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GROUP THEORY OF α-CLUSTER CONFIGURATIONS

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Dedicado a mi estimado y querido Maestro Marcos Moshinsky, quien me infectó con la enfermedad de los grupos.

ABSTRACT: The α-cluster ansatz for α-like nuclei leads to the introduction of a group of permutations of the clusters. It proves to be a useful tool to study the ansatz and to simplify the antisymmetrization of α-cluster functions. The method is applied to the calculation of two body matrix elements. The algebraic part of this problem is completely solved. The number of coefficients that appear is significantly reduced and all coefficients are given either explicitly or implicitly through a generating function.

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1. INTRODUCTION

The cluster representation of nuclei[1,2] is today a vast field extending into nuclear structure and nuclear reactions[3,4]. The permutational structure of the general cluster ansatz has been investigated in several papers[5-9]. In every respect the theory has gone far beyond the simple concepts of α-clustering. Yet the α-cluster is the most tightly bound and therefore physically most likely structure to appear in a nucleus, particularly as α-clustering is strongly favored by supermultiplet theory. Actually α-clustering in α-like nuclei is briefly discussed in ref. 10 and some qualitative insight is obtained. Also there has been recently considerable interest in quartets[11] and other excitations related to α-cluster structure[12] in highly excited α-like compounds. First microscopic calculations of α-^{16}O scattering are performed[13].

All this interest seems to justify an analysis of the permutational structure of the α-cluster ansatz within the general framework given in refs. 5, 6 and 7 (henceforth denoted by I, II, III). We shall therefore proceed to such an analysis in the present paper and we shall make extensive use of results and notations given in I, II and III. The paper will consist of two parts.

In chapter 2 we shall discuss in detail the special properties of α-cluster functions that were briefly touched upon in ref. 10.

As an application we discuss in chapter 3 the calculation of two-body matrix elements between symmetry adapted functions. Important simplifications are obtained and part of the algebraic coefficients evaluated using the properties discussed in chapter 2. To complete this result and make its application practical in chapter 4 a generating function for all algebraic coefficients not given in chapter 3 is constructed, using results on nonorthogonal orbitals[14]. Thus a complete procedure to obtain the algebraic coefficients appearing for two-body matrix elements between symmetry adapted α-cluster functions is presented.

2. THE α-CLUSTER ANSATZ

We shall treat α-like nuclei in the framework of the nuclear cluster model[1,2]. Therefore we make for the $n = 4k$ particle orbital wave function an ansatz in terms of α-clusters as

$$\Psi = \prod_{i=1}^{k} \varphi_i(n_i) \quad \chi(R_1 \ldots R_{k-1}) \ . \tag{2.1}$$

Here n_i is a set of four nucleons forming the ith α-cluster, φ_i is a function of the relative vectors of these four particles and is assumed symmetric under permutations within the set n_i. χ is a function of the $k-1$ relative vectors between the sets of nucleons forming the k clusters. Thus Ψ must be symmetric, i.e. a basis of the unit IR (irreducible representation) with respect to the subgroup H of $S(n)$ of all permutations within the sets n_i. H may be expressed as a direct sum of $S(4)$ groups

$$H = \sum_{i=1}^{k} \oplus S(4) \subset S(n) \ . \tag{2.2}$$

The unit IR of H consists of unit IR's of the $S(4)$ groups and we shall denote it by $\{4\}^k$ (distinguish from $\{4^k\}$ which denotes an IR of $S(n)$).

As we are interested first of all in the permutational aspects of the problem we rewrite Ψ as

$$\Psi = |\alpha^n \{4\}^k) \tag{2.3}$$

emphasizing the properties with respect to H and including all other characteristics of the function in α^n that denotes an orbital configuration of n nucleons.

From the wavefunction Ψ we have to construct a symmetry adapted wavefunction, i.e. a basis for an IR of $S(n)$, in order to be able to combine the latter with a spin-isospin function to an antisymmetric trial-function as described in I. To construct the symmetry adapted functions we proceed as proposed in III chapter 2 by induction. The multiplicity of any IR f of $S(n)$ in the representation induced from the IR $\{4\}^k$ of H is determined by Littlewoods rules for the outer product[15]. Because of the Pauli principle no partititon f with more than 4 columns is admissable and we find therefore that $f = \{4^k\}$ is the only IR of $S(n)$ that is possible and the corresponding multiplicity is 1. This implies that the multiplicity labels introduced in III are superfluous in our case, and can take only one value. Particularly their choice as intermediate partitions (III, eq. (2.6)) resulting from the induction from $j < k$ clusters to an IR of $S(4j)$ leads to the unique partitions $\{4^j\}$. In agreement with III eq. (2.5) the symmetry adapted state may then be written as

$$\alpha^n(\{4\}^k)\ \{4^k\}\ t) = c^{\{4^k\}}_{t,\{4\}^k} \mid \alpha^n\{4^k\}) \tag{2.4}$$

where t is a row index for the IR $\{4^k\}$ of $S(n)$, and $c^{\{4^k\}}_{t,\{4\}^k}$ is a Young operator (as defined in II eq. A.6), that ensures the right transformation properties of the state (2.4) under permutations from $S(n)$.

According to the reciprocity theorem uniqueness of induction implies uniqueness of subduction. Thus a state of the type

$$\mid \{4^k\}\ \ \{4\}^k > \tag{2.5}$$

with $\{4\}^k$ denoting a row of the IR $\{4^k\}$ of $S(n)$ is uniquely defined (a fact implicitly used in the Young operator in eq. (2.4)).

We now revert to the chain of groups given in eq. (2.2). It has been noted previously[10] that it is possible to insert a rather significant group into this chain. This group is a semidirect product of H with the group $\mathcal{S}(k)$ that permutes the sets of particles forming the α-clusters. The new chain reads

$$H \subset H \wedge \mathcal{S}(k) \subset S(n)\ . \tag{2.6}$$

The semidirect product properties are easily checked and H is found to be the normal subgroup in the semidirect product. For completeness we note that this semidirect product is actually a wreth product[16] but we shall not make use of this fact.

We are interested in the IR $\{4\}^k$ of H and $\{4^k\}$ of $S(n)$. From the properties of the semidirect product we find that no element of $\mathcal{S}(k)$ and therefore of $H \wedge \mathcal{S}(k)$ can change the IR of H. On the other hand the multiplicity of the IR $\{4\}^k$ of H in the IR $\{4^k\}$ of $S(n)$ is 1. Therefore no element of $\mathcal{S}(k)$ or $H \wedge \mathcal{S}(k)$ can transform the state eq. (2.5) into a different state but must reproduce it almost with a phase. The state eq. (2.5) is thus not only a basis for the unit IR of H but also for a one dimensional IR of $H \wedge \mathcal{S}(k)$. Calculation proves the phase to be + 1 (see ref. 17) and the state eq. (2.5) to be a basis of the unit IR of $H \wedge \mathcal{S}(k)$ and by consequence of $\mathcal{S}(k)$.

If we wish to use the chain of groups (2.6) rather than (2.2) to simplify our approach to α-like nuclei we may proceed in two quite distinct ways.

One would be to find a primitive cluster function which would not only form a basis for the unit IR of H but rather for the unit IR of $H \wedge \mathcal{S}(k)$.

Such a procedure involves additional assumptions about the explicit form of the trial function. For some trial functions such an approach may prove useful. Also in ref. 10 it was shown that some qualitative information about the wave function may be obtained in this way.

The other approach relies on the fact that the states eq. (2.5) belong to the unit of IR of $H \wedge \mathcal{E}^3(k)$. From III we can see that such states do occur in bra and ket of some algebraic coefficients that appear in the two-body matrix elements of symmetry adapted states. We thus expect that these coefficients simplify if the above mentioned properties are exploited. In the next chapter we shall follow this line to simplify two-body matrix elements without further assumptions about the trial functions.

3. TWO-BODY MATRIX ELEMENTS

In I it was shown how to obtain the matrix element of a two-body operator between antisymmetric states from orbital and spin-isospin matrix elements. The latter may be obtained according to the procedures described in refs. 18, 19 and become particularly simple for all α-like nuclei as the partition describing the spin-isospin state belongs to the unit IR of $SU(4)$.

Similarly to III eq. (4.3) the orbital two-body matrix element is given by

$$(\bar{\alpha}^n(\{4\}^k)\{4^k\}f'f'' \,\|\, T(n-1,n) \,\|\, \alpha^n(\{4\}^k)\{4^k\}f'f'')\,|f'|^{-1/2}$$

$$= (\bar{\alpha}^n\{4\}^k \,|\, c^{\{4^k\}}_{\{4\}^k, f'r'f''}\, T(n-1,n)\, c^{\{4^k\}}_{f'r'f'',\{4\}^k} \,|\, \alpha^n\{4\}^k)\ , \quad (3.1)$$

where the states eq. (2.4) have been used. f' defines an IR of $S(n-2)$, $|f'|$ its dimension, r' its row index and f'' an IR of $S(2)$ referring to the last two particles. Together $f'r'f''$ define a row index of the IR $\{4^k\}$ of $S(n)$.

We shall now proceed in analogy to III chapter 4 and try as far as possible to separate the interacting particles. For this purpose we have to find a DC (double coset) decomposition of the Young operators with respect to to H and $S(n-2) \oplus S(2)$. Such a decomposition will contain matrix elements of the DC generators that have either in bra or ket a state of the type (2.5). At this point we shall therefore try to take advantage of the fact that elements of $H \wedge \mathcal{E}^3(k)$ act trivially on such a state. In order to do this we shall not

make the choice of DC generators proposed in III eq.(3.9). Instead we shall choose the generator Y_q of the DC q of $S(n)$ with respect to $S(n-2) \oplus S(2)$ to the left and H to the right in the form

$$Y_q = y_\mu \; s_{\mu,g} \; , \tag{3.2}$$

where the pair of indices μ and g will span the range of q i.e. all DC. We require $s_{\mu,g}$ to be an element of $\mathfrak{S}(k)$ and y_μ will turn out to be the DC generator of $S(n)$ decomposed with respect to $S(n-2) \oplus S(2)$ to the left and $H \wedge \mathfrak{S}(k)$ to the right. In order to find the permutations y_μ and $s_{\mu,g}$, and to determine the range of the indices μ and g, we have to consider the possible DC symbols for decomposition with respect to $S(n-2) \oplus S(2)$ and H. According to III eq. (3.6) they are $2 \times k$ matrices essentially of two types: One contains in the second row and ith column a 2 and the rest of the second row contains zeros. The other contains in the second row a 1 in the ith and jth column. Again the remaining elements of the second row are zero. The two types of DC symbols are shown in fig. 1.

1			i			k
4	·	· ·	2	4	· · ·	4
0	·	· ·	2	0	· · ·	0

Fig. 1a

1		i			j			k
4	· · ·	3	4	· · ·	3	4	· · ·	4
0	· · ·	1	0	· · ·	1	0	· · ·	0

Fig. 1b

Fig. 1. The two types of DC symbols for the decomposition of $S(n)$ with respect to $S(n-2) \oplus S(2)$ and H.

We propose for the generators (3.2) the following form, that has only two values for the index μ. For $\mu = 1$ we obtain

$$y_1 = e\,,\ s_{1,g} = (n_i, n_k);\ \ i = 1\ldots k,\ \ g = 1\ldots k\,, \tag{3.3a}$$

where the index g could be set identical to the index i. For $\mu = 2$ we find

$$y_2 = \begin{pmatrix} n-4 & n-3 & n-2 & n-1 \\ n-3 & n-2 & n-1 & n-4 \end{pmatrix}\,, \tag{3.3b}$$

$$s_{2,g} = (n_i, n_{k-1})\,(n_j, n_k);\ \ i < j = 2\ldots k,\ g = 1\ldots k(k-1)/2\,,$$

where the index g stands for the possible index pairs i, j. While y_μ are some permutations from $S(n)$, e denoting the unit permutation, the symbols (n_i, n_j) imply transpositions of $\mathfrak{S}(k)$ interchanging the particle sets making up the clusters i and j. Comparison of (3.3a) with fig. 1a and (3.3b) with fig. 1b show that actually for every DC exactly one generator is defined.

No further simplification would be possible by choosing $s_{\mu,g} \in H \wedge \mathfrak{S}(k)$ instead of $s_{\mu,g} \in \mathfrak{S}(k)$. In order to see this we have to consider, that in the semidirect product for $h, h' \in H$, $s \in \mathfrak{S}(k)$, $hs = ss^{-1}hs = sh'$ holds. On the other hand for the DC defined by the symbols fig. 1a, b possible applications of elements of H *to the right have been* considered. Therefore the choice of $s_{\mu,g} \in \mathfrak{S}(k)$ is no restriction and the maximal reduction is found with the definition eq. (3.3a, b) for y_1 and y_2. We may conclude that these are actually the generators for the DC of $S(n)$ when decomposed with respecto to $S(n-2) \oplus S(2)$ to the left and $H \wedge \mathfrak{S}(k)$ to the right.

For the bra, i.e. the decomposition of $S(n)$ with respect to H on the left and $S(n-2) \oplus S(2)$ on the right we obtain immediately the generators

$$\overline{Y}_{\underline{q}} = s_{\underline{\mu, g}}\, y_\mu^{-1} \tag{3.4}$$

with the same ranges for the indices as in eq. (3.3a, b).

Using these results we obtain for the DC decomposition of the Young operator

$$c^{\{4^k\}}_{f'r'f'',\{4\}^k} = \frac{|\{4^k\}|\,(n-2)!\,2(4!)^k}{|f'|\;n!} \sum_{\mu} m^{-1}(y_\mu)\, c^{f'}_{r',a'(\mu)}\, c^{f''}$$

$$<\{4^k\} f' a'(\mu) f'' | y_\mu | \{4^k\}\{4\}^k > y_\mu \sum_g s_{\mu,g}\, c^{\{4\}^k} \quad . \tag{3.5}$$

Note the difference between this expression and III eq. (4.5). We obtain our expression from the latter by using the DC generators defined in 3.3a, b. We then are able to apply the $s_{\mu,g}$ to the ket of the matrix element of the DC generator trivially. The algebraic coefficients as well as the remaining Young operator of $S(n-2)$ become independent of the index g and only depend on the indices f' and μ. As for $a'(\mu)$ we have to choose it as a row index of f' in such a way that no sum over row indices occurs in eq. (3.5). As y_1 and y_2 actually are chosen to be two of the DC generators used in III eq. (4.5) we may just pick the corresponding $a'(q)$ that appear in the term with $s_{\mu,g} = e$ where the two expressions coincide. We find

$$a'(1) = \{4\}^{k-1}\{2\}$$

$$a'(2) = \{4\}^{k-2}\{3\}^2 \tag{3.6}$$

actually describing the unit IR of the subgroups of $S(n-2)$:

$$H'_1 = [\sum_{i=1}^{k-1} \oplus S(4)] \oplus S(2) \subset S(n-2)$$

$$H'_2 = [\sum_{i=1}^{k-2} \oplus S(4)] \oplus S(3) \oplus S(3) \subset S(n-2) \quad . \tag{3.7}$$

That the coefficients $m(y_\mu)$ do not depend on g follows immediately from comparing their definition III eq. (3.13) with the DC symbols of fig. 1.

Subduced from the IR $\{4^k\}$ of $S(n)$ only two possible pairs $f' = \{4^{k-1}2\}$, $f'' = \{2\}$ and $f' = \{4^{k-2}3^2\}$ $f' = \{11\}$ occur as IR of $S(n-2)$ and $S(2)$. But while in $f' = \{4^{k-1}2\}$ both $a'(1)$ and $a'(2)$ are contained only $a'(2)$ is contained in $f' = \{4^{k-2}3^2\}$. Therefore only three different non-zero matrix

elements of y_μ can occur in eq. (3.5). These may be reduced to $SU(2)$ coefficients and are evaluated in the appendix. The results are listed in table 1.

TABLE 1

The matrix elements $<\{4^k\}\, f'\alpha'(\mu)\, f'' \,|\, y_\mu \,|\, \{4^k\}\{4\}^k>$ for all possible values of $f'f''$ and μ.

$f'\ f''$ \ μ	1	2
$\{4^{k-1}2\}\{2\}$	1	$-\sqrt{3/8}$
$\{4^{k-2}3^2\}\{11\}$	0	$\sqrt{5/8}$

We can use the decomposition eq. (3.5) for the Young operators as well as their contraction properties discussed in II eq. (A.9) to obtain for the two body matrix element eq. (3.1)

$$(\overline{\alpha}^n(\{4\}^k)\{4^k\}\, f'f'' \,\|\, T(n-1,n) \,\|\, \alpha^n(\{4\}^k)\{4^k\}\, f'f'')\,|f'|^{-\frac{1}{2}}$$

$$= \frac{|\{4^k\}|^2 (4!)^{2k}((n-2)!)^2\, 2^2}{(n!)^2 \qquad |f'|^2} \sum_{\overline{\mu}\,\mu} m^{-1}(\underline{y}^{-1}_{\overline{\mu}})\; m^{-1}(y_\mu) \tag{3.8}$$

$$<\{4^k\}\{4\}^k \,|\, \underline{y}^{-1}_{\overline{\mu}} \,|\, \{4^k\}\, f'\alpha'(\overline{\mu})\, f''><\{4^k\}\, f'\alpha'(\mu)\, f'' \,|\, y_\mu \,|\, \{4^k\}\{4\}^k>$$

$$\sum_{\overline{g}\,g} (\overline{\alpha}^n\{4\}^k \,|\, \underline{s}_{\overline{\mu},\overline{g}}\, \underline{y}^{-1}_{\overline{\mu}}\, T(n-1,n)\, c^{f'}_{\alpha'(\overline{\mu}),\alpha'(\mu)}\, c^{f''} y_\mu s_{\mu,g} \,|\, \alpha^n\{4\}^k) \ .$$

We use the notation of pointed brackets for purely algebraic coefficients and round brackets for matrix elements involving the trial function eq. (2.3) that will require explicit integration.

If we compare this result with III eq. (4.7) we find important simplifications. The algebraic coefficients only depend on the indices $\overline{\mu}$ and μ that take at most two values and are independent of $\overline{g}$ and g. Furthermore

they are given in table 1. The remaining Young operator $c^{f'}_{\alpha'(\bar{\mu}),\alpha'(\mu)}$ is independent of $\bar{g}$, g and for the two matrix elements that actually may occur in eq. (3.1) only essentially four different operators of this type can occur. If we use the approach of III their number would have been 27 for ^{12}C and 76 for ^{16}O.

Note that we have made no assumptions about the wave functions beyond the fact that they are constructed of α-clusters. By consequence the number of integrals occurring is not reduced by the procedure. At this stage though it is easy to see what additional advantage we obtain if the primitive functions would allow trivial application of elements of $H \wedge \mathfrak{S}(k)$ as suggested at the end of chapter 2. We could apply $s_{\underline{\mu},\underline{g}}$ and $s_{\mu,g}$ to the bra and ket in the last line of eq. (3.8) and the sum over $\bar{g}$ and g would be replaced by multiplication with the range of these sums given in eq. (3.3a, b).

The Young operator $c^{f'}_{\alpha'(\bar{\mu}),\alpha'(\mu)}$ may obviously again be decomposed into DC with respect to $H'_{\underline{\mu}}$ and H'_{μ} as their elements can be trivially applied to the left and the right of the Young operator in eq. (3.8). If we perform this decomposition as well as the trivial one of $c^{f''}$ we may rewrite (3.7) using the results of III eq. (4.9) as

$$(\bar{\alpha}^n(\{4\}^k)\{4^k\}\, f'f'' \,\|\, T(n-1,n) \,\|\, \alpha^n(\{4\}^k)\{4^k\}\, f'f'')$$

$$= \frac{|\{4^k\}|^2 (4!)^{2k}(n-2)!\,2}{(n!)^2\,|f'|^{1/2}} \sum_{\bar{\mu},\mu} \langle \{4^k\}\{4\}^k \,|\, y^{-1}_{\underline{\mu}} \,|\, \{4^k\}\, f'\alpha'(\bar{\mu})\, f'' \rangle$$

$$\langle \{4^k\}\, f'\alpha'(\mu)\, f'' \,|\, y_{\mu} \,|\, \{4^k\}\{4\}^k \rangle \sum_{K'K''} R'(f',\bar{\mu},\mu,K')\, R''(f'',\bar{\mu},\mu,K'')$$

$$\sum_{\bar{g}\,g} (\bar{\alpha}^n\{4\}^k \,|\, s_{\underline{\mu},\underline{g}}\, y^{-1}_{\underline{\mu}}\, T(n-1,n)\, Z'_{K'}\, Z''_{K''}\, y_{\mu} s_{\mu,g} \,|\, \bar{\alpha}^n\{4\}^k) \quad . \qquad (3.9)$$

Here $Z'_{K'}$ are the DC generators of $S(n-2)$ with respect to $H'_{\underline{\mu}}$ and similarly $Z''_{K''}$ are generators of $S(2)$ with respect to $H''_{\underline{\mu}}$ and H''_{μ} where $H''_1 = S(2)$ and $H''_2 = S(1) \oplus S(1)$. The expression

$$R''(f'',\bar{\mu},\mu,K'') = \langle f'' \,|\, Z''_{K''} \,|\, f'' \rangle\, m^{-1}(Z''_{K''}) \tag{3.10}$$

can be easily evaluated and is given in table 2.

TABLE 2

The non-zero coefficients $R''(f'',\bar{\mu},\mu,K'')$ and their indices

$\bar{\mu},\mu$	f''	K''	R''
1,1	{2}	1	1/2
1,2	{2}	1	1
2,1	{2}	1	1
2,2	{2}	1	1
2,2	{2}	2	1
2,2	{11}	1	1
2,2	{11}	2	−1

Remember that the dependence on $\bar{\mu}$ and μ is implicit in the subgroups defining the DC and actually appears in $m(Z''_{K''})$. The expression

$$R'(f',\bar{\mu},\mu,K') = \langle f'\alpha'(\bar{\mu}) \,|\, Z'_{K'} \,|\, f'\alpha'(\mu) \rangle\, m^{-1}(Z'_{K'}) \tag{3.11}$$

remains finally to be evaluated. Note that the coefficients eq. (3.11) again do not depend on $\bar{g}$ and g and thus their number is greatly reduced as compared to III eq. (4.9). It would now seem natural to proceed to their reduction

and evaluation using the fact that $k-1$ or $k-2$ $S(4)$ groups are still contained in H'_μ. This must allow similar methods as used up to now involving the groups $H'_1 \wedge \mathfrak{S}(k-1)$ and $H'_2 \wedge \mathfrak{S}(k-2)$ and eventually the interchange of the two sets of three particles in H'_2 could also be considered. While such an approach certainly reduces the number of different coefficients of the type given in eq. (3.11) it seems that the most complicated matrix elements involving exchanges of nucleons between all clusters remain and would be difficult to treat. We shall therefore use a different way by deriving in the next chapter a generating function for the coefficients appearing in eq. (3.11).

4. A GENERATING FUNCTION FOR DOUBLE COSET MATRIX ELEMENTS

To obtain a generating function for the coefficients $R(f',\overline{\mu},\mu,K')$ we note that their dependence on the DC K' is the same as for a normalization matrix element between states induced to the symmetry f' from primitive $n-2$ nucleon functions that belong to the unit IR of $H'_{\underline{\mu}}$ and H'_μ respectively for bra and ket. This may be seen from III eq. (4.9) by choosing $n''=0$ to obtain a normalization matrix element. It would be possible to calculate the $\overline{\mu},\mu$ and f' dependence correctly by comparing III eq. (4.9) for the normalization case with eq. (3.9) but we still would have to check the phase conventions implicit in the generating function with those used in the appendix to calculate the matrix elements of $y_{\underline{\mu}}$. We shall therefore use a different method to obtain the dependence on $\overline{\mu},\mu$ and f' simultaneously with correct phases at the end of this chapter and shall concentrate on a generating function for the K' dependence. This allows us to omit all over factors that may appear.

The methods of I, II, III and the present paper do not make use of the explicit form of the cluster ansatz but only of the fact that it belongs to the unit IR of some subgroup of the group of permutations of all particles. For our $n-2$ particle problem an alternative ansatz fulfilling this condition could be chosen as follows. Take a set of k non-orthogonal single particle functions ψ_i, $i=1\ldots k$ and choose a product function with occupation numbers $w_i^1 = 4,\ldots 4,2$ or $w_i^2 = 4,\ldots 4,3,3$ as

$$|\alpha^{n-2}\,\sigma'(\mu)) = \prod_{i=1} \psi_i^{w_i^\mu} \tag{4.1}$$

where the first set of occupation numbers leads to the unit IR $a'(1)$ of H_1' and the second to the unit IR $a'(2)$ of H_2'.

In ref. 9 it was discussed that the normalization matrix element using the ansatz (4.1) must be a homogeneous polynomial in the overlaps of the orbitals

$$u_{ij} = \int \psi_i^* \psi_j \quad . \tag{4.2}$$

Each term in this polynomial must have the form

$$\prod_{i,j=1}^{k} (u_{ij})^{d_{ij}(K')} \tag{4.3}$$

Here the $d_{ij}(K')$ are the elements of the DC symbol characterizing the DC K' as given in III eq. (3.6). The coefficients of this polynomial must have the K' dependence of $R'(f',\overline{\mu},\mu,K')$ and therefore it can be used as a generating function for these coefficients, if we obtain it explicitly.

In ref. 14 this polynomial was obtained exploiting the single particle properties of the function eq. (4.1). In this approach the symmetry adapted states are characterized by Gelfand patterns determining an IR and a row index of $GL(k)$. The IR of $GL(k)$ is uniquely determined by the partition f'. For the three states we are interested in, the row is determined by the weight of the state i. e. by the occupation numbers w_i^{μ}, which are fixed by the $a'(\mu)$. For simplicity we may therefore use f' to indicate also the IR of $GL(k)$ and $a'(\mu)$ as a row index for this IR. The Gelfand patterns for the three states read as follows:

$$|\{4^{k-1}2\}\ a'(1)) \longrightarrow \left| \begin{array}{ccccccc} 4 & 4 & \ldots\ldots & 4 & 2 \\ & 4 & \ldots\ldots & 4 & 4 \\ & & 4 & \ldots\ldots & 4 \\ & & & \cdot & \cdot \\ & & & & 4 \end{array} \right\rangle \tag{4.4a}$$

$$|\{4^{k-1}2\}\ a'(2)) \longrightarrow \left| \begin{array}{ccccccc} 4 & 4 & \ldots\ldots & 4 & 2 \\ & 4 & \ldots\ldots & 4 & 3 \\ & & 4 & \ldots\ldots & 4 \\ & & & \cdot & \cdot \\ & & & & 4 \end{array} \right\rangle \tag{4.4b}$$

$$|\{4^{k-2}3^2\}\,a'(2)\rangle \longrightarrow \left|\begin{array}{ccccccc} 4 & 4 & \ldots\ldots & 4 & 3 & 3 & \\ & 4 & \ldots\ldots & 4 & 3 & & \\ & & 4 & \ldots\ldots & 4 & & \\ & & & \cdot & \cdot & & \\ & & & & 4 & & \end{array}\right\rangle \tag{4.4c}$$

The states eq. (4.4a, c) are of highest weight and a state proportional to the one eq. (4.4b) may be obtained by applying to the state eq. (4.4a) a lowering operator proportional to a single generator of $GL(k)$[14, 19].

With the connection to Gelfand states established in eqs. (4.4a, b, c) we can now use the result of ref. 14 (eq. 6.11) that gives the Wigner D-matrix element of $GL(k)$

$$D^{f'}_{a'(\bar{\mu}),a'(\mu)}(|u_{ij}|^{-1}) \tag{4.5}$$

for the normalization matrix element. Here $|u_{ij}|$ stands for the $k \times k$ matrix of single particle overlaps u_{ij}. The whole expression is a real function, actually a homogeneous polynomial in u_{ij}. This polynomial may be obtained if we use Louck's result[20] that it is the same polynomial in u_{ij}, which we find for a double Gelfand state[19] in terms of Bose creation operators.

In the present case there are two practical methods to obtain these polynomials, neglecting any all over factors.

If the weight determines a Gelfand state uniquely, this state is proportional to a Weyl state[21] of the same weight. We may proceed to obtain it as indicated in ref. 21 by filling the creation operators in the Young diagram corresponding to f' and then symmetrizing rows and antisymmetrizing columns with respect to one of the indices. If at least one of the Gelfand patterns is of highest weight we get a simple product of determinants.

An alternative procedure starts from the fact that states of highest weight in both patterns are known[19] simple products of determinants. The case where either one or both patterns are of the type eq. (4.4b) can be obtained by applying lowering operators to the polynomial where both patterns are of the type eq. (4.4a). This is easy because, as mentioned earlier[19], the only operators we need are simple generators, namely $C^{k\,k-1}$ and $C_{k\,k-1}$.

The result which we obtain up to a factor from both procedures is listed in table 3. The determinant notation

$$\Delta^{i}_{j} = u_{ij}\,,\quad \Delta^{ii'}_{jj'} = u_{ij}u_{i'j'} - u_{ij'}u_{i'j}\quad \text{etc.} \tag{4.6}$$

is used. We have thus given homogeneous polynomials in u_{ij} which are proportional to the normalization matrix elements of the $n-2$ particle problem and by consequence to the generating function we seek.

TABLE 3

The generating function proportional to $D^{f'}_{\alpha'(\overline{\mu}),\alpha'(\mu)}$ and the coefficients $R'(f',\overline{\mu},\mu,1)$ for the relevant values of $\overline{\mu},\mu$ and f'.

$\overline{\mu},\mu$	f'	$\sim D$		R'
1,1	$\{4^{k-1}2\}$	$(\Delta^{1\ldots k}_{1\ldots k})^2$	$(\Delta^{1\ldots k-1}_{1\ldots k-1})^2$	$\dfrac{1}{(4!)^{k-1}\cdot 2}$
1,2	$\{4^{k-1}2\}$	$(\Delta^{1\ldots k}_{1\ldots k})^2$	$\Delta^{1\ldots k-1}_{1\ldots k-1}\;\Delta^{1\ldots k-1}_{1\ldots(k-2)k}$	$\dfrac{\sqrt{1/6}}{(4!)^{k-2}3!2}$
2,2	$\{4^{k-1}2\}$	$(\Delta^{1\ldots k}_{1\ldots k})^2$	$[\Delta^{1\ldots k-1}_{1\ldots k-1}\;\Delta^{1\ldots(k-2)k}_{1\ldots(k-2)k}]$ $+\,[\Delta^{1\ldots k-1}_{1\ldots(k-2)k}\;\Delta^{1\ldots(k-2)k}_{1\ldots\ldots k-1}]$	$\dfrac{1}{(4!)^{k-2}(3!)^2}$
2,2	$\{4^{k-2}3^2\}$	$(\Delta^{1\ldots k}_{1\ldots k})^3$	$\Delta^{1\ldots k-2}_{1\ldots k-2}$	$\dfrac{1}{(4!)^{k-2}(3!)^2}$

To determine the proportionality factor depending on $\overline{\mu},\mu,f'$ we proceed as follows. The DC containing the unit permutation e is denoted by $K'=1$. All four generating functions listed in table 3 yield a $+1$ as coefficient for this DC. Thus the correct generating function is obtained by multiplying the one given in table 3 by $R(f',\overline{\mu},\mu,1)$ i.e. the correct coefficient for the DC $K'=1$. This coefficient is a matrix element of the permutation e divided by $m(Z_1')$. The evaluation is done in the appendix using again the reduction to $SU(2)$ coefficients, which ensures a choice of phases consistent with the one for the matrix elements of y_μ. The coefficients $R(f',\overline{\mu},\mu,1)$ are listed in table

3 together with the generating functions, and thus the correct generating function for the coefficients $R'(f', \overline{\mu}, \overline{\mu}, K')$ is obtained.

5. CONCLUSIONS

We have found the group of cluster permutations $\mathcal{S}(k)$ and its semi-direct product $H \wedge \mathcal{S}(k)$ with the group H of internal permutations of the clusters to be powerful tools to handle the α-cluster ansatz. Actually we were able to simplify enormously by its use, the coefficients which we need in two body matrix elements between symmetry adapted α-cluster functions. We indicate how to obtain all coefficients in a way that is practical for computation. Yet there are a number of interesting open questions.

We did not use the techniques of cluster permutations at all points but avoided them at one stage by introducing the generating function. We mentioned at that point that no simplification of the most complicated coefficients is to be expected, but a reduction of the number of coefficients would occur if we used cluster permutations. It would be useful if this aspect could somehow be included in the way we use the generating function.

We also treated the possibility of using cluster functions that belong from the outset to the unit IR of $H \wedge \mathcal{S}(k)$ very marginally. Yet in special models like e.g. the harmonic oscillator cluster model[3] such a procedure looks promising.

Finally there is the important question to what extent the presented methods can be generalized. Consider a case of $k-1$ α-clusters and one cluster of less than four particles. This is certainly an interesting structure as it leads to the lowest supermultiplets for many light nuclei. We can see immediately that a group $\mathcal{S}(k-1)$ of α-cluster permutations can be formed and we could proceed in close analogy with the steps presented in this paper. A number of complications will appear. The matrix elements we have to evaluate will be connected rather with $SU(3)$ than with $SU(2)$ and the multiplicity appearing in the subduction of $S(n-2)$ will complicate the generating function. We can thus conclude that the method of this paper is useful whenever many α-clusters appear in the ansatz but at the same time difficulties arising from non α-clusters are considerable.

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APPENDIX

To evaluate the matrix elements of y_μ we consider that in a state

$$|\{4^k\} f' a'(\mu) f''> = |\{4^k\} a'(\mu) f''> \qquad (A.1)$$

the IR f' of $S(n-2)$ and therefore the states are completely determined by the IR $\{4^k\}$ of $S(n)$ and the one dimensional IR $a'(\mu) f''$ of $H'_\mu \oplus S(2)$.

We now consider any group G fulfilling the condition $H'_\mu \oplus S(2) \subset G \subset S(n)$. The IR of G to which (A.1) is a basis vector again must be uniquely determined and we may introduce the label for this IR as an additional although spurious quantum number in the state eq. (A.1). For our purposes it is convenient to consider the chain

$$H'_\mu \oplus S(2) \subset S(n-8) \oplus S(6) \oplus S(2) \subset S(n-8) \oplus S(8) \subset S(n) \qquad (A.2)$$

The IR of the intermediate groups to which the state eq. (A.1) belongs are $\{4^{k-2}\}$ for $S(n-8)$ and $\{4^2\}$ for $S(8)$. The IR of $S(2)$ is naturally again f'' and determines the IR F' of $S(6)$ to be $\{42\}$ for $f'' = \{2\}$ and $\{33\}$ for $f'' = \{11\}$. Similarly we may introduce a group $S(n-8) \oplus S(8)$ in the chain $H \subset S(n)$. The state $|\{4^k\}\{4\}^k>$ is characterized by the IR $\{4^{k-2}\}\{4^2\}$ of $S(n-8) \oplus S(8)$.

We write the newly found labels although they are superfluous and omit f' in the matrix element of y_μ. The phases that might be introduced for the states containing these additional quantum numbers are chosen to be + 1. We then find

$$<\{4^k\}\, f'\, a'(\mu)\, f'' \,|\, y_\mu \,|\, \{4^k\}\{4\}^k> = <\{4^k\}\, a'(\mu)\, f'' \,|\, y_\mu \,|\, \{4^k\}\{4^k\}>$$

$$= <\{4^k\}\{4^{k-2}\}\{4^2\}\, F'\, a'(\mu)\, f'' \,|\, y_\mu \,|\, \{4^k\}\{4^{k-2}\}\{4^2\}\{4\}^k>$$

$$= <\{4^2\}\, F'\, b'(\mu)\, f'' \,|\, y_\mu \,|\, \{4^2\}\{4\}^2> \quad . \tag{A.3}$$

In the last line we used the fact that the permutation y_μ involves only the last 8 particles. $b'(\mu)$ contains the last two components of $a'(\mu)$, i.e. $b'(1) = \{4\}\{2\}$ characterizes the unit IR of $S(4) \oplus S(2)$ and $b'(2) = \{3\}\{3\}$ the unit IR of $S(3) \oplus S(3)$. The matrix element of y_μ is reduced in eq. (A.3) to a matrix element of the two cluster case. These matrix elements are evaluated in II eqs. (A.1,3) in terms of gj symbols of $SU(2)$ and thus may be considered as known. The results are listed in table 1.

To calculate the matrix element of the unit permutation that appears in $R(f', \overline{\mu}, \mu, 1)$ we introduce again a group in the chain $H'_\mu \subset S(n-2)$ characterizing bra and ket. A useful chain is

$$H'_\mu \subset S(n-8) \oplus S(6) \subset S(n-2) \quad . \tag{A.4}$$

The state $|f'\, a'(\mu)>$ belongs to the IR $\{4^{k-2}\}$ of $S(n-8)$ and to an IR F' of $S(6)$ which is $\{42\}$ for $f' = \{4^{k-1}2\}$ and $\{33\}$ for $f' = \{4^{k-2}3^2\}$.

Using these results we find again setting eventual phases = + 1

$$<f'\, a'(\overline{\mu}) \,|\, e \,|\, f'\, a'(\mu)>$$

$$= <f'\, \{4^{k-2}\}\, F'\, a'(\overline{\mu}) \,|\, f'\, \{4^{k-2}\}\, F'\, a'(\mu)> \tag{A.5}$$

$$= <F'\, b'(\overline{\mu}) \,|\, F'\, b'(\mu)> \quad .$$

The $b'(\mu)$ are defined as above. We thus obtain a two cluster matrix element, which we evaluate again using II (A.1, 3). This ensures consistent phases and we can obtain the coefficients $R'(f', \overline{\mu}, \mu, 1)$ listed in table 3.

REFERENCES

1. K. Wildermuth, Th. Kanellopoulos, Nucl. Phys. 7 (1958) 150
2. K. Wildermuth and W. McClure, *Cluster representations of nuclei*, *Springer Tracts in Modern Physics* 41 (Springer, Berlin, 1966)
3. P. Kramer and D. Schenzle, Nucl. Phys. (in print) (1973)
4. H.H. Hackenbroich, *General treatment of reactions*, in *Clustering Phenomena in Nuclei*, IAEA, Vienna (1969)
5. P. Kramer and T.H. Seligman, Nucl. Phys. A123 (1969) 161
6. P. Kramer and T.H. Seligman, Nucl. Phys. A136 (1969) 545
7. P. Kramer and T.H. Seligman, Nucl. Phys. A186 (1972) 49
8. P. Kramer, *Permutation group in light nuclei*, in *Proc. of the XV Solvay Congress*, (Brussels, 1970)
9. T.H. Seligman, *Gruppentheoretische Untersuchungen zum Clustermodell der Atomkerne*, BMBW-FB K 69-42 Karlsruhe (1969)
10. H. Hutzelmeyer, P. Kramer, T.H. Seligman, in *Clustering Phenomena in Nuclei*, IAEA, Vienna (1969) p. 327
11. A. Arima and V. Gillet, Ann. of Physics, 66 (1971) 117
12. W. Gruhle, Th. Bauer, H.H. Hackenbroich, T.H. Seligman (to be published)
13. W. Sünkel and K. Wildermuth, Phys. Letts. 41B (1972) 439
14. M. Moshinsky and T.H. Seligman, Ann. of Physics 66 (1971) 311
15. M. Hamermesh, *Group theory and its applications to physical problems*, (Addison-Wesley, Reading, 1962)
16. G. de B. Robinson and A. Kerber, private communication
17. P. Kramer, Z. Phys. 216 (1968) 68
18. J.P. Elliott, J. Hope and H.A. Jahn, Phil. Trans. Roy. Soc. A246 (1953) 241
19. P. Kramer and M. Moshinsky, *Harmonic oscillators and nuclear structure*, in *Group theory and its applications*, ed. E.M. Loebl, (Academic Press New York, 1968) and refs. given therein
20. J.P. Louck, Amer. J. Phys. 38 (1970) 3
21. T.H. Seligman, J. Math. Phys. 13 (1972) 876

RESUMEN

El ansatz de cúmulos α para núcleos tipo α, lleva a la introducción de un grupo de permutaciones de los cúmulos. Resulta útil estudiar este ansatz y la simplificación de la antisimetrización de las funciones de cúmulos α. El método se aplica al cálculo de elementos de matriz de dos cuerpos. La parte algebráica de este problema está completamente resuelta. El número de coeficientes que aparecen se reduce significativamente, y se dan todos los coeficientes, ya sea explícitamente o implícitamente a través de una función generadora.