e^\tau q^{m-1}(1-e^{-m\tau})$. Since a linear combination of g's will produce a linear combination of γ_τ's the group multiplication of R being the addition of functions, the Taylor expansion of a general $g(q)$ will yield

$$\gamma_\tau(q) = e^\tau q^{-1} [g(q) - g(e^{-\tau} q)] \quad , \tag{2.18}$$

which has the right boundary conditions and satisfies (2.15). The general point transformation can thus be generated by (2.10) as

$$q \to q' = \varphi_\tau(q) = \chi^{-1}(\chi(q) - \tau) \tag{2.19a}$$

$$p \to p' = \psi_\tau(q,p) = p\, [\dot{\varphi}_\tau(q)]^{-1} + \gamma_\tau(q) \quad , \tag{2.19b}$$

where we use the definition (2.12b) and

$$\gamma_\tau(q) = [f(q)\, \dot{\varphi}_\tau(q)]^{-1}\, [g(q) - g(\varphi_\tau(q))] \quad , \tag{2.20}$$

which can be seen to satisfy (2.15).

3. THE GROUP OF QUANTUM POINT TRANSFORMATIONS

3.1 The Quantization Process

One can describe the quantization of a system characterized by a set of observables as a scheme by which we can associate to each function $f(q,p)$ in $\mathcal{S}$ relevant for the system, one element $\mathbf{F}(\mathbf{Q},\mathbf{P})$ of $\overline{\mathbb{W}}$ the universal enveloping algebra of the Heisenberg-Weyl algebra $\mathbb{W}$:

$$[\mathbf{Q},\mathbf{P}] = i\mathbf{H}, \quad [\mathbf{Q},\mathbf{H}] = 0, \quad [\mathbf{P},\mathbf{H}] = 0, \tag{3.1}$$

i.e. the set of all formal sums and products of the generators $\mathbf{Q}$, $\mathbf{P}$ and $\mathbf{H}$, including formal infinite series[19]. We further ask for the element $\mathbf{F}(\mathbf{Q},\mathbf{P})$ to be hermitean under the scalar product defined on the Heisenberg-Weyl group W generated from (3.1), which essentially reduces to the statement that $\mathbf{Q}$, $\mathbf{P}$ and $\mathbf{H}$ be taken hermitean. When we speak of a "quantization scheme" we mean a scheme for associating to *every* function $f(q,p)$, polynomial or series in q and p, an $\mathbf{F}(\mathbf{Q},\mathbf{P})$. The quantization scheme is not unique[20] and, it may be argued, it should be physically irrelevant, at least in the framework of the physical systems amenable to solution à la Schrödinger. This is certainly the case for up-to-second order operators in $\mathbf{Q}$ and $\mathbf{P}$; however, it has been found that the dynamical algebras, classically equivalent through a point transformation[6] may *not* be unitarity equivalent on the quantum level[9]: they belong to different representations of the algebra, a feature directly traceable to the fact that the quantization scheme is not invariant under general canonical transformations and thus the values of the Casimir operators (or order higher than two in $\mathbf{Q}$ and $\mathbf{P}$) need not be equal, even though the dynamical algebras are the same. Thus, if we want to avoid "paradoxes" of this kind, we must follow the maxim *quantize once and only once*.

Keeping this in mind, we can choose the quantization scheme proposed in Ref. 4 out of simplicity on group-theoretical arguments, the "symmetrization" rule which is defined through

$$q^m p^n \Rightarrow \tfrac{1}{2}\{\mathbf{Q}^m \mathbf{P}^n + \mathbf{P}^n \mathbf{Q}^m\}\,, \tag{3.2}$$

and represent $\mathbf{Q}$ and $\mathbf{P}$ in the Schrödinger realization

$$\mathbf{Q} f(q) = q f(q), \qquad \mathbf{Q} : q \quad , \tag{3.3a}$$

$$\mathbf{P} f(q) = -i\hbar \frac{\partial}{\partial q} f(q), \qquad \mathbf{P} : -i\hbar \frac{\partial}{\partial q} \quad , \tag{3.3b}$$

$$\mathbf{H} f(q) = \hbar f(q), \qquad \mathbf{H} : \hbar \quad , \tag{3.3c}$$

where the colon separates the abstract operator from its realization, on the space of infinitely differentiable functions $f(q)$ of compact support. These are dense in the space of square-integrable functions on the real line with measure dq. The domain of the operators (3.3) can then be enlarged through adjunction to the space of continuous linear functionals. We do not have to explore the full freedom in choosing non-Schrödinger realizations of the algebra (3.1) since the Stone-vonNeumann theorem asserts that[21] all are unitarily equivalent to (3.3). Indeed, through the use of the unitary representations of canonical transformations we should be able to explore this freedom in a systematic fashion.

3.2 Quantum Canonical Transformations

We now want to build a proper definition for a canonical transformation in Quantum Mechanics as a mapping of $\mathbb{W}$ on itself, i.e.

$$\boldsymbol{Q} \to \boldsymbol{Q}' = \Phi(\boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{H}) , \tag{3.4a}$$

$$\Phi: \ \boldsymbol{P} \to \boldsymbol{P}' = \Psi(\boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{H}) , \tag{3.4b}$$

$$\boldsymbol{H} \to \boldsymbol{H}' = \Omega(\boldsymbol{H}) , \tag{3.4c}$$

such that the functions Φ, Ψ and Ω include a specification of the order of the arguments, and the commutation relations of the algebra (3.1) be preserved, i.e.

$$[\Phi(\boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{H}), \ \Psi(\boldsymbol{Q}, \boldsymbol{P}, \boldsymbol{H})] = i\Omega(\boldsymbol{H})$$

$$[\Phi, \Omega] = 0 , \quad [\Psi, \Omega] = 0 , \tag{3.4d}$$

and such that the domains of $\boldsymbol{Q}'$, $\boldsymbol{P}'$ and $\boldsymbol{H}'$ be the same as that of $\boldsymbol{Q}$, $\boldsymbol{P}$ and $\boldsymbol{H}$. As $\boldsymbol{H}'$ is still in the centre of $\mathbb{W}$, it cannot but be a function of $\boldsymbol{H}$ only.

A representation of $\mathbb{W}$ will yield a representation of (3.4) if we take the latter to be a similarity transformation of $\mathbb{W}$ as

$$\boldsymbol{Q}' = c_q \boldsymbol{A} \boldsymbol{Q} \boldsymbol{A}^{-1} , \tag{3.5a}$$

$$\boldsymbol{P}' = c_p \boldsymbol{A} \boldsymbol{P} \boldsymbol{A}^{-1} , \tag{3.5b}$$

$$\boldsymbol{H}' = c_h \boldsymbol{H} , \tag{3.5c}$$

where we assume the existence of a left inverse $\mathbf{A}^{-1}$ for every element $\mathbf{A}$ considered, which evidently satisfies (3.4) when the numbers c_q, c_p and c_h are related as

$$c_q \, c_p = c_h \, , \tag{3.5d}$$

and thence, for the whole of $\overline{\mathbb{W}}$

$$\boldsymbol{F}'(\boldsymbol{Q},\boldsymbol{P}) = \boldsymbol{F}(\boldsymbol{Q}',\boldsymbol{P}') = \boldsymbol{F}(\mathbf{A}\boldsymbol{Q}\mathbf{A}^{-1}, \mathbf{A}\boldsymbol{P}\mathbf{A}^{-1}) = \mathbf{A}\boldsymbol{F}(\boldsymbol{Q},\boldsymbol{P})\,\mathbf{A}^{-1} \; . \tag{3.6}$$

Furthermore, if the set of transformations (3.4) is to form a group, a two-sided inverse $\mathbf{A}^{-1}$ must exist.

3.3 Unitary Quantum Canonical Transformations

The very general realization (3.5) is, besides the assumption of the existence of an $\mathbf{A}^{-1}$ for every $\mathbf{A}$ considered, not very satisfying since the transformation (3.5) does not in general preserve the hermiticity of the elements of $\mathbb{W}$, unless $\mathbf{A}$ be unitary (with respect to the measure on the Heinsenberg-Weyl group), i.e. the existence of $\mathbf{A}^{-1} = \mathbf{A}^{\dagger}$ is assured. Preservation of hermiticity thus leads us from the group of general transformations (3.4) to a more restricted group, "parametrized" in a different way, of *unitary* canonical transformations Q, given by

$$\begin{aligned}
\boldsymbol{F} \xrightarrow{\boldsymbol{Z}} \boldsymbol{F}' &= e^{i\tau'\boldsymbol{Z}}\,\boldsymbol{F}\,e^{-i\tau'\boldsymbol{Z}} \\
&= \boldsymbol{F} + i\tau'\,[\boldsymbol{Z},\boldsymbol{F}] + \frac{(i\tau')^2}{2!}\,[\boldsymbol{Z},[\boldsymbol{Z},\boldsymbol{F}]] + \ldots \\
&= \exp(i\tau'\,\boldsymbol{Z}_{\mathrm{com}})\,\boldsymbol{F} \; ,
\end{aligned} \tag{3.7a}$$

where the hermitean element $\boldsymbol{Z}(\boldsymbol{Q},\boldsymbol{P}) \in \overline{\mathbb{W}}$ generates a Q transformation on $\boldsymbol{F} \in \mathbb{W}$ labelled by $\boldsymbol{Z}$, where we have defined[22] the operator $\boldsymbol{Z}_{\mathrm{com}}$ associated to $\boldsymbol{Z}$ as

$$\mathbf{Z}_{\text{com}}\,\mathbf{F} = [\mathbf{Z}, \mathbf{F}] \ , \tag{3.7b}$$

a generalization of (2.3b) in (2.4). The linearity of (3.7b) allows us to write (3.6) as we did with (2.5). Furthermore, the pseudo-Lie algebra of infinitesimal generators of Q is given by the hermitean elements of $\mathbb{W}$ themselves with the Lie bracket

$$[\mathbf{Z}_{1\text{com}}, \mathbf{Z}_{2\text{com}}] = [\mathbf{Z}_1, \mathbf{Z}_2]_{\text{com}} \ , \tag{3.7c}$$

a generalization of (2.6) which can be proven by applying it to any $\mathbf{F} \in \overline{\mathbb{W}}$ and using the Jacobi identity. We expect that a representation of the hermitean elements of $\mathbb{W}$ as (3.3) will provide a unitary representation of Q.

We should remark here that while we will indeed only consider elements of Q, particular non-unitary canonical transformations, with $\mathbf{A} = \mu(Q)$, function only of $\mathbf{Q}$, in a basis where $\mathbf{Q}$ is diagonal, are being used to provide the passage between local and non-local measures preserving the hermiticity of operators in $\mathbb{W}$ which generate the dynamical algebra of a system. The price paid is the loss of hermicity of the operators $\mathbf{Q}$ and $\mathbf{P}$ themselves[9].

What is the relation between K and Q, the classical and quantum groups of canonical transformations? There being an infinity of quantization schemes, to every $z \in \mathscr{S}$ of order higher than second in q and p will correspond in general more than one hermitean operator $\mathbf{Z} \in \overline{\mathbb{W}}$. Hence, to every one-parameter subgroup in K will correspond one *or more* one-paramter subgroups in Q. There is thus in general a many-to-one correspondence between the elements of Q and K.

Assume, however, that we have chosen a definite quantization scheme so that out of K we build a unique subset of Q, so that to every one-parameter subgroup of K generated by a function z corresponds a one-parameter subgroup of Q generated by the corresponding, unique, operator $\mathbf{Z}$. In this way we construct a one-to-one mapping between the elements of the generating algebras of K and the subset of Q which can be extended, at least locally, to the group. This, however, is still not an isomorphism between the algebras (nor the groups), since we must still demand that under the Lie bracket operation, the one-to-one correspondence be preserved, i.e. that the Poisson bracket (2.6) for K keep the correspondence with the commutator (3.7c) for Q. This clearly does not hold in general. We conclude therefore that K is neither locally nor globally isomorphic with Q, not even within a definite quantization scheme subset.

We can restrict ourselves, however, to those subgroups of K and Q whose generators *do* maintain the correspondence between Poisson brackets and commutators. Indeed, we have a slightly larger freedom in that if a commutator differs from the quantization of a Poisson bracket in an additive term, function of $\boldsymbol{H}$ only, the correspondence is still assured, since the generators of Q act through the commutator operation (3.7b). These subgroups are precisely those of inhomogeneous symplectic and point transformations[4]. The classical and quantum versions of these subgroups will be isomorphic.

3.4 Extended Symplectic Transformations

The group of linear automorphisms[23] of the Heisenberg-Weyl algebra $\mathbb{W}$ (3.1) is given by

$$\begin{bmatrix} \boldsymbol{Q} \\ \boldsymbol{P} \\ \boldsymbol{H} \end{bmatrix} \longrightarrow \begin{bmatrix} \boldsymbol{Q}' \\ \boldsymbol{P}' \\ \boldsymbol{H}' \end{bmatrix} = \begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & g \end{bmatrix} \begin{bmatrix} \boldsymbol{Q} \\ \boldsymbol{P} \\ \boldsymbol{H} \end{bmatrix} , \tag{3.8a}$$

such that

$$g = [ad - bc]^{-1} . \tag{3.8b}$$

The upper-left 2×2 submatrix is, for $g = 1$, the group of real symplectic transformations $Sp(2, R)$ considered in Ref. 1. This is multiplied in direct product by the subgroup of dilatations $Dil(g)$ with $g = 0$, and multiplied in semi-direct product by the subgroup $T(2)$ of "translations" $\boldsymbol{Q} \to \boldsymbol{Q} + e\boldsymbol{H}$, $\boldsymbol{P} \to \boldsymbol{P} + f\boldsymbol{H}$. This is further multiplied in semi-direct product by the representatives of the two disconnected pieces of the dilatation group $g > 0$ and $g < 0$, a $C(2)$ group of two elements. The group of automorphisms of $\mathbb{W}$ is thus the extended symplectic group

$$C(2) \otimes [T(2) \otimes (Sp(2, R) \otimes Dil)] . \tag{3.9}$$

There is clearly an isomorphism between the $Isp(2, R) \simeq T(2) \otimes Sp(2, R)$ subgroup of (3.9) and the group (2.7) of classical inhomogeneous symplectic transformations. Under these restricted canonical transformations, classical and quantum observables and operators of up-to-second order are mapped among themselves and hence, for these transformations on these observables, Classical and Quantum Mechanics follow each other.

3.5 Quantum Point Transformations

The subset of Q of elements of the form

$$\mathbf{Q} \to \mathbf{Q}' = \Phi(\mathbf{Q}) \tag{3.10a}$$

$$\mathbf{P} \to \mathbf{P}' = \Psi(\mathbf{Q}, \mathbf{P}, \mathbf{H}) \tag{3.10b}$$

$$\mathbf{H} \to \mathbf{H}' = \Omega(\mathbf{H}) \tag{3.10c}$$

in analogy with Section 2.4, will constitute the group of quantum point transformations if it preserves the $\mathbb{W}$ algebra (3.1). We can translate the results of Sections 2.4 and 2.5 to the quantum case[4, 24] because both the classical generator functions (2.10) and the functions involved in (3.10) i.e. (2.12c) and (2.14) are of the form $z(q,p) = pF(q) + G(q)$. Under quantization in the symmetrization scheme (3.2) we associate the operators

$$\mathbf{Z}(\mathbf{Q},\mathbf{P}) = \tfrac{1}{2}\left\{\mathbf{P}F(\mathbf{Q}) + F(\mathbf{Q})\mathbf{P}\right\} + G(\mathbf{Q}) \ , \tag{3.11}$$

and it is easy to check that the Poisson bracket and commutator of two quantities –classical or quantum– of the form (3.11) is again of that form. As $F(q)$ and $G(q) \in \mathcal{S}$ they are differentiable and hence $F(\mathbf{Q})$ and $G(\mathbf{Q})$ are formally so, defining $\dot{F}(\mathbf{Q})$ and $\dot{G}(\mathbf{Q})$ as

$$[F(\mathbf{Q}), \mathbf{P}] = i\mathbf{H}\dot{\mathbf{F}}(\mathbf{Q}) \ . \tag{3.12}$$

The expansion series (3.7) for point transformations generated by $\mathbf{Z}$ acting on $\mathbf{Q}$ and $\mathbf{P}$, is then identical to (2.4) generated by z acting on q and p, with $\tau'\mathbf{H} = \tau\mathbf{1}$. The group of quantum point transformations (3.10) is therefore isomorphic to the group P of classical point transformations, and will be denoted by the same letter. Without further computation we can thus state that (3.11) in (3.7) generates a point transformation (3.10) in P with

$$\boldsymbol{Q}' = \Phi_{\tau'}(\boldsymbol{Q}) = X^{-1}(X(\boldsymbol{Q}) - \tau' \boldsymbol{H}) \ , \tag{3.13a}$$

$$\boldsymbol{P}' = \Psi(\boldsymbol{Q}, \boldsymbol{P}) = \tfrac{1}{2}\{\boldsymbol{P}[\dot{\Phi}_{\tau'}(\boldsymbol{Q})]^{-1} + [\dot{\Phi}_{\tau'}(\boldsymbol{Q})]^{-1}\boldsymbol{P}\} + \Gamma_{\tau'}(\boldsymbol{Q}) \ , \tag{3.13b}$$

$$\boldsymbol{H}' = \boldsymbol{H} \ , \tag{3.13c}$$

where upper-case greek-lettered functions are identified with their classical lower-case counterparts[25] in (2.12) and (2.15). Using (3.12) it is easy to show that (3.13) indeed leaves the algebra $\mathbb{W}$ invariant. When writing down these equations we have glossed over as to the kind of elements of $\mathbb{W}$ which can be used. We repeat the caveat given in the Introduction: We ask our point transformations to be one-to-one invertible, infinitely differentiable mappings of the whole phase-space onto itself. We further assume that we are working in a representation (where $\boldsymbol{Q}$ is diagonal, for instance) where we can give meaning to the possible appearance of formal power series in $\boldsymbol{Q}$. For operators of the kind (3.13) under point transformations, therefore, Classical and Quantum Mechanics follow each other. We should note, however, that even though we have proven that the classical and quantum versions of the subgroups of inhomogeneous symplectic and point transformations are isomorphic, the *composition* of one symplectic and one point transformation may lie outside both subgroups. Similarly, the correspondence will break down if we apply point transformations to observables other than (3.13) (i.e. (2.8), for example) or symplectic transformations to observables other than (2.8) (i.e. (3.13), for example).

3.6 The Quantization Scheme under Canonical Transformations

Consider two or more systems related through a canonical transformation on the classical level. This is the case of the three elementary one-dimensional svstems considered in Ref. 6. It is a question of central interest to know whether this correspondence can be carried over into Quantum Mechanics through a single quantization scheme. The answer is, in general, no. In order to show this it is sufficient to give a counterexample.

Consider the observable $f = qp^2$ and its corresponding hermitean operator $\boldsymbol{F} = \tfrac{1}{2}\{\boldsymbol{Q}\boldsymbol{P}^2 + \boldsymbol{P}^2\boldsymbol{Q}\}$ under the symmetrization scheme (3.2). Now consider a classical canonical transformation (2.19) with $\gamma_\tau \equiv 0$ for simplicity and a fixed τ, and its quantization

$$f \to f' = \varphi(q)\,[\dot{\varphi}(q)]^{-2}\,p^2 \equiv \theta(q)\,p^2$$

$$\Rightarrow \tfrac{1}{2}\,\{\Theta(\boldsymbol{Q})\,\boldsymbol{P}^2 + \boldsymbol{P}^2\Theta(\boldsymbol{Q})\}\ . \tag{3.14a}$$

Lastly, consider the hermitean operator obtained from $\boldsymbol{F}$ through (3.13), again, with $\Gamma_{\boldsymbol{T}} \equiv 0$. This is

$$\begin{aligned} \boldsymbol{F} \to \boldsymbol{F}' &= (1/8)(\Phi\,\{\Upsilon\boldsymbol{P} + \boldsymbol{P}\Upsilon\}^2 + \{\Upsilon\boldsymbol{P} + \boldsymbol{P}\Upsilon\}^2\,\Phi) \\ &= \tfrac{1}{2}\,\{\Theta(\boldsymbol{Q})\boldsymbol{P}^2 + \boldsymbol{P}^2\Theta(\boldsymbol{Q})\} + \tfrac{1}{4}\boldsymbol{H}^2(4\dot{\Phi}\Upsilon\dot{\Upsilon} + 2\Phi\Upsilon\ddot{\Upsilon} + 3\Phi\dot{\Upsilon}^2)\ , \end{aligned} \tag{3.14b}$$

where $\Phi \equiv \Phi(\boldsymbol{Q})$ and $\Upsilon = (\dot{\Phi})^{-1}$. This is obviously an operator different from (2.24a), although they both have the same classical limit f.

When *is* the quantization scheme preserved? It is preserved for up-to-second order functions under symplectic transformations and for functions of the kind $pf(q) + g(q)$ under point transformations. It appears that no other general cases exist.

The importance of the non-invariance of the quantization scheme under general canonical transformations, is that the Casimir operators of the dynamical algebra of two classically related systems, being of order in general higher than second in q and p, may not be equal quantum mechanically, so that the quantum systems belong to different irreducible representations of the algebra[9].

4. UNITARY REPRESENTATIONS OF THE GROUP OF QUANTUM POINT TRANSFORMATIONS

4.1 Two Eigenbases

Consider the eigenbasis of the operator $\boldsymbol{Q}$ in (3.3) $\{\,|q\rangle,\ q \in (-\infty, \infty)\}$ i.e.

$$\boldsymbol{Q}\,|q\rangle = q\,|q\rangle\ , \tag{4.1a}$$

orthonormal in the Dirac sense

$$<q|q'> = \delta(q-q') \ , \tag{4.1b}$$

and complete in $\mathcal{L}^2(-\infty,\infty)$

$$\int_{-\infty}^{\infty} |q> dq <q| = \mathbf{1} \ , \tag{4.1c}$$

so that

$$<q|\psi> = \psi(q), \ \ \psi \in \mathcal{S} \ . \tag{4.1d}$$

Consider now the general (unitary) point transformation (2.15)–(3.13), defining a new bases

$$|q') \equiv |\varphi^{-1}(q')> = |q>, \ \ q' = \varphi(q) \ , \tag{4.2a}$$

where φ^{-1} is the inverse function of φ (corresponding to the operator function Φ), both in $\mathcal{S}$. This is an eigenbasis of $\mathbf{Q}' = \Phi(\mathbf{Q})$ since

$$\mathbf{Q}'|q') = \Phi(\mathbf{Q})|\varphi^{-1}(q')> = \varphi(\varphi^{-1}(q'))|\varphi^{-1}(q')> = q'|q') \ , \tag{4.2b}$$

orthogonal and complete in $\mathcal{L}^2(-\infty,\infty)$, and

$$(q'|\psi> = <\varphi^{-1}(q')|\psi> = \psi(\varphi^{-1}(q')) = \psi(q) \ . \tag{4.2c}$$

The scalar product in $\mathcal{L}^2(-\infty,\infty)$ can be written in terms of either basis as

$$<\psi_1|\psi_2> = \int_{-\infty}^{\infty} dq\, \psi_1(q)^* \psi_2(q)$$

$$= \int_{\varphi(-\infty)}^{\varphi(\infty)} [\dot{\varphi}(\varphi^{-1}(q'))]^{-1} dq'\, \psi_1(\varphi^{-1}(q'))^* \psi_2(\varphi^{-1}(q')) \ , \tag{4.3}$$

where $\dot{\varphi}$ is the derivative of φ with respect to its argument. It will prove convenient to define a positive definite weight function ω_φ so that the measure in (4.3) be

$$dq = \omega_\varphi(q')\, dq' \ , \tag{4.4a}$$

where

$$\omega_\varphi(q') = \left| \dot{\varphi}(\varphi^{-1}(q')) \right|^{-1} = \left| \dot{\overline{\varphi^{-1}}}(q') \right| \tag{4.4b}$$

takes care, through the absolute value, to keep in (4.3) the lower integration limit $\varphi(-\infty)$ smaller than the upper limit $\varphi(\infty)$ by exchanging them whenever φ is a decreasing function of q. All integrals will henceforth be from $-\infty$ to ∞ and the limits may be dropped from the notation. By $\dot{\overline{\varphi^{-1}}}$ we mean the derivative of φ^{-1} with respect to its argument. The characteristics of the mapping φ imply that ω_φ will not vanish over q.

The basis $\{|q)\}$ is not normalized in the sense (4.1b) but, φ being monotonous

$$(q_1 | q_2) = \delta(\varphi^{-1}(q_1) - \varphi^{-1}(q_2)) = [\omega_\varphi(q_1)]^{-1} \delta(q_1 - q_2) \ , \tag{4.5a}$$

while completeness in $\mathcal{L}^2(-\infty,\infty)$ is phrased as

$$\int |q)\, \omega_\varphi(q)\, dq\, (q| = \mathbf{1} \ . \tag{4.5b}$$

We can normalize the basis $\{|q)\}$ however, to form an orthonormal (in the sense of Dirac) basis

$$|q>' = |q)\, \omega_\varphi(q)^{\frac{1}{2}} \equiv \Phi |q> \tag{4.6a}$$

orthogonal and complete in the same sense as (4.1) but eigenbasis of $\mathbf{Q}'$. Formally $\Phi \in P$ is the point transformation which maps the basis $\{|q>\}$ on the basis $\{|q>'\}$, in this notation

$$\mathbf{Q}' = \Phi \mathbf{Q} \Phi^{-1} = \Phi(\mathbf{Q}) \ . \tag{4.6b}$$

4.2 The Transformation Matrices

In order to find the transformation matrix

$$\| <q_1 | \Phi | q_2> \| = \| <q_1 | q_2>' \| \tag{4.7a}$$

between the original and the transformed bases, we inclose $\mathbf{Q}' = \Phi(\mathbf{Q})$ in the bracket $<q_1|, |q_2>$, obtaining

$$<q_1 | \mathbf{Q}' | q_2>' = q_2 <q_1 | q_2>' = \varphi(q_1) <q_1 | q_2>' \tag{4.7b}$$

hence the transformation bracket $<q_1 | q_2>'$ must have the form

$$<q_1 | q_2>' = \delta(q_2 - \varphi(q_1)) \, \mu_\varphi(q_2) \ , \tag{4.7c}$$

where μ_φ is an as yet undetermined function of q_2. We have asked the point transformation (3.13) to be unitary, that is, if $\{|q>\}$ is a complete orthonormal basis, so must $\{|q>'\}$ be, thus placing all eigenbases of operators related through point transformations on the same footing.

Hence, let

$$\begin{aligned} \delta(q_1 - q_2) &= {}'<q_1 | q_2>' = \int {}'<q_1 | q_3> dq_3 <q_3 | q_2>' \\ &= \int \delta(q_1 - \varphi(q_3)) \, \mu_\varphi(q_1)^* \, dq_3 \, \delta(q_2 - \varphi(q_3)) \, \mu_\varphi(q_2) \\ &= | \dot{\varphi}(\varphi^{-1}(q_1)) |^{-1} \, | \mu_\varphi(q_1) |^2 \, \delta(q_1 - q_2) \ , \end{aligned} \tag{4.7d}$$

where we have used (4.4b) and (4.7c). The point transformation $\Phi \in P$ will thus have a unitary representation in the basis $\{|q>\}$ when, in (4.7c)

$$|\mu_\varphi(q)|^2 = [\omega_\varphi(q)]^{-1} \ . \tag{4.8}$$

This determines μ_φ up to a phase.

A similar analysis for the transformation φ^{-1} allows us to write

$$'\langle q_2 | q_1 \rangle = \delta(q_1 - \varphi^{-1}(q_2))\, \mu_{\varphi^{-1}}(q_1) \ , \tag{4.9a}$$

which can be directly compared with (4.7c) written as its complex conjugate

$$'\langle q_2 | q_1 \rangle = \delta(q_1 - \varphi^{-1}(q_2))\, [\mu_\varphi(q_2)]^{-1} \ , \tag{4.9b}$$

so we conclude that

$$\mu_\varphi(q)\, \mu_{\varphi^{-1}}(\varphi^{-1}(q)) = 1 \ . \tag{4.10}$$

We shall now explore the problem of a consistent choice of phases.

4.3 The Multiplier Representation

Let Φ_1 and Φ_2 be two point transformation elements of P. Their composition is

$$q \xrightarrow{\Phi_1} q' \xrightarrow{\Phi_2} q'' = \varphi_2(\varphi_1(q)) = \varphi_3(q) \tag{4.11a}$$

and similarly

$$\mathbf{Q}'' = \Phi_2(\Phi_1(\mathbf{Q})) = \Phi_2(\mathbf{Q}') = \Phi_3(\mathbf{Q}) \ . \tag{4.11b}$$

The corresponding bases $\{|q\rangle\}$, $\{|q\rangle'\}$ and $\{|q\rangle''\}$ relate as

$$\delta(q_3 - \varphi_3(q_1))\, \mu_{\varphi_3}(q_3) = \langle q_1 | q_3 \rangle'' = \int \langle q_1 | q_2 \rangle' \, dq_2 \, '\langle q_2 | q_3 \rangle''$$

$$= \int \delta(q_2 - \varphi_1(q_1))\, \mu_{\varphi_1}(q_2)\, dq_2\, \delta(q_3 - \varphi_2(q_2))\, \mu_{\varphi_2}(q_3)$$

$$= \mu_{\varphi_1}(\varphi_1(q_1))\, \mu_{\varphi_2}(q_3)\, \delta(q_3 - \varphi_2(\varphi_1(q_3))) \ , \tag{4.12}$$

that is, we have the relation

$$\mu_{\varphi_1}(q')\mu_{\varphi_2}(q'') = \mu_{\varphi_3}(q'') \tag{4.13a}$$

and hence if Φ_0 is the identity transformation in P

$$\mu_{\varphi_0}(q) = 1 \ . \tag{4.13b}$$

Hence the function μ_φ has the properties of a multiplier[26, 27].

4.4 A Principal Series Property

The transformation matrix of $\Phi \epsilon P$ in the basis $\{|q>\}$ is

$$<q_1|\Phi|q_2> = <q_1|q_2>' =$$

$$= \delta(q_2 - \varphi(q_1))\mu_\varphi(q_2) = \delta(q_1 - \varphi^{-1}(q_2))\mu_{\varphi^{-1}}(q_1) \ , \tag{4.14}$$

and is unitary if (4.8) is satisfied.

We can examine the transformation undergone by the coordinates of a vector ψ. Indeed, under Φ,

$$<q|\psi> \overset{\Phi}{\rightarrow} {}'<q|\psi> = \int {}'<q|q_1> dq_1 <q_1|\psi>$$

$$= \int \delta(q_1 - \varphi^{-1}(q)) \, [\mu_\varphi(q)]^{-1} dq_1 \psi(q_1) \ , \tag{4.15a}$$

where we have used (4.9b). Hence, a function over q transforms as

$$\psi(q) \overset{\Phi}{\rightarrow} \Phi\psi(q) = [\mu_\varphi(q)]^{-1} \psi(\varphi^{-1}(q)) \ . \tag{4.15b}$$

In this way we have a multiplier representation of the group of quantum point transformations P on the space $\mathcal{L}^2(-\infty, \infty)$. We recall that if μ is a multiplier, so is any power of it. In particular, we can satisfy the condition of unitarity (4.8) proposing

$$\mu_\varphi(q) = [\omega_\varphi(q)]^{-\frac{1}{2}+i\lambda} \tag{4.16}$$

$$= [\omega_\varphi(q)]^{-\frac{1}{2}} \exp[i\lambda \ln \omega_\varphi(q)] \ , \tag{4.16}$$

for λ real. This has the advantage of establishing a direct connection with the principal series multiplier representations for semisimple groups[27] in terms of the Jacobian (4.4), i.e.

$$|dq/dq'| = \omega_\varphi(q') \ . \tag{4.17}$$

Two representations with different λ's differ thus by a q-dependent phase factor.

4.5 Ray Transformations and the Heisenberg-Weyl Group.

We want now to explore the freedom we have in (3.3) to add any function $\Gamma(\boldsymbol{Q})$ to $\boldsymbol{P}$ and still have a canonically conjugate operator to $\boldsymbol{Q}$, i.e. the ray transformation[25] $\boldsymbol{Q}, \boldsymbol{P} \to \boldsymbol{Q}, \boldsymbol{P} + \Gamma(\boldsymbol{Q})$ in R (Section 2.5) in terms of the multiplier representation introduced in the last section.

We can choose the realization (3.3) (the Schrödinger representation) of the Heisenberg-Weyl algebra as basic. When integrated to the group, $\boldsymbol{P}$ generates translations T_r by r as

$$\psi(q) \to T_r \psi(q) = \exp(ir\boldsymbol{P}/\hbar)\, \psi(q) = \psi(q+r) \ . \tag{4.18}$$

If we now add to $\boldsymbol{P}$ a function $\Gamma(\boldsymbol{Q})$ through a ray transformation in R, the integration to the group will be given by

$$\psi(q) \to T_r^{\Gamma}\psi(q) = \exp(ir[\boldsymbol{P} + \Gamma(\boldsymbol{Q})]/\hbar)\, \psi(q)$$

$$= \nu(q, r)\, \psi(q+r) \tag{4.19}$$

where the function $\nu(q, r)$ is a multiplier. As $T_0^{\Gamma} = \mathbf{1}$ we must have $\nu(q, 0) = 1$.

We can relate $\nu(q, r)$ with the generators of the transformation through

$$\left.\frac{\partial \nu(q,r)}{\partial r}\right|_{r=0} = \frac{i}{\hbar}\gamma(q) \,, \tag{4.21}$$

where as in Section 3, γ is the function corresponding to the operator function Γ. It is also straightforward to check that the translation group multiplication implies the multiplier composition law

$$\nu(q,r_1)\,\nu(q+r_1, r_2) = \nu(q, r_1 + r_2) = \nu(q, r_2)\,\nu(q + r_2, r_1) \;. \tag{4.22}$$

We can propose a general form which will satisfy (4.22) introducing the complex parameter σ as

$$\nu^{\sigma}(q,r) = [\rho(q+r)/\rho(q)]^{\sigma} \tag{4.23}$$

which, upon replacement in (4.21) relates ρ and γ through

$$\gamma(q) = -i\hbar\sigma \frac{d}{dq} \ln \rho(q) \,, \tag{4.24}$$

$$\rho(q) = \exp\left(i[\sigma\hbar]^{-1}\int dq\,\gamma(q)\right) \,. \tag{4.24b}$$

If in addition we ask the transformation (4.19) to be unitary, we come to the restriction that σ in (4.24) must be pure imaginary and hence ν^{σ} is a phase depending on $\sigma = i\tau$ i.e., for $g(q) = \ln \rho(q)$,

$$\begin{aligned}\psi(q) \to T_r^{\Gamma}\psi(q) &= \exp\left(ir[\mathbf{P} + \tau\Gamma(\mathbf{Q})]/\hbar\right)\psi(q) \\ &= \exp\left(i\tau[g(q+r) - g(q)]\right)\psi(q+r)\end{aligned} \tag{4.25a}$$

where

$$\gamma(q) = \hbar \frac{d}{dq} g(q) \,. \tag{4.25b}$$

Thus for $g(q)$ independent of q, $\Gamma(\mathbf{Q}) \equiv 0$, while if $g(q)$ is linear in q, $\gamma(q)$ = constant, and the phase factor in (4.25a) is q-independent. In general, however, the phase will be q-dependent.

4.6 The Conjugate Operators

The representation of $\mathbf{P}$ as a differential operator in terms of the eigenbasis space q' of $\mathbf{Q}'$ is the one obtained writing $\mathbf{P}$ through (3.3b) using the chain rule,

$$\mathbf{P} : \; -i\hbar \frac{d}{dq} = -i\hbar\epsilon \left[\omega_\varphi(q')\right]^{-1} \frac{d}{dq'} \tag{4.26a}$$

where the sign $\epsilon = +1$ is used for a monotonically increasing function φ and $\epsilon = -1$ for a decreasing one. The operator (4.26) continues to he hermitean under the scalar product (4.3) for the basis $\{|q)\}$. We can then pass to a new basis $\{|q>'\}$, orthonormal in the sense of Dirac, and then $\mathbf{P}$ will be represented as

$$\mathbf{P} : \; -i\hbar\epsilon \left[\omega_\varphi(q')\right]^{-\frac{1}{2}} \frac{d}{dq'} \left[\omega_\varphi(q')\right]^{-\frac{1}{2}} , \tag{4.26b}$$

which is manifestly hermitean under the measure dq'. Similarly, the representation of $\mathbf{P}'$ in (3.13) as a differential operator in the eigenbasis spaces q' of $\mathbf{Q}'$ and q of $\mathbf{Q}$ is

$$\mathbf{P}' : \; -i\hbar \frac{d}{dq'} = -i\hbar\epsilon\omega_\varphi(\varphi^{-1}(q)) \frac{d}{dq} \tag{4.27a}$$

which, when normalized, is represented by

$$\begin{aligned} \mathbf{P}' : \; & -i\hbar\epsilon \left[\omega_\varphi(\varphi^{-1}(q))\right]^{-\frac{1}{2}} \frac{d}{dq} \left[\omega_\varphi(\varphi^{-1}(q))\right]^{\frac{1}{2}} \\ & = -i\hbar\epsilon \, \tfrac{1}{2} \left\{\omega_\varphi(\varphi^{-1}(q)) \frac{d}{dq} + \frac{d}{dq}\omega_\varphi(\varphi^{-1}(q))\right\} \qquad (4.27b) \\ & = -i\hbar\epsilon\omega_\varphi(\varphi^{-1}(q)) \frac{d}{dq} - i\hbar\epsilon \, \tfrac{1}{2}\dot{\omega}_\varphi(\varphi^{-1}(q)) , \end{aligned}$$

hermitean under dq. This is to be compared with (3.13) for $\Gamma \equiv 0$. An implementation of the chain rule can thus be seen to be equivalent to the choice of the point transformation generated by (2.10) with $g \equiv 0$ and hence $\Gamma \equiv 0$. The addition of a purely imaginary function in the last equality in (4.27b) is to be understood as due to a change in the measure, and not as a ray transformation.

4.7 The Representation in the Conjugate Basis

The eigenbasis $\{|p>\}$ of $\mathbf{P}$, the operator conjugate to $\mathbf{Q}$, fulfilling

$$\mathbf{P}\,|p> = p\,|p> \ , \qquad p \in (-\infty, \infty) \ , \tag{4.28a}$$

is well known to be related, when properly normalized, with the eigenbasis of $\mathbf{Q}$ through

$$<q\,|\,p> = (2\pi\hbar)^{-\frac{1}{2}} \exp\,(ipq/\hbar) \ , \tag{4.28b}$$

when in the Schrödinger representation (3.3), orthogonal and complete in $\mathcal{L}^2(-\infty,\infty)$, and similar relations hold for the normalized eigenbasis $\{|q>'\}$ and $\{|p>'\}$ of $\mathbf{Q}'$ and $\mathbf{P}'$. Use of (4.4b), (4.7c), (4.9a), (4.16) and the completeness of the bases yields

$$<q\,|\,p>' = (2\pi\hbar)^{-\frac{1}{2}} \left[\omega_{\varphi}(\varphi(q))\right]^{-\frac{1}{2}+i\lambda} \exp\,(ip\varphi(q)/\hbar) \tag{4.29a}$$

$$'<q\,|\,p> = (2\pi\hbar)^{-\frac{1}{2}} \left[\omega_{\varphi^{-1}}(\varphi^{-1}(q))\right]^{-\frac{1}{2}+i\lambda} \exp\,(ip\varphi^{-1}(q)/\hbar) \tag{4.29b}$$

and

$$\begin{aligned} <p_1|\,\Phi\,|p_2> &= <p_1\,|\,p_2>' = \\ &= (2\pi\hbar)^{-1}\int dq\, \exp\,(-ip_1 q/\hbar)\left[\omega_{\varphi}(\varphi(q))\right]^{-\frac{1}{2}+i\lambda} \exp\,(ip_2\varphi(q)/\hbar) \\ &= (2\pi\hbar)^{-1}\int dq\, \exp\,(-ip_1\varphi^{-1}(q)/\hbar)\left[\omega_{\varphi^{-1}}(\varphi^{-1}(q))\right]^{-\frac{1}{2}+i\lambda} \exp\,(ip_2 q/\hbar) \ , \end{aligned} \tag{4.30}$$

is the representation of the point transformation Φ in the conjugate basis.

It is to be remarked that (4.30) is an analogue (p being continuous rather than discrete) of Bargmann's integral formula (with $\varphi_\tau(q) = 2 \arctan (e^\tau \tan [q/2])$, compare with the form (2.12c)) for the matrix elements of the principal series of $SO(2,1)$ group[26, 27], the Bargmann $d^{\lambda}_{mm'}(\tau)$ functions. The multiplier phase given by λ in (4.16) is thus seen to be of determinant importance A supplementary series of representations can be obtained letting the additive function to $\boldsymbol{P}'$ in (4.27b) to be a real function stemming from a true ray transformation and allowing the completeness relation to read

$$\iint |p> dp\Omega(p,p')\, dp' <p'| = \mathbf{1} , \tag{4.31}$$

thereby introducing a non-local measure in the conjugate p-space. This problem is under unvestigation[9].

5. CONCLUSION

We have used the Heisenberg-Weyl algebra and its realizations to frame the role of canonical transformations in Quantum Mechanics, leading to the mapping of a physical problem into a mathematically simpler system with a linear or constant-density spectrum with a known dynamical algebra, and we have found unitary principal-series representations for the group of point transformations. These have been shown to be essentially a continuous generalized analogue of the integral formulae obtained for the representation matrix elements of non-compact classical groups as groups of deformations (point transformations) of homogeneous spaces as spheres, hyperboloids and similar coset spaces.

The usefulness of the formalism presented here must be justified, however, with concrete applications to definite pairs of "physical" systems and their corresponding dynamical algebras and groups. This will be done in future publications where the systems to be related will be the point rotor, the harmonic oscillator and the (pseudo-) Coulomb potential. All three are classically related by pairs through canonical and point transformations and exhibit $SO(2,1)$ as their dynamical group. That "not all is well" quantum mechanically, is obvious from the fact that their spectra are different. A thorough understanding of the role of canonical transformations in Quantum Mechanics should account for these differences as stemming from the fact that these systems belong to different representations of the dynamical algebra and provide the proper quantum passage from one system to another.

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RESUMEN

Se presentan la definición y propiedades de las transformaciones canónicas clásicas y cuánticas, desde el punto de vista de un programa actual de investigación, que se lleva a cabo sobre el papel de las transformaciones canónicas en Mecánica Cuántica. Se exploran los grupos y las álgebras infinitesimales correspondientes. El subgrupo de transformaciones puntuales (es decir transformaciones canónicas entre pares de observables conjugadas, donde una observable de cada par es función sólo de una del otro par), es de interés especial, ya que es un grupo de transformaciones para el que la mecánica clásica y la mecánica cuántica dan resultados iguales. Sólo el grupo de transformaciones simplécticas inhomogéneas tiene características similares. Las transformaciones puntuales se tratan en detalle y se construye una serie principal de representaciones unitarias.