# BASIS FUNCTIONS FOR THE NUCLEAR THREE-BODY PROBLEM*

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ABSTRACT: Closed algebraic expressions are given for complete (but non-orthogonal) sets of functions, adequate for the quantum mechanical description of the translationally invariant orbital motion of three identical particles. The functions appear as linear combinations of hyperspherical harmonics with good permutational and orbital angular momentum symmetries. Both the two-dimensional and the three-dimensional problems are discussed.

## I. INTRODUCTION

The subject of the nuclear three-body problem has attracted the attention of a great number of investigators, both theoreticians and experimentalists, in the last few years. On the theoretical side of the problem, attempts have been made to obtain the wave function of the ground state of the bound system by a variational method. There are two basic requirements that any

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trial wave function should satisfy. One is that it must depend on two relative vectors (elimination of the motion of the center of mass); in this connection a suitable pair of relative vectors are the Jacobi vectors that will be defined in Section II. The second requirement is the observance of the exclusion principle; this can be accomplished by working with antisymmetric functions that are an appropriate sum of products of an orbital function with a definite permutational symmetry, times a spin-isospin function with the conjugate symmetry.

A complete orthonormal set of orbital functions permutationally adapted, was constructed by M. Moshinsky[1], in terms of harmonic oscillator functions. Even if the oscillator functions seem to be doing well in nuclear physics, it would be desirable to have other sets of symmetry adapted orbital functions at one's disposal. An alternative that has been widely discussed in the literature, consists of orbital functions which are a product of a hyper-radial function depending on the six-dimensional radius, times a hyperspherical harmonic depending on five angles.

In this paper we shall implement a method, originally proposed by Dragt[2], that will allow us to obtain, starting from the results of ref. (1) for oscillator functions, general formulas for hyperspherical harmonics with good permutational symmetry. This analysis will be presented in Section IV. We think that the essence of the method can be more easily appreciated by presenting the detailed calculations at a more elementary level. For this reason we discuss in Sections II and III what we call a model problem: a system of three bodies on a plane, and deduce for this simpler problem all the results that have a correspondence in the real problem. The two-dimensional problem was analyzed long ago by Smith[11], but he did not discuss the permutational aspect of the problem, in the case of identical particles.

## II. THE MODEL PROBLEM

The internal motion of three particles on a plane can be described by the two Jacobi relative vectors $\dot{r}_1$, $\dot{r}_2$, defined in terms of the physical position vectors $r_1, r_2, r_3$ as

$$\dot{r}_1 = (1/\sqrt{2})(r_1 - r_2) \; ; \quad \dot{r}_2 = (1/\sqrt{6})(r_1 + r_2 - 2r_3) \; . \qquad \text{(II.1)}$$

Then the Schrödinger equation for the internal motion, supposing only twobody interactions, reads

$$\left[-\frac{\hbar^2}{2m}\sum_{\mu s}\frac{\partial^2}{\partial \dot{x}_{\mu s}^2}+V_{12}+V_{23}+V_{31}\right]\Psi=E\Psi\ , \tag{II.2}$$

where $\dot{x}_{\mu s}$, $\mu=1,2$, are the two cartesian components of the Jacobi vector $\dot{r}_s$ $(s=1,2)$. It is generally impossible to find exact solutions of eq. (II.2) for physically interesting potentials $V_{ij}$. Therefore, we shall try to find an approximate ground state eigenfunction and eigenvalue of (II.2) by means of the variation method. The completely antisymmetric trial function is constructed as a linear combination of products of an orbital function $\Phi(\dot{r}_1, \dot{r}_2)$ times a spin-isospin function $G(\sigma^1\sigma^2\sigma^3\tau^1\tau^2\tau^3)$:

$$\Psi=\sum_\nu C_\nu^f\sum_r(-)^r\,\Phi_\nu(fr)\,G(\tilde{f}\tilde{r}) \tag{II.3}$$

Here $f=\{f_1f_2f_3\}$ a partition of the number 3, and $r=(r_3r_2r_1)$ the Yamanouchi symbol[3], denote the permutational symmetry of a function; it is known[3] that $\Phi$ and $G$ must possess conjugate (or associate) permutational symmetries, we indicate the symmetry conjugate to $fr$ by $\tilde{f}\tilde{r}$. The $C_\nu$ in eq. (II.3) play the role of variational parameters and they are determined by the usual method of solving the secular problem associated with eq. (II.2). The spin-isospin part of the trial function is well known, (see for instance, ref. (4)), so we shall discuss in what follows only the orbital part $\Phi(r_1, r_2)$.

Let us introduce four-dimensional hyperspherical coordinates $(\rho,\beta,\phi_1,\phi_2)$ for the Jacobi vectors, in the following way

$$\rho=\sqrt{\dot{r}_1^2+\dot{r}_2^2}\ ,\quad \dot{r}_1=\rho\cos(\beta/2)\ ,\quad \dot{r}_2=\rho\sin(\beta/2) \tag{II.4}$$

$$\dot{x}_{1s}+i\,\dot{x}_{2s}=\dot{r}_s e^{i\phi_s}\ ,\quad s=1,2 \tag{II.5}$$

the range of the angles being $0\leqslant\beta\leqslant\pi$, $0\leqslant\phi_s\leqslant 2\pi$. We shall select for $\Phi(\dot{r}_1, \dot{r}_2)$ a function of the type

$$\Phi(\dot{r}_1,\dot{r}_2)=R(\rho)\,Y(\beta,\phi_1,\phi_2)\ , \tag{II.6}$$

where the angular function is a hyperspherical harmonic and satisfies the equation[5]

$$-\left[4\frac{\partial^2}{\partial\beta^2} + 4\cot\beta\frac{\partial}{\partial\beta} + \frac{1}{\sin^2\frac{\beta}{2}}\frac{\partial^2}{\partial\phi_2^2} + \frac{1}{\cos^2\frac{\beta}{2}}\frac{\partial^2}{\partial\phi_1^2}\right]Y = K(K+2)\,Y\,. \tag{II.7}$$

The solutions of this equation can be expressed in terms of the representation functions[6] of the group $SU(2)$,

$$Y_{Km_1m_2}(\phi_1,\beta,\phi_2) = \sqrt{\frac{K+1}{8\pi^2}}\;\mathcal{D}^{K/2}_{\frac{m_1+m_2}{2},\frac{m_1-m_2}{2}}(\phi_1+\phi_2,\beta,\phi_1-\phi_2)$$

$$= \sqrt{\frac{K+1}{8\pi^2}}\;e^{i(m_1\phi_1+m_2\phi_2)}\,d^{K/2}_{\frac{m_1+m_2}{2},\frac{m_1-m_2}{2}}(\beta)\;. \tag{II.8}$$

These hyperspherical harmonics constitute a complete orthonormal set in the following sense

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{2\pi} Y^*_{K'm_1'm_2'}\,Y_{Km_1m_2}\sin\beta\,d\beta\,d\phi_1\,d\phi_2 = \delta_{KK'}\,\delta_{m_1m_1'}\,\delta_{m_2m_2'}$$

$$K = 0,1,2,\ldots\;;\quad m_1, m_2 = K, K-2, K-4,\ldots, -K\;. \tag{II.9}$$

One of the advantages of the orbital functions, as given in eq. (II.6), is that the internal kinetic energy operator is diagonal with respect to them in the quantum numbers $K, m_1, m_2$ ; namely[5]

$$\sum_{\mu s}\frac{\partial^2}{\partial \dot{x}^2_{\mu s}}R(\rho)\,Y_{Km_1m_2} = Y_{Km_1m_2}\left[\frac{d^2R}{d\rho^2} + \frac{3}{\rho}\frac{dR}{d\rho} - \frac{K(K+2)R}{\rho^2}\right]\,. \tag{II.10}$$

Another important advantage, is that they allow an efficient evaluation of the matrix elements of the interaction potential through the use of angular momentum techniques. For instance, in the simple case of a central potential,

$V_{12}(|r_1 - r_2|) = V_{12}(\sqrt{2}\rho \cos(\beta/2))$, we can make an expansion in a series of Legendre polynomials (a multipole expansion)

$$V_{12}(\sqrt{2}\rho \cos\frac{\beta}{2}) = \sum_l f_l(\rho)\, d^l_{00}(\beta) \qquad \text{(II.11)}$$

and then the matrix elements of $V_{12}$ are expressed as a finite sum of integrals of a product of three $d(\beta)$ functions, for which a closed formula exists[6]. As for the radial functions themselves, in the case of bound three-body systems, a convenient choice would be a set of orthonormal functions decaying to zero as $\rho$ goes to infinity; for instance, Laguerre functions[7].

From the definition of $\rho$ and of the Jacobi vectors, it is clear that $\rho$ is invariant under permutations of the three particles. Therefore the permutational properties of the orbital function $\Phi(\dot{r}_1, \dot{r}_2)$ will be wholly contained on the hyperspherical harmonics. Thus the main problem to be solved now is the construction of linear combinations of $Y_{Km_1m_2}$ possessing a good permutation symmetry. We shall do this in the next section, following the idea of Dragt[2] in which one starts working with harmonic oscillators, but then this restriction is removed and general results are obtained.

## III. THE MODEL PROBLEM. HYPERSPHERICAL HARMONICS WITH GOOD PERMUTATION SYMMETRY.

We shall begin with an analysis of harmonic oscillator functions of the two Jacobi vectors $\dot{r}_1, \dot{r}_2$, in which case enforcing the permutational symmetry is relatively easy[1]. As the analysis to follow is so similar to that presented in ref. (1), we shall only sketch the main steps. We choose a system of units such that $m = \omega = \hbar = 1$.

The harmonic oscillator states can be expressed very conveniently in terms of a polynomial function of the creation operators

$$\dot{\eta}_s = (1/\sqrt{2})(\dot{r}_s - i\dot{p}_s) \qquad s = 1, 2 \qquad \text{(III.1)}$$

acting on the ground state $|0\rangle$; we can also define the annihilation operators $\dot{\xi}_s = \dot{\eta}_s^+$. For our problem it is rather more useful to introduce the "spherical" components of the vectors, namely $\dot{\eta}_{\sigma s}$, $\sigma = +, -$, defined as

$$\dot{\eta}_{\pm s} = (1/\sqrt{2})(\dot{\eta}_{1s} \pm i\dot{\eta}_{2s}) \ . \qquad \text{(III.2)}$$

The two particle oscillator states are then

$$|\dot{N}_1\dot{m}_1\dot{N}_2\dot{m}_2> = \dot{A}(\dot{\eta}_{+1})^{\frac{\dot{N}_1+\dot{m}_1}{2}}(\dot{\eta}_{-1})^{\frac{\dot{N}_1-\dot{m}_1}{2}}(\dot{\eta}_{+2})^{\frac{\dot{N}_2+\dot{m}_2}{2}}(\dot{\eta}_{-2})^{\frac{\dot{N}_2-\dot{m}_2}{2}}|0> \qquad \text{(III.3)}$$

where $\dot{A}$ is a normalization coefficient whose value is

$$\dot{A} = (-)^{\frac{1}{2}(\dot{N}_1+\dot{N}_2-\dot{m}_1-\dot{m}_2)}\left[\left(\frac{\dot{N}_1+\dot{m}_1}{2}\right)!\left(\frac{\dot{N}_1-\dot{m}_1}{2}\right)!\left(\frac{\dot{N}_2+\dot{m}_2}{2}\right)!\left(\frac{\dot{N}_2-\dot{m}_2}{2}\right)!\right]^{-\frac{1}{2}} \qquad \text{(III.4)}$$

The states (III.3) correspond to a number of quanta of energy $\dot{N}_s$ and an angular momentum $\dot{m}_s$ for the $s$th oscillator ($s = 1, 2$). From a group-theoretical standpoint they belong to a basis for an irreducible representation of the chain of groups $U(4) \supset U(2) \times U(2) \supset O(2) \times O(2)$. It is clear that the function of $\dot{\eta}_{\sigma s}$ in eq. (III.3) is a polynomial only if $\dot{N}_s = 0, 1, 2, \ldots$ and $\dot{m}_s = \dot{N}_s, \dot{N}_s - 2, \ldots, -\dot{N}_s$.

Following the analysis of ref. (1), we introduce a particular combination of creation operators

$$\ddot{\eta}_1 = (1/\sqrt{2})(-i\dot{\eta}_1 + \dot{\eta}_2) \ , \qquad \ddot{\eta}_2 = (1/\sqrt{2})(i\dot{\eta}_1 + \dot{\eta}_2) \ . \qquad \text{(III.5)}$$

If we denote the states (III.3) as $P(\dot{N}_1\dot{m}_1\dot{N}_2\dot{m}_2)|0>$, let us consider a set of polynomials $P(N_1 m_1 N_2 m_2)$ with exactly the same structure as the previous $P$, but only constructed in terms of $\ddot{\eta}_s$ instead of $\dot{\eta}_s$. Then we find that the permutations (1, 2) and (1, 2, 3) have the following effect on the polynomials in $\ddot{\eta}_s$:

$$(1,2)\ P(N_1 m_1 N_2 m_2) = P(N_2 m_2 N_1 m_1) \qquad \text{(III.6)}$$

$$(1,2,3)\ P(N_1 m_1 N_2 m_2) = \omega^{N_1 - N_2} P(N_1 m_1 N_2 m_2)\ ;\ \omega = \exp(\tfrac{2}{3} i\pi) \ . \qquad \text{(III.7)}$$

Using projection operators of $S(3)$, as in ref. (1), we deduce that the polynomial

$$(1/\sqrt{2})\,[P(N_1 m_1 N_2 m_2) + (-)^{\epsilon} P(N_2 m_2 N_1 m_1)] \qquad \text{(III.8)}$$

has the permutational symmetry defined by the partition $\{f_1 f_2 f_3\}$ and Yamanouchi symbol $(r_3 r_2 r_1)$ indicated in the following table

| $\epsilon$ | $N_1 - N_2$ | $\{f\}$ | $(r)$ |
|---|---|---|---|
| 0 | 1, 2 mod 3 | {21} | (211) |
| 1 | 1, 2 mod 3 | {21} | (121) |
| 0 | 0 mod 3 | {3} | (111) |
| 1 | 0 mod 3* | {111} | (321) |

(III.9)

To express the polynomial (III.8) in terms of Jacobi vectors, we proceed as in ref. (1), and arrive to the result

$$P(N_1 m_1 N_2 m_2) =$$

$$= \sum_{\dot{N}_1 \dot{m}_1 \dot{N}_2 \dot{m}_2} \langle \dot{N}_1 \dot{m}_1 \dot{N}_2 \dot{m}_2 \,|\, N_1 m_1 N_2 m_2 \rangle (-i)^{N_i} P(\dot{N}_1 \dot{m}_1 \dot{N}_2 \dot{m}_2), \qquad \text{(III.10)}$$

where the bracket is a harmonic oscillator transformation bracket[8] (commonly referred to in the literature as Moshinsky brackets). In Appendix A we shall show that for the two-dimensional problem the Moshinsky brackets are just a product of two representation functions $d^j_{m,m'}(\pi/2)$ of the group $SU(2)$, namely

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* When $N_1 = N_2$ and $m_1 = m_2$, only the symmetric state exists, and then the multiplicative factor in (III.8) should be $\frac{1}{2}$ instead of $1/\sqrt{2}$.

$$\langle \dot{N}_1 \dot{m}_1 \dot{N}_2 \dot{m}_2 | N_1 m_1 N_2 m_2 \rangle = \delta_{N_1+N_2,\, \dot{N}_1+\dot{N}_2}\, \delta_{m_1+m_2,\, \dot{m}_1+\dot{m}_2} \times$$

$$\times\, d^{\frac{1}{4}(N_1+N_2+m_1+m_2)}_{\frac{1}{4}(N_1-N_2+m_1-m_2),\, \frac{1}{4}(\dot{N}_1-\dot{N}_2+\dot{m}_1-\dot{m}_2)}(\pi/2) \times$$

$$\times\, d^{\frac{1}{4}(N_1+N_2-m_1-m_2)}_{\frac{1}{4}(N_1-N_2-m_1+m_2),\, \frac{1}{4}(\dot{N}_1-\dot{N}_2-\dot{m}_1+\dot{m}_2)}(\pi/2) \,. \tag{III.11}$$

Using properties of the $d$ functions[6] we deduce the symmetry relation

$$\langle \dot{N}_1 \dot{m}_1 \dot{N}_2 \dot{m}_2 | N_1 m_1 N_2 m_2 \rangle = (-)^{\dot{N}_1} \langle \dot{N}_1 \dot{m}_1 \dot{N}_2 \dot{m}_2 | N_2 m_2 N_1 m_1 \rangle \;,$$

and therefore the oscillator functions with definite permutational symmetry which are given in (III.8) can be written in general as

$$\Phi(N_1 m_1 N_2 m_2,\, fr) =$$

$$= \sqrt{2} \sum_{\dot{N}_1 \dot{m}_1 \dot{N}_2 \dot{m}_2} \langle \dot{N}_1 \dot{m}_1 \dot{N}_2 \dot{m}_2 | N_1 m_1 N_2 m_2 \rangle (-)^{\frac{1}{2}(\dot{N}_1+\epsilon)}\, P(\dot{N}_1 \dot{m}_1 \dot{N}_2 \dot{m}_2) | 0 \rangle \;, \tag{III.12}$$

where the relation of $f$, $r$ with $N_1$, $N_2$, $\epsilon$ is given in table (III.9), and the summation over $\dot{N}_1$ runs only over even values when $\epsilon = 0$, or over odd values when $\epsilon = 1$.

We have thus in eq. (III.12) the complete explicit solution for oscillator states with good permutational symmetry. Let us see how can we use these results to obtain the hyperspherical harmonics of Section II, with definite permutational symmetry. For this purpose let us note that there is an alternative expression for the two particle oscillator states, namely

$$|NK\dot{m}_1\dot{m}_2> =$$

$$\sum_{\nu} B_{\nu}(\dot{\eta}_{+1})^{\dot{m}_1}(\dot{\eta}_{+2})^{\dot{m}_2+\nu}(\dot{\eta}_{-2})^{\nu}(\dot{\eta}_{+1}\dot{\eta}_{-1}+\dot{\eta}_{+2}\dot{\eta}_{-2})^{\frac{1}{2}(N-\dot{m}_1-\dot{m}_2)^{-\nu}}\ |0> \tag{III.13}$$

with the coefficient $B_{\nu}$ being given by

$$B_{\nu} = \frac{(-)^{\frac{N-K}{2}+\nu}\left(\frac{K+\dot{m}_1+\dot{m}_2}{2}+\nu\right)!}{\nu!(m_2+\nu)!\left(\frac{K-\dot{m}_1-\dot{m}_2}{2}-\nu\right)!}\left[(K+1)\left(\frac{K-\dot{m}_1-\dot{m}_2}{2}\right)!\left(\frac{K-\dot{m}_1+\dot{m}_2}{2}\right)!\right]^{\frac{1}{2}} \times$$

$$\times\left[\left(\frac{N-K}{2}\right)!\left(\frac{N+K}{2}+1\right)!\left(\frac{K+\dot{m}_1+\dot{m}_2}{2}\right)!\left(\frac{K+\dot{m}_1-\dot{m}_2}{2}\right)!\right]^{-\frac{1}{2}}. \tag{III.14}$$

These states correspond to a number $N = \dot{N}_1 + \dot{N}_2$ of quanta of energy, an orbital angular momentum $\dot{m}_s$ for the $s$th oscillator ($s = 1, 2$), and a value $K(K+2)$ of the operator on the LHS of eq. (II.7)(which is the square of the generalized four-dimensional angular momentum). From a group-theoretical viewpoint the states (III.13) belong to a basis for an irreducible representation of the chain of groups $U(4) \supset O(4) \supset O(2)\times O(2)$. From the two previous equations it can be seen that the function of $\dot{\eta}_{\sigma s}$ in eq. (III.13) will be a polynomial only when $N = 0, 1, 2, \ldots$; $K = N, N-2, \ldots, 1$ or $0$; and $\dot{m}_1 \pm \dot{m}_2 = K, K-2, \ldots, -K$.

Taking the scalar product of (III.3) with (III.13) we obtain the transformation coefficient between the oscillator states classified by $U(2)\times U(2)$ and by $O(4)$, namely

$$(N_1m_1N_2m_2|NKm_1'm_2') = \delta_{m_1m_1'}\delta_{m_2m_2'}\delta_{N,N_1+N_2}\times$$

$$\times\sum_s \frac{(-)^s\left(\frac{K+m_1+m_2}{2}+s\right)!\left(\frac{N-m_1-m_2}{2}-s\right)!}{s!\,(m_2+s)!\left(\frac{K-m_1-m_2}{2}-s\right)!\left(\frac{N_2-m_2}{2}-s\right)!}\times \tag{III.15}$$

$$\times(-)^{\frac{K-m_1-m_2}{2}}\left[\frac{(K+1)\left(\frac{K-m_1-m_2}{2}\right)!\left(\frac{K-m_1+m_2}{2}\right)!\left(\frac{N_1+m_1}{2}\right)!\left(\frac{N_2+m_2}{2}\right)!\left(\frac{N_2-m_2}{2}\right)!}{\left(\frac{N-K}{2}\right)!\left(\frac{N+K}{2}+1\right)!\left(\frac{K+m_1+m_2}{2}\right)!\left(\frac{K+m_1-m_2}{2}\right)!\left(\frac{N_1-m_1}{2}\right)!}\right]^{\frac{1}{2}}.$$

Using this coefficient we can express the states on the RHS of eq. (III.12) in terms of $O(4)$ states, i.e.,

$$\Phi(N_1 m_1 N_2 m_2, fr) =$$

$$= \sqrt{2} \sum_{\dot{N}_1 \dot{N}_2 \dot{m}_1 \dot{m}_2} (-)^{\frac{\dot{N}_1+\epsilon}{2}} < \dot{N}_1 \dot{m}_1 \dot{N}_2 \dot{m}_2 | N_1 m_1 N_2 m_2 > \times$$

$$\times (NK\dot{m}_1 \dot{m}_2 | \dot{N}_1 \dot{m}_1 \dot{N}_2 \dot{m}_2) | NK\dot{m}_1 \dot{m}_2 > . \qquad \text{(III.16)}$$

If we transform now $| \dot{N}_1 \dot{m}_1 \dot{N}_2 \dot{m}_2 >$ back to the Jacobi configuration space vectors, we obtain a result of this sort

$$\Phi(N_1 m_1 N_2 m_2, fr) = \sum_K R_{NK}(\rho) \, Z_{Kfr}^{N_1 m_1 N_2 m_2}(\phi_1, \beta, \phi_2). \qquad \text{(III.17)}$$

The angular function $Z$ is defined as

$$Z_{Kfr}^{N_1 m_1 N_2 m_2}(\phi_1, \beta, \phi_2) = \sum_{\dot{m}_1 \dot{m}_2} C_{\dot{m}_1 \dot{m}_2}^{N_1 m_1 N_2 m_2}(Kfr) \, Y_{K\dot{m}_1 \dot{m}_2}(\phi_1, \beta, \phi_2)$$

$$\text{(III.18)}$$

with

$$C_{\dot{m}_1 \dot{m}_2}^{N_1 m_1 N_2 m_2}(Kfr) =$$

$$\sum_{\dot{N}_1 \dot{N}_2} < \dot{N}_1 \dot{m}_1 \dot{N}_2 \dot{m}_2 | N_1 m_1 N_2 m_2 > (-)^{\frac{\dot{N}_1+\epsilon}{2}} (NK\dot{m}_1 \dot{m}_2 | \dot{N}_1 \dot{m}_1 \dot{N}_2 \dot{m}_2) \qquad \text{(III.19)}$$

At this point we follow the reasoning of ref. (2). It is shown there that the four dimensional Laplacian operator $\nabla_4^2$ acting on a harmonic oscillator wave function with $N$ quanta of energy: $\Psi_{N, K, \dot{m}_1 \dot{m}_2}$, produces a multiple of $\Psi_{N-2, K, \dot{m}_1, \dot{m}_2}$. From this fact, and from the identity

$$\rho^2 \nabla_4^2 = \rho \frac{\partial}{\partial \rho}\left( \rho \frac{\partial}{\partial \rho} + 2\right) - \Lambda^2 \tag{III.20}$$

where $\Lambda^2$ is the operator on the left hand side of (II.7), it follows that all the $Z$ with given values of $K, f, r$ but with $N = N_1 + N_2 = K, K+2, K+4, \ldots$ are proportional to each other. We can thus obtain a minimal set of $Z$ if we just include those $Z$ with $N = N_1 + N_2 = K$. This set is still overcomplete because some $Z$ functions in it are proportional to other $Z$ with $K = N - 2, N - 4, \ldots, 1$ or $0$, which occurred earlier in a step-by-step counting procedure. A rule to obtain a complete system is to delete for $K = N_1 + N_2$ all sets $(N_1 m_1 N_2 m_2)\,\{f\}$ for which the set $(N_1 - 1, m_1 - 1, N_2 - 1, m_2 + 1)\,\{f\}^2$ occurred at an earlier step when $K = N_1 + N_2 - 2$.

Thus we have in eq. (III.18) the solution we were looking for: the functions defined by eq. (III.18) constitute a complete system of hyperspherical harmonics with good quantum numbers $K, f, r$, if we consider only the cases of $N_1 + N_2 = K$ and the sets $(N_1 m_1 N_2 m_2)\,\{f\}$ obeying the rule of the preceeding paragraph.

When $N = N_1 + N_2 = K$, the transformation coefficient of (III.15) simplifies to a monomial and the detailed formula for the coefficient appearing in (III.18) is

$$C^{N_1 m_1 N_2 m_2}_{\dot{m}_1 \dot{m}_2}(K/r) = (-)^{\frac{\dot{m}_1 + \epsilon}{2}} \delta_{m_1 + m_2, \dot{m}_1 + \dot{m}_2} \sum_{\dot{N}_1 \dot{N}_2} \delta_{N_1 + N_2, \dot{N}_1 + \dot{N}_2} \begin{pmatrix} \frac{K + \dot{m}_1 + \dot{m}_2}{2} \\ \frac{\dot{N}_1 + \dot{m}_1}{2} \end{pmatrix}^{\frac{1}{2}} \times$$

$$\times \begin{pmatrix} \frac{K - m_1 - m_2}{2} \\ \frac{\dot{N}_1 - \dot{m}_1}{2} \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} K \\ \frac{K + m_1 - m_2}{2} \end{pmatrix}^{\frac{1}{2}} d^{\frac{1}{4}(N_1 + N_2 + m_1 + m_2)}_{\frac{1}{4}(N_1 - N_2 + m_1 - m_2),\, \frac{1}{4}(\dot{N}_1 - \dot{N}_2 + \dot{m}_1 - \dot{m}_2)}\left(\frac{\pi}{2}\right) \times$$

$$\times\, d^{\frac{1}{4}(N_1 + N_2 - m_1 - m_2)}_{\frac{1}{4}(N_1 - N_2 - m_1 + m_2),\, \frac{1}{4}(\dot{N}_1 - \dot{N}_2 - \dot{m}_1 + \dot{m}_2)}\left(\frac{\pi}{2}\right) . \tag{III.21}$$

We recall that when the $\epsilon$ of table (III.9) is 0 the summation over $\dot{N}_1$ runs over even values, and when $\epsilon = 1$ the summation is over $\dot{N}_1$ odd.

## IV. THE THREE-DIMENSIONAL PROBLEM

The steps followed in Section III to obtain hyperspherical harmonics with good permutational symmetry for use in the quantum mechanical problem of three bodies in a plane, can be repeated now for the physically realistic three-body problem in space. We shall give next the essential results, stressing the parallelism with the corresponding results of Section III.

The harmonic oscillator functions in the Jacobi coordinates $\dot{r}_1$, $\dot{r}_2$, and with definite permutational symmetry $\{f\}(r)$, orbital angular momentum $L$ and projection $M$, were obtained by Moshinsky et al[1]. They are given by

$$\Phi(n_1 l_1 n_2 l_2 LMfr) = \sum_{\dot{n}_1 \dot{l}_1 \dot{n}_2 \dot{l}_2} (-)^{\dot{n}_1} B(\dot{l}_1, n_1 l_1 n_2 l_2, fr) \langle \dot{n}_1 \dot{l}_1 \dot{n}_2 \dot{l}_2 L \,|\, n_1 l_1 n_2 l_2 L \rangle \times$$

$$\times \; [\Psi_{\dot{n}_1 \dot{l}_1}(\dot{r}_1) \Psi_{\dot{n}_2 \dot{l}_2}(\dot{r}_2)]_{LM} \; . \qquad \text{(IV.1)}$$

On the right hand side of this formula, $B$ is essentially a phase factor given in ref. (1); the next bracket is a harmonic oscillator transformation bracket[8] (i. e. a Moshinsky bracket) for whose evaluation tables or computer programs are available; and the last factor is the vector coupled product of two oscillator functions in the Jacobi vectors $\dot{r}_1$, $\dot{r}_2$.

These vector coupled functions correspond to a classification scheme by the chain of groups

$$U(6) \supset U(3)^{(1)} \times U(3)^{(2)} \supset O(3)^{(1)} \times O(3)^{(2)} \supset O(3) \supset O(2) \; . \qquad \text{(IV.2)}$$

In analogy with the two-dimensional problem we can introduce oscillator functions corresponding to a classification by the chain of groups

$$U(6) \supset O(6) \supset O(3)^{(1)} \times O(3)^{(2)} \supset O(3) \supset O(2) \; . \qquad \text{(IV.3)}$$

We shall denote these functions as[9]

$$\Psi_{NKl_1l_2LM}(\vec{r}_1, \vec{r}_2) = R_{NK}(\vec{\rho})\, Y_{Kl_1l_2LM}(\alpha, \theta_1, \phi_1, \theta_2, \phi_2)$$

$$= R_{NK}(\rho)\, F_{Kl_1l_2}(\alpha)\, [Y_{l_1}(\theta_1, \phi_1)\, Y_{l_2}(\theta_2, \phi_2)]_{LM}\ , \tag{IV.4}$$

where $\vec{r}_1 = \rho \cos\alpha$, $\vec{r}_2 = \rho \sin\alpha$, and $(\theta_s, \phi_s)$ are the polar angles of $\vec{r}_s$. $s = 1, 2$. The range of the indices is $N = 0, 1, 2, \ldots$; $K = N, N-2, N-4, \ldots, 1$ or 0, and $K \geqslant l_1 + l_2 \geqslant L \geqslant |l_1 - l_2|$. We call the $Y_{Kl_1l_2LM}$ hyperspherical harmonics for the spatial three-body problem; the explicit form of $F(\alpha)$ is given in ref. 9.

We must now calculate the transformation coefficient between the states in the chains (IV.2) and (IV.3), namely $(NKl_1l_2|n_1l_1n_2l_2)$. Just as in the two-dimensional problem however, we shall not need the most general coefficients of this type but only those with $K = N$; these particular coefficients were calculated by Raynal and Revai[10], we shall introduce them later in eq. (IV.7).

Passing then in eq. (IV.1) from $U(3)^{(1)} \times U(3)^{(2)}$ states to $O(6)$ states, we arrive at a result of this sort

$$\Phi(n_1l_1n_2l_2LM, fr) = \sum_K R_{NK}(\rho)\, Z^{n_1l_1n_2l_2}_{KLMfr}(\alpha, \theta_1, \phi_1, \theta_2, \phi_2) \tag{IV.5}$$

whe.. the angular function $Z$ is defined as

$$Z^{n_1l_1n_2l_2}_{KLMfr}(\alpha, \theta_1, \phi_1, \theta_2, \phi_2) =$$

$$= \sum_{\dot{l}_1\dot{l}_2} B(\dot{l}_1, n_1l_1n_2l_2fr)\, C^{n_1l_1n_2l_2}_{\dot{l}_1\dot{l}_2}(KLfr)\, Y_{K\dot{l}_1\dot{l}_2LM}(\alpha, \theta_1, \phi_1, \theta_2, \phi_2) \tag{IV.6}$$

with

$$C^{n_1l_1n_2l_2}_{\dot{l}_1\dot{l}_2}(KLfr) = \sum_{\dot{n}_1\dot{n}_2} (-)^{\dot{n}_1}(NK\dot{l}_1\dot{l}_2|\dot{n}_1\dot{l}_1|\dot{n}_2\dot{l}_2) < \dot{n}_1\dot{l}_1\dot{n}_2\dot{l}_2L|n_1l_1n_2l_2L > . \tag{IV.7}$$

By a reasoning similar to that following eq. (III.19), we obtain an (over-) complete set of angular functions with good quantum numbers $K, L, M, f, r$ if we take all those $Z$ of (IV.6) with $2n_1 + l_1 + 2n_2 + l_2 = N = K$; and in order to obtain a complete set of independent functions we follow the rule of taking into account only those quartets $(n_1 l_1 n_2 l_2)$ for which $l_1 + l_2 = L + \frac{1}{2}\,[1 - (-)^{N-L}]$.

For the case $N = K$, taking the coefficient $(NK\,|\,\dot{n}_1\dot{n}_2)_{\dot{l}_1\dot{l}_2}$ from ref. 10, the $C$ of eq. (IV.7) becomes

$$C^{n_1 l_1 n_2 l_2}_{\dot{l}_1 \dot{l}_2}(KLfr) = \sum_{\dot{n}_1 \dot{n}_2} \binom{\frac{K-l_1+l_2+1}{2}}{\dot{n}_1}^{\frac{1}{2}} \binom{\frac{K+\dot{l}_1-\dot{l}_2+1}{2}}{\dot{n}_2}^{\frac{1}{2}} \times$$

$$\times \binom{K+1}{\frac{K-\dot{l}_1-\dot{l}_2}{2}}^{-\frac{1}{2}} \langle \dot{n}_1\dot{l}_1\dot{n}_2\dot{l}_2 L \,|\, n_1 l_1 n_2 l_2 L \rangle \,. \qquad \text{(IV.8)}$$

Thus from eqs. (IV.6), (IV.8), and the two rules of the last paragraph, we obtain a complete set of six-dimensional hyperspherical harmonics with good permutational symmetry. It should be mentioned that the functions in this set which have the same quantum numbers $N_1, N_2, K, L, M, f, r$ and differ only in $l_1$, $l_2$, are in general, not orthogonal ($N_s = 2n_s + l_s$).

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It is my great pleasure to dedicate this paper to Prof. M. Moshinsky, as a small token of gratitude to the person who has been most influential in my career as a physicist.

# APPENDIX A

In this appendix we shall derive formula (III.11) for the two-dimensional harmonic oscillator transformation brackets (i.e., Moshinsky brackets). In the oscillator state given in eq. (III.3) the vectors $\dot{\eta}_1$, $\dot{\eta}_2$ play the role of position vectors of each oscillator. The transformation to (normalized) relative $\eta_1$ and center of mass $\eta_2$ coordinates would be given by

$$\dot{\eta}_1 = \frac{1}{\sqrt{2}}(\eta_1 + \eta_2) \quad ; \quad \dot{\eta} = \frac{1}{\sqrt{2}}(-\eta_1 + \eta_2) \; . \tag{A.1}$$

Substituting this into eq. (III.3) and expanding the binomials

$$|\dot{N}_1 \dot{m}_1 \dot{N}_2 \dot{m}_2 > = \dot{A}\, 2^{-\frac{\dot{N}_1 + \dot{N}_2}{2}} \sum_{pqrs} (-)^{r+s} \binom{\frac{\dot{N}_1 + \dot{m}_1}{2}}{p} \times$$

$$\times \binom{\frac{\dot{N}_1 - \dot{m}_1}{2}}{q} \binom{\frac{\dot{N}_2 + \dot{m}_2}{2}}{r} \binom{\frac{\dot{N}_2 - \dot{m}_2}{2}}{s} \times (\eta_{+1})^{\frac{\dot{N}_1 + \dot{m}_1}{2} - p + r} \times$$

$$\times (\eta_{-1})^{\frac{\dot{N}_1 - \dot{m}_1}{2} - q + s} (\eta_{+2})^{\frac{\dot{N}_2 + \dot{m}_2}{2} + p - r} (\eta_{-2})^{\frac{\dot{N}_2 - \dot{m}_2}{2} + q - s} |0> . \tag{A.2}$$

Taking the scalar product of this state with another state $|N_1 m_1 N_2 m_2 >$ which is constructed exactly as (III.3) but only with vectors $\eta_s$ instead of $\dot{\eta}_s$, we obtain

$$<\dot{N}_1\dot{m}_1\dot{N}_2\dot{m}_2|N_1m_1N_2m_2> = 2^{-\frac{N_1+N_2}{2}}\,\delta_{N_1+N_2,\,\dot{N}_1+\dot{N}_2}\,\delta_{m_1+m_2,\,\dot{m}_1+\dot{m}_2}\,\times$$

$$\times\prod_{s=1}^{2}\left[\left(\frac{N_s+m_s}{2}\right)!\left(\frac{N_s-m_s}{2}\right)!\left(\frac{\dot{N}_s+\dot{m}_s}{2}\right)!\left(\frac{\dot{N}_s-\dot{m}_s}{2}\right)!\right]^{\frac{1}{2}}\times$$

$$\times\sum_r(-)^r\left[r!\left(\frac{N_2+M_2}{2}-r\right)!\left(\frac{N_1+m_1-\dot{N}_1-\dot{m}_1}{2}+r\right)!\left(\frac{\dot{N}_1+\dot{m}_1}{2}-r\right)!\right]^{-1}\times$$

$$\times\sum_s(-)^s\left[s!\left(\frac{N_2-m_2}{2}-s\right)!\left(\frac{N_1-m_1-\dot{N}_1+\dot{m}_1}{2}+s\right)!\left(\frac{\dot{N}_1-\dot{m}_1}{2}-s\right)!\right]^{-1}. \qquad (A.3)$$

Comparing this result with the expression[6] for the $d^j_{m,\dot{m}}(\beta)$ representation functions of $SU(2)$, we obtain eq. (III.11). The fact that $N_s \pm m_s$ and $\dot{N}_s \pm \dot{m}_s$ are even numbers ensures that the three indices of a $d$ are simultaneoulsy integers or semi-integers.

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RESUMEN

Se presentan expresiones algebraicas cerradas para conjuntos de funciones completas (pero no ortogonales), adecuados para la descripción cuántica del movimiento orbital, translacionalmente invariante, de tres partículas idénticas. Las funciones aparecen como combinaciones lineales de armónicos hiperesféricos con buenas simetrías permutacionales y de momento angular orbital. Se discuten tanto los problemas en dos como en tres dimensiones.