LIE ALGEBRAS, EXACTLY SOLUBLE SHELL MODELS AND THEORIES OF COLLECTIVE MOTION*

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ABSTRACT: We define an exactly soluble shell model by prescribing as
Hamiltonian a polynomial in the generators of a Lie Algebra,
and by choosing the algebra to be sufficiently simple that dia-
gonalizations carried out by computer can be interpreted as
the solution for a large system of Fermions. Up to now models
based on $SU(2) \times SU(2) \times \ldots$, and $R(5)$ have been studied most
extensively. The main interest of these studies is the excep-
tionally pure form in which such systems manifest collective
degrees of freedom, interpretable in limiting cases as vibrations
and rotations. Consequently, the aim of much work on these
models is the comparison of various widely employed approxi-
mate theories of collective motion with the exact solutions. In
addition to the above, our major interest has been the develop-
ment of a distinct scheme which has its genesis both, in the
equations of motion methods of modern many body theory, and

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in the tenets of the "current algebra" approach to elementary
particles. In this method both kinematical and dynamical
commutators and other algebraic relations are turned into sum
rules for observables, the whole constituting a closed set of
inhomogeneous non-linear equations defining the properties
of a collective subset of states of the system studied. Physical
significance, accuracy, and relation to other methods are illus-
trated.

1. BACKGROUND

Just over a decade ago, A. Kerman and the writer introduced a new
method in the theory of collective motion to which we attached the unprepos-
sessing name, Generalized Hartree-Fock Approximation, GHFA[1], which was
later amended to the equally unattractive appellation, Self-Consistent Core-
Particle Coupling Method[2]. Since the work to be described in detail in the
present paper is integral with this earlier approach, and in particular, since
both are aimed at a solution, ultimately, of the same class of problems, let
us begin by stressing this unity.

We work with an underlying Hamiltonian of the form

$$H = \sum_{\alpha\beta} t_{\alpha\beta} a_\alpha^\dagger a_\beta + \tfrac{1}{4} \sum V_{\alpha\beta\gamma\delta} a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma , \qquad (1)$$

where $a_\alpha (a_\alpha^\dagger)$ are destruction (creation) operators for shell model orbitals and
H may in principle be the complete many body Hamiltonian (in which event t
is the kinetic energy and V the observed two-body force), or else, and more
commonly in present day application's, H is a shell-model Hamiltonian as
this term is understood in either atomic or nuclear physics.

In Fig. 1, we indicate schematically a typical physical situation
which we endeavor to study. The low-lying states of four neighboring nuclei,
two even and two odd are designated by the symbols I, J, K, and L respective-
ly, and A is the number of particles. In addition to the energies of these
states, we have experimental information about the matrix elements of various
kinds of operators connecting these states. These would include, for ex-
ample, "vertical" or nucleon number-conserving transition amplitudes of the
form

$$< K' \,|\, Q \,|\, K > \;=\; \sum_{\alpha\beta} Q_{\alpha\beta} < K' \,|\, a^{\dagger}_{\alpha} a_{\beta} \,|\, K > \,, \tag{2}$$

which comprise electromagnetic and weak decays. Neighboring nuclei are connected by direct single-particle pickup and stripping reactions in which such quantities as

$$v_J(\alpha K) \;=\; < J \,|\, a_{\alpha} \,|\, K > \,, \tag{3}$$

$$u^{*}_{J}(\alpha I) \;=\; < J \,|\, a^{\dagger}_{\alpha} \,|\, I > \,, \tag{4}$$

are measured, respectively. In a two-particle direct pickup reaction one measures such quantities as

$$< I \,|\, P \,|\, K > \;=\; \sum_{\alpha\beta} P_{\alpha\beta} < I \,|\, a_{\alpha} a_{\beta} \,|\, K > \,. \tag{5}$$
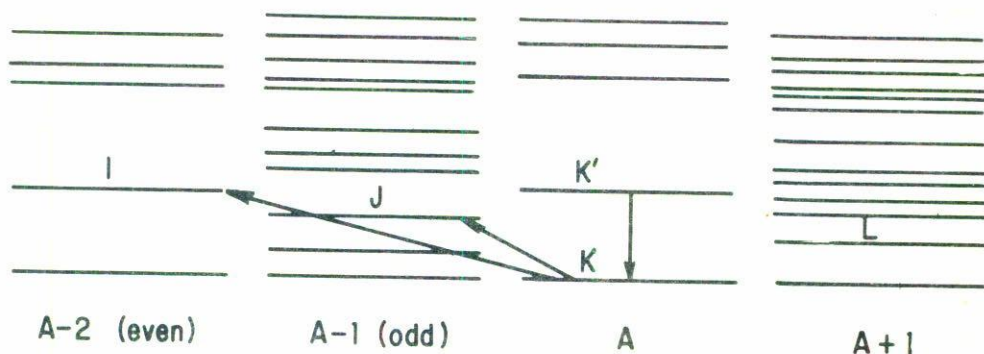

Fig. 1. Schematic representation of transition matrix elements within a nucleus or between neighboring nuclei.

Our basic approach is to ask if we can develop a direct calculus for observable matrix elements of the type exemplified by Eqs. (2)–(5) and if in the formulation of such a scheme, some concept of collective motion will play an essential role. Toward this end we employ a completely quantum mechanical and abstract definition of collective motion which we can illustrate for the fixed nucleus A of Fig. 1. Let $|\, K^{-}_1 > \ldots |\, K_p >$ be a chosen set of states and let $Q_1 \ldots Q_{\nu}$ be a set of operators of the type (2) (for example) and $\nu \ll p$. Then we assume that sum rules involving matrix elements of

products of the operators Q taken between states of the set $K>$ are sensibly exhausted within the same set, thus

$$< K \mid Q_\rho Q_\lambda \mid K' > = \sum_{K''} < K \mid Q_\rho \mid K'' > < K'' \mid Q_\lambda \mid K' > . \qquad (6)$$

That is, we have closure within a subspace of the many-body space. In the development of the theories to be described, or alluded to below, we have shown that in suitable limiting cases this definition implies the classical concept of collective motion as the coherent motion of many particles describable by a few degrees of freedom (translation, rotation, vibration).

For quite a long time our view, as developed between 1962 and 1967[1], was that the basic calculus should be for the quantities v_J and u_J^* of Eqs. (3) and (4) which in nuclear physics are called (single particle) coefficients of fractional parentage (CFP). (For example the quantity $u_J^*(\alpha I)$ measures the "parentage" (= overlap) of the odd-nucleus state $\mid J >$ in the antisymmetrical direct product state $a_\alpha^\dagger \mid I >$. Our reasons for emphasizing these quantities was two-fold. First, all other quantities are, in principle, expressible in terms of these by means of sum rules. For example, we have

$$< K' \mid a_\alpha^\dagger a_\beta \mid K > = \sum_j v_j(\beta K) \, v_J^*(\alpha K') , \qquad (7)$$

and by an extension of the definition of collective motion, it may be supposed that for a suitable choice of indices α, β the number of states $\mid J >$ that contribute meaningfully to (7) is also sharply limited. Of course the theoretical development that ensues, can apply only if the experimental evidence from observed transitions is consonant with the assumptions made, but this is widely true.

A second reason for preferring these quantities is that in avoiding the explicit construction of antisymmetrized wave functions, we must nevertheless verify, insofar as it is possible, that the amplitudes (2)-(5) that we compute by approximation schemes are not in contradiction with the Pauli principle. We have, for instance, the sum rule

$$< K \mid [a_\alpha^\dagger a_\beta + a_\beta a_\alpha^\dagger] \mid K' > = \delta_{\alpha\beta} \delta_{KK'}$$

$$= \sum_J v_J(\beta K') \, v_J^*(\alpha K) + \sum_L u_L(\beta K) \, u_L^*(\alpha K') , \qquad (8)$$

which must be imposed on any solution or satisfied by it.

We cannot give any details of the scheme of calculation for the CFP which emerged from these considerations, but it is important to summarize its elements which are both dynamical and kinematical:

(i) From the equation of motion

$$[a_\alpha, H] = \Sigma t_{\alpha\beta} a_\beta + \tfrac{1}{2} \Sigma V_{\alpha\beta\gamma\delta} a_\beta^\dagger a_\delta a_\gamma , \qquad (9)$$

using sum rules, we obtain non-linear equations for the CFP. These can be shown to be eigenvalue equations for the energies of the states of an odd nucleus, provided the energies of the associated even nucleus are known. The latter, however, can also be expressed by means of equations which give a second dynamical aspect, namely the conditions

$$(ii) \quad W_K \delta_{KK'} = <K|H|K'> . \qquad (10)$$

By applying sum rules to the right hand side of these equations, the diagonal elements provide a *definition* needed in the evaluation of (9), whereas the off-diagonal elements, where they are not automatically satisfied because of angular momentum conservation or some such symmetry property, provide *additional* non-linear dynamical equations. It can be shown that by satisfying (9) and (10) one is diagonalizing the Hamiltonian within the subspace of states considered.

(iii) The kinematical constraints provided by Eqs. (8) guarantee that we have a theory of (approximate) fermions.

(iv) Additional useful relations follow from expressing the relevant constants of the motion as sum rules.

Considerable effort has been devoted to the development of the scheme outlined above. It was shown early on, that it contained as limiting cases many widely used approximations such as Hartree–Fock, BCS, Random Phase Approximation, the Cranking Model for Rotations and the various semi-phenomenological core-particle coupling schemes applied to the analysis of the spectra of odd nuclei. Even this part of the program is not really complete, though it has not been pursued recently.

We may remark parenthetically, that in the special case that the Hamiltonian H is a polynomial in the generators of a Lie Algebra, then the

GHFA provides a scheme for calculating a set of CFP connecting irreducible representations. So far, however, we have never followed through on this observation.

Finally we set ourselves the task of applying the scheme numerically to real nuclei. This program has had limited success[3] but has not been pursued recently. We are, however, on the verge of picking it up again in consequence of the development of new insights and new optimism, based on the experience gained in the intervening period with the ideas that form the main burden of this presentation.

2. THE ALGEBRAIC METHOD FOR EVEN NUCLEI

We were led to seek a modified approach, by our conviction that technical difficulties encountered in the GHFA, stemmed from the need to deal simultaneously with the properties of even and of odd nuclei, and that we would do better to deal first with the even problem. This strong desire stimulated the realization that the need to satisfy the Pauli principle is, at least superficially, no impediment to such a program. The previous remark will be obvious to group theorists. Thus as long as we deal only with pairs of Fermion operators $a_\alpha^\dagger a_\beta$, or $a_\alpha a_\beta$, $a_\alpha^\dagger a_\beta^\dagger$ or both, then the general Hamiltonian (1) can be written, in more than one way, as a polynomial in a suitable Lie Algebra. Then the specification of the Casimir invariants of the algebra completely specifies the representation, including of course its Fermion character. Unfortunately, the transformation of this realization into a practical scheme of calculation for real nuclei, where the Lie Algebras are formidable, was not immediately evident. Thus, although we have applied the GHFA, albeit in a most imperfect form, to the properties of many isotopes of Sn,[3] the new method has been applied (successfully) mainly to toy models easily studied exactly by computer. For these exactly soluble models we have obtained impressive results. In the long run, we believe that we shall be able to apply our methods to "real nuclei".

Let us now summarize the elements of our scheme just as we did for the GHFA. This will provide not only a useful comparison with the former, but also summarize our experience to date with a growing set of exactly soluble models. The problem is to find a set of algebraic equations determining matrix elements of operator pairs, among the chosen collective states of neighboring even nuclei. For then we can compute all transition matrix elements among the states in question, and a fortiori, from the manner

in which the scheme will be developed, also the energies.

To realize this aim let X_ρ be the above set of fermion pair operators, which either conserve the number of fermions or change the number by two. H can be considered a polynomial in the X_ρ, in general, in many ways. We assume that this choice has been made. (For simple models with separable interactions, such as the conventional pairing plus quadrupole-quadrupole interaction model, it is the obvious one). We then wish to diagonalize the Hamiltonian, within the subspace of collective states, and find its eigen-values and "eigenvectors". But we do not follow the conventional road of setting up a basis of antisymmetric states, and then diagonalizing the Hamiltonian within this basis.

Our basic tool is the sum rule (6) applied to the generators X_ρ themselves. But these sum rules cannot be true in a realistic model for all choices of generators X_ρ, since if they were, the entire space spanned by the X_ρ and the collective space would be synonymous. The latter is true, however, for our exactly soluble simplified models. We shall assume, in what follows, that the Hamiltonian has been expressed in terms of a subset of the X_ρ which permit the straightforward application of sum rules. We now proceed in analogy with the listing given for the GHFA:

(i) The first dynamical condition, the analogue of (9), is that the eigenvalues of H be correctly given in the collective subspace. We have found that there are three convenient, more or less equivalent expressions of this condition. For example, let S be any polynomial in the generators X_ρ which commutes with all the operators (other than H) specifying $|K>$. The relation

$$< K | [S, H] | K > = 0 , \qquad (11)$$

leads, upon utilization of (6), to a sum rule which may be viewed as our version of the Schrödinger equation. On the other hand, if S is of the form

$$S = \sum_i T_i T_i , \qquad (12)$$

where i is an appropriate tensor index and T_i may be a generator, we may use instead of (11), the "equations of motion"

$$< K | [T_i, H] | K' > = [W(K') - W(K)] < K | T_i | K' > . \qquad (11')$$

It is not difficult to show that (11) is a consequence of the set (11'). That (11') brings in new variables, the energy differences $[W(K') - W(K)]$, is not necessarily a disadvantage, since we want these numbers anyway. The third form of this condition consists in the computation of the eigenvalue $W(K)$ in the obvious manner, by use of (6)

$$W(K) = <K|H|K> = W[K'; <K'|X_\rho|K''>] . \tag{13}$$

The Rayleigh-Ritz principle is then applied under the assumption that the $<K|X_\rho|K'>$ may be considered variational parameters. We thus use the condition

$$\delta W[K; <K'|X_\rho|K''>] = 0 . \tag{14}$$

We have utilized all these conditions in one application or another.

(ii) The condition

$$<K|H|K'> = 0, \qquad K \neq K' \tag{15}$$

remains part of the scheme but now is a polynomial in the matrix elements of the X_ρ rather than of the $a_\alpha, a_\beta^\dagger$.

(iii) The kinematical constraints are of two kinds. We have sum rules based on the commutators

$$[X_\rho, X_\sigma] = \sum_\tau C_{\rho\sigma\tau} X_\tau , \tag{16}$$

and we have specification of the representation by means of the Casimir relations

$$<K|C|K'> = \delta_{KK'}\lambda , \tag{17}$$

where C is one of the Casimir invariants and λ is its eigenvalue for the representation of interest. An alternative to (17) is also being utilized and may prove useful for practical cases: The X_ρ may be divided into two sets $\{Y_\rho, Z_\sigma\}$. From the Y_ρ we may construct the basis, and pass from any basis vector to

any other. It follows that the remaining Z_σ are expressible as polynomials in the Y_ρ and it may happen that for a *subset* of all the states, these polynomials contain a convergence factor so that a few low order terms suffice for a good numerical approximation to matrix elements of Z_σ. Altogether instead of a set of non-linear equations for the single particle CFP, we have equations for the matrix elements of the X_ρ.

3. EXAMPLES

This method has been applied to a number of models, which we shall describe briefly, giving representative results.

(a) The model of MGL[4,5]. In this model, which is of two equal but nondegenerate shells, the lower is one completely filled in the independent particle limit. The residual interaction is monopole-monopole force. The Hamiltonian in this case is a quadratic polynomial in the generators of $SU(2)$, so that the model is almost irreducibly simple. Nevertheless it is rich enough to contain, in the limit of a large number of particles, a phase transition from a monopole vibrational spectrum for weak interaction, to a nearly degenerate doublet structure in the strong coupling limit. We have an algebraic solution which gives accurately the properties of the first four states for any value of the coupling strength.
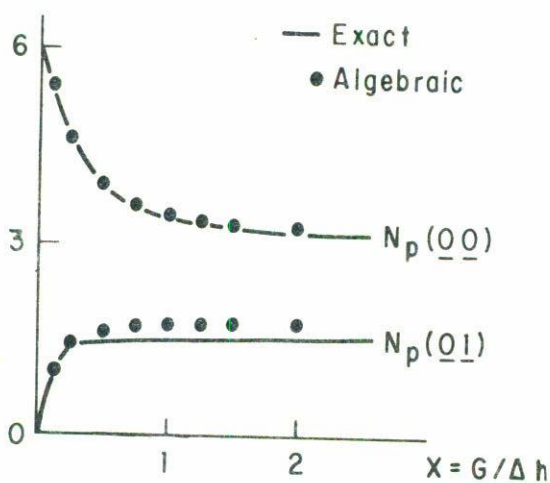

Fig. 2. Comparison of exact and algebraic method results for a two-level pairing model. The levels are labelled p and q. Level q(unperturbed) lies 2 MeV above level p and each carries $j = (11/2)$. Results are given as a fraction of the ratio of the pairing strength to the single particle splitting for $N = 8$ particles. Number conservation and blocking have been carefully taken into account.
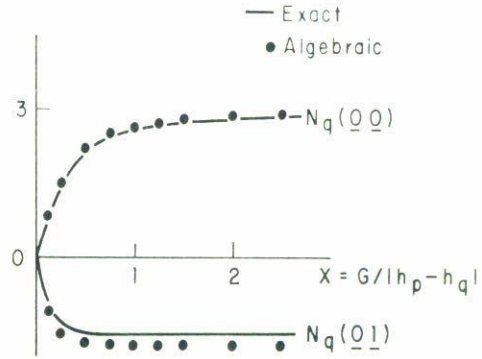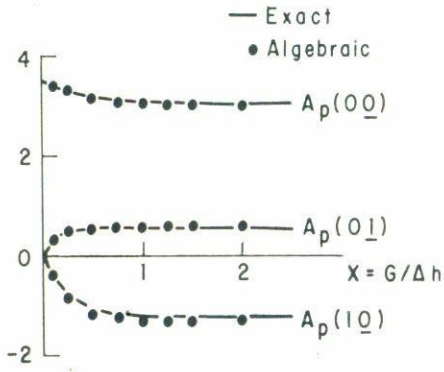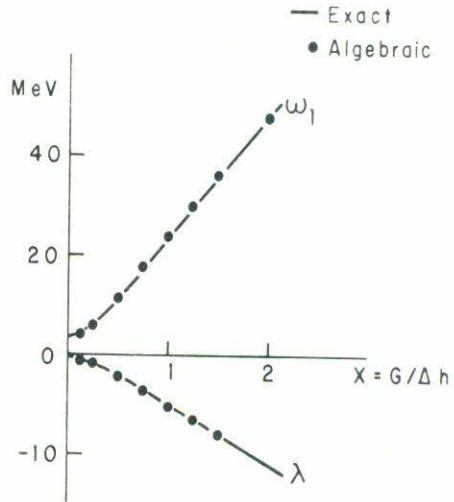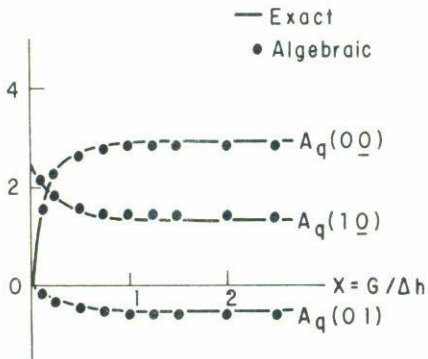
FIG. 2

Fig. 2. Continued

(b) Several levels interacting via a pairing force, where the usual first treatment employs the BCS approximation. In this model we have $SU(2) \times SU(2) \times \ldots$ (one for each single particle level). In Fig. 2, we illustrate our best results[6] for a two-level model and in Fig. 3 for a three level model. In the former instance our states $|K>$ are two in number, the superconducting ground state and the lowest lying excited state of the same seniority. For the latter case we utilize the ground state and two excited states. Correspondingly for n levels, we could construct a theory of similar accuracy with $n-1$ excited states.

(c) Somewhat more amusing and novel than the previous models is one recently proposed possessing the symmetry of the algebra $R(5)$ which we describe in more detail. We consider $2(2p + 1)$ degenerate orbitals labeled $m, \pm 1$ (Fermion destruction, creation operators, $a_{m,\pm 1}$, $a_{m,\pm 1}{}^{\dagger}$). In this model, m plays the role of degeneracy label and ± 1 means a positive or negative unit of angular momentum with respect to a fixed axis. With $\Omega = p + \frac{1}{2}$, the following ten pairing and multipole operators span the algebra of $R(5)$:

$$A_0^{\dagger} = (A_0)^{\dagger} = (2\Omega)^{-\frac{1}{2}} \sum_m (-1)^{p-m} a_{m,+1}^{\dagger} a_{-m,-1}^{\dagger} , \qquad (18a)$$

$$A_+^{\dagger} = (A_+)^{\dagger} = \frac{1}{2} (\Omega)^{-\frac{1}{2}} \sum_m (-1)^{p-m} a_{m,+1}^{\dagger} a_{-m,+1}^{\dagger} , \qquad (18b)$$

$$A_-^{\dagger} = (A_-)^{\dagger} = \frac{1}{2} (\Omega)^{-\frac{1}{2}} \sum_m (-1)^{p-m} a_{m,-1}^{\dagger} a_{-m,-1}^{\dagger} , \qquad (18c)$$

$$B_+ = (B_-)^{\dagger} = (2\Omega)^{-\frac{1}{2}} \sum_m a_{m,+1}^{\dagger} a_{m,-1} , \qquad (18d)$$

$$J_0 = \sum_m [a_{m,+1}^{\dagger} a_{m,+1} - a_{m,-1}^{\dagger} a_{m,-1}] , \qquad (18e)$$

$$N = \sum_m [a_{m,+1}^{\dagger} a_{m,+1} + a_{m,-1}^{\dagger} a_{m,-1}] . \qquad (18f)$$

The definitions imply that p is half-integral. The operators $A^{\dagger}$ create two particles and, according to subscript, zero or ± 2 units of angular momentum, The remaining operators conserve particle number, B_+ raising the angular momentum, J_0 measuring it, and N measuring the particle number.
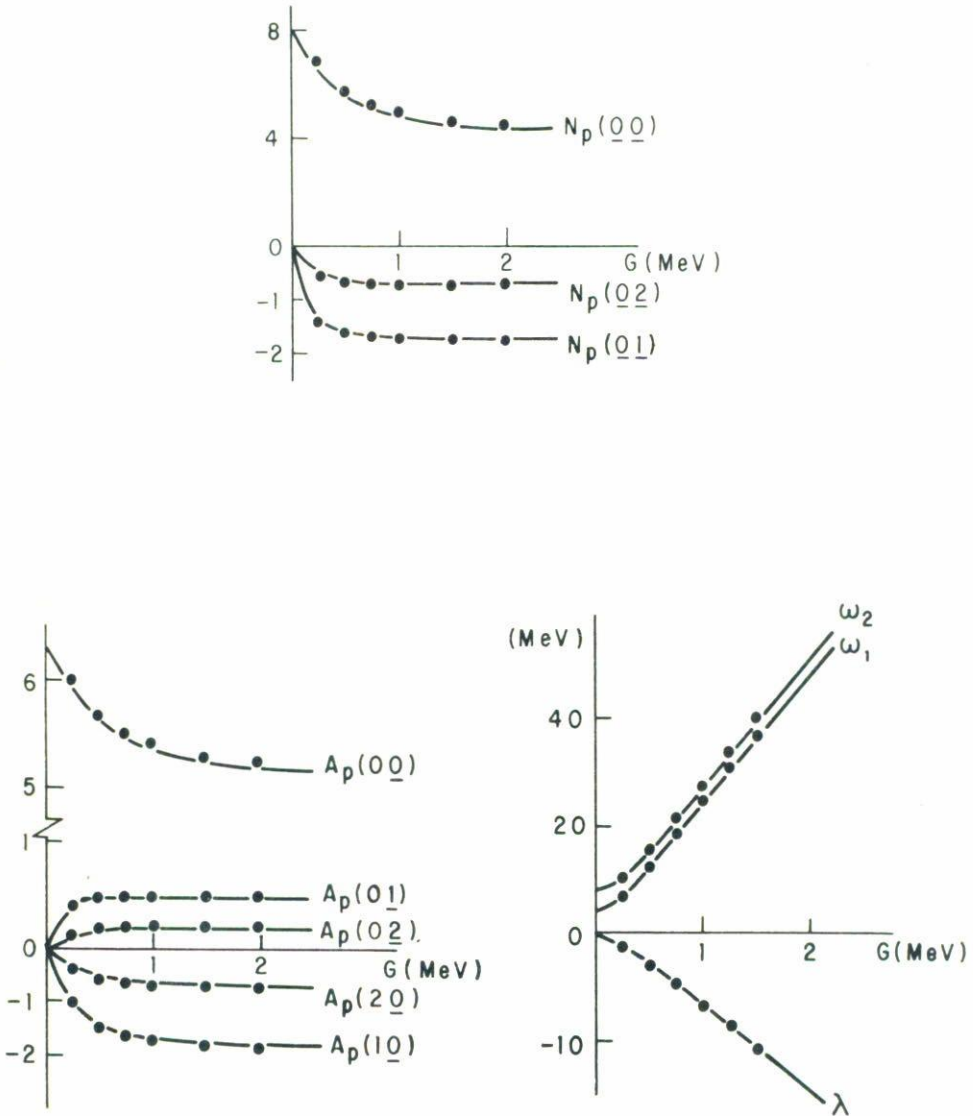
Fig. 3. Same results for a three level model. Level q lies 2 MeV above p and 2 MeV below level r. Also $j_p = (23/2)$, $j_q = j_r = 11/2$ and $N = 8$. The approximation is number conserving, but does not include blocking.

We study the Hamiltonian[7]

$$H = -2G\Omega\, A_0^\dagger A_0 - F\,\{B_+, B_-\}\;, \tag{19}$$

which commutes with J_0. Exact solutions are obtained starting from a convenient set of (non-orthogonal) basis vectors

$$\left|\, N\, J_0\, n_0 \right> \equiv (A_+)^m (A_-^\dagger)^n (A_0^\dagger)^{n_0} \left|\, 0 \right>\;,$$

$$m + n + n_0 = \tfrac{1}{2}N\;, \quad m - n = \tfrac{1}{2}J_0\;. \tag{20}$$

Fig. 4 shows typical spectra, obtained for $\Omega = 24$ and $N = 12$ particles. For $x = (F/4G\Omega) = 0$, the spectrum is nearly harmonic near the ground state, the excitation energies depending only on the seniority v

$$W_{N,v+2} - W_{N,v} = 2G\Omega\left[1 - (v/2\Omega)\right]\;. \tag{21}$$

In the other extreme limit $x^{-1} = 0$ we have another $SU(2)$ group characterized by the three operators $J_\pm^1 = (2\Omega)^{\frac{1}{2}}\, B_\pm$, $J_0^1 = \tfrac{1}{2}J_0$ and

$$W_{J,J_0} = -(F/8\Omega)\left[J(J+2) - J_0^2\right]\;, \tag{22}$$

where J is the maximum allowed value of J_0. Thus the value of J is the maximum angular momentum to be found in the given band, higher bands starting with values of J diminished by multiples of four. In general for $x < 1$, the spectrum resembles that of a two-dimensional quadrupole vibrator. For $x > 1$ there is a transition to a series of rotational bands.

In this model, we have applied[8] the algebraic method to the first two layers of states comprising in the vibrational region the ground state $J_0 = 0$, the one phonon states $J_0 = \pm 2$, and the two phonon states $J_0 = \pm 4, 0'$. In the deformed region these become the members $0, \pm 2, \pm 4$ of the ground state rotational band and $0'$ the band head for the second rotational band. Comparison of exact and approximate results for a system $N = 22$, $\Omega = 44$ is carried out in Figs. 5, 6, 7 for the excitation energies and in Figs. 8, 9 for the matrix elements and is, on the whole, gratifying.
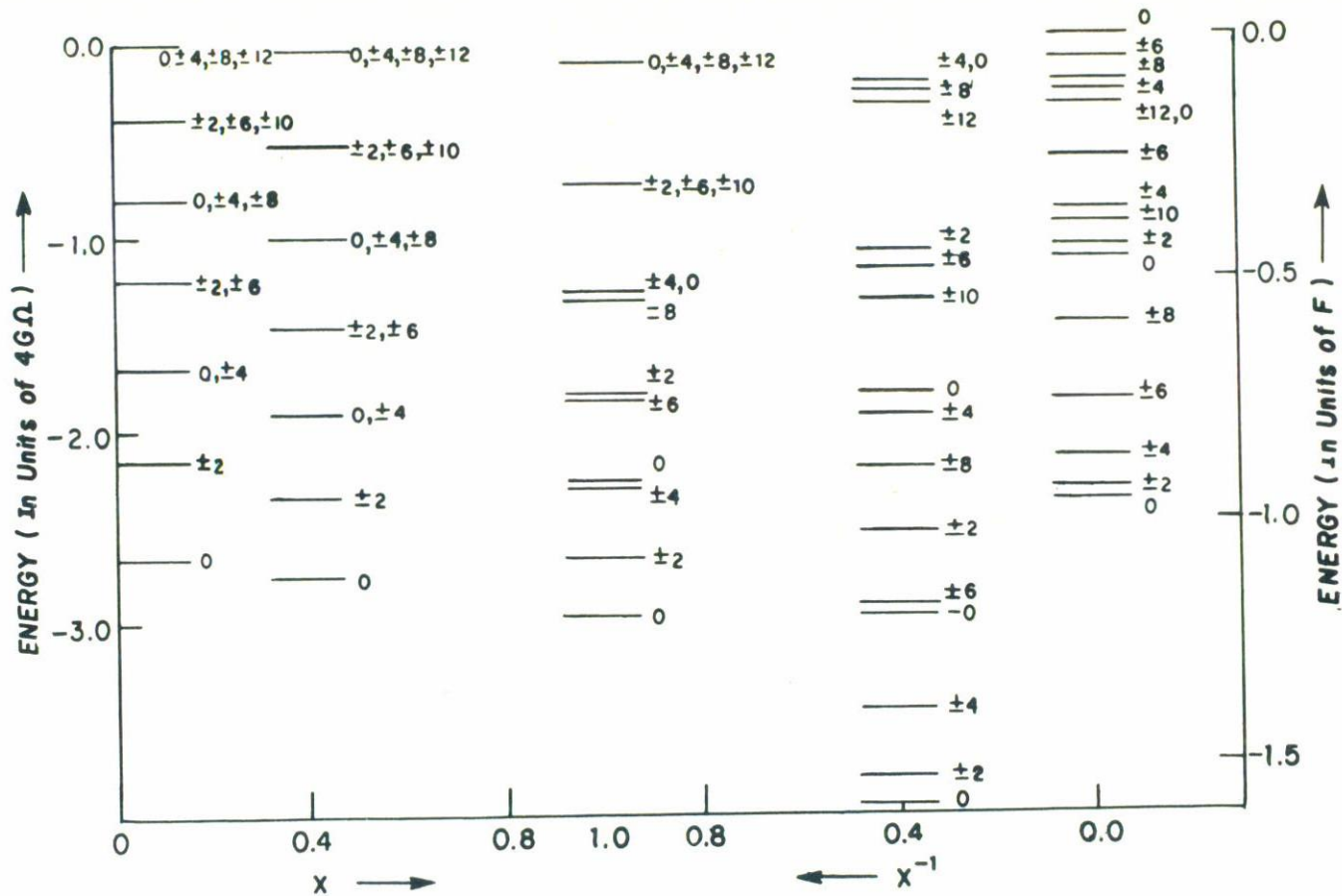
Fig. 4.   Energy levels of the Hamiltonian, Eq. (19) for various values of the ratio $(x = F/4G\Omega)$. Notice that at $x = 1$, the scale of the abscissa has been changed to x^{-1}. For $x \leq 1$ the energy scale on the left applies and for $x > 1$, the energy scale on the right is appropriate. The levels correspond to the system $\Omega = 22$, $N = 12$.
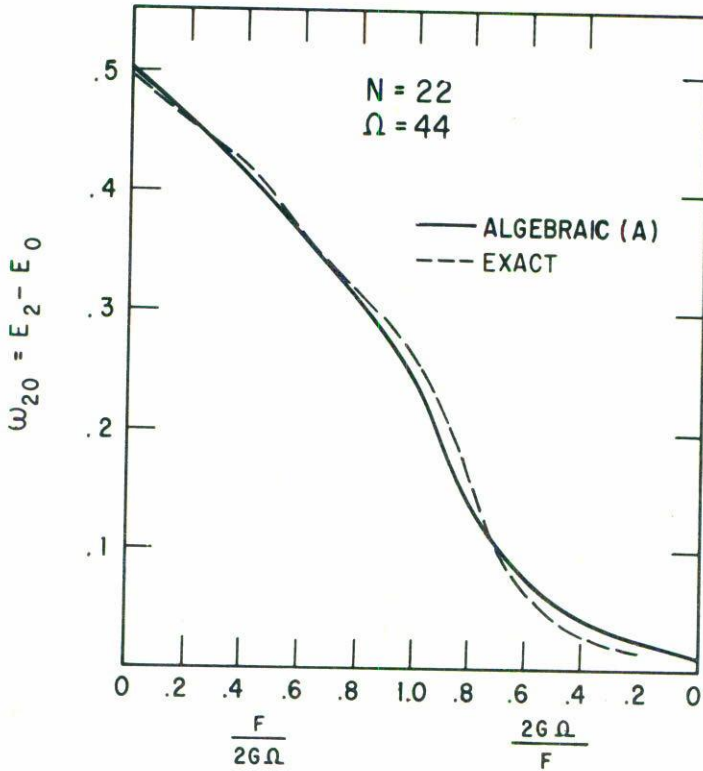
Fig. 5. Excitation energy ω_{20} of the first excited states of angular momentum ± 2 as a function of the relative strength $(F/2G\Omega)$ of the quadrupole to the pairing interaction for the system defined by $\Omega = 44$, $N = 22$. The exact calculations are compared with the result of an algebraic approximation. Note here, as in all succeeding figures, the change in scale about $F = 2G\Omega$.

(d) Wavefunctions. For this model and by obvious extension for the others we have considered, we have constructed a class of state vectors[9] with which it is quite practical to work, and which turn out to be good approximations to the algebraic treatment, in the restricted sense of permitting treatment of the ground state band. The nominally three-parameter generating state

$$| xy_+y_- \rangle = \exp [xA_0^\dagger + y_+ A_+^\dagger + y_- A_-^\dagger] | 0 \rangle \tag{23}$$

conserves neither particle number nor angular momentum. However, two of the parameters may be chosen to yield preassigned average N and J_0 and
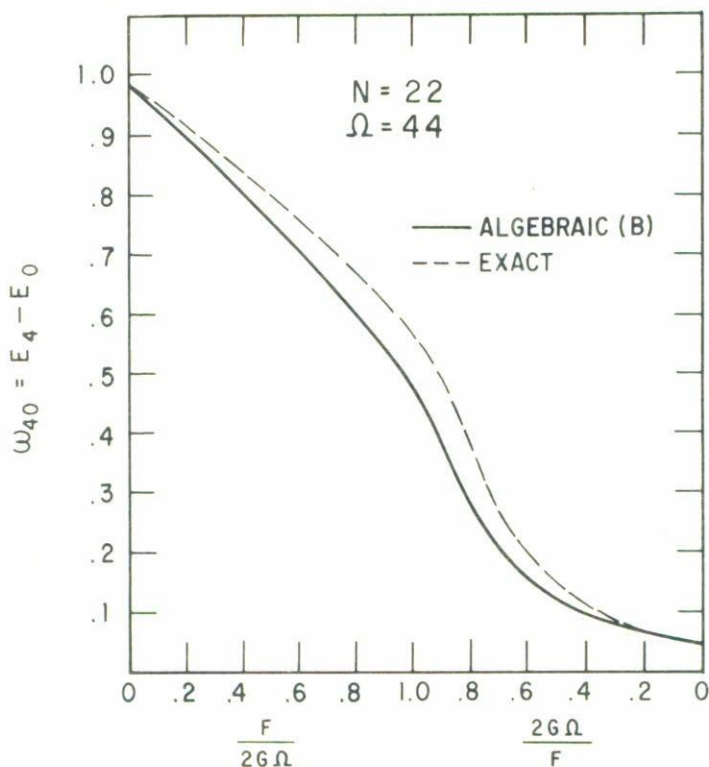
Fig. 6.   Excitation energy of the first states ±4 calculated both exactly and by
          an algebraic approximation. Cf. caption of Fig. 5.

the third to minimize the energy. We have shown, representing the algebra
of $R(5)$ by means of polynomials in the parameters and their derivatives, that
we can convert this calculation into an approximate version of the algebraic
method. This was shown both theoretically and by numerical comparison.
It also suggests itself that we should be able to approach more closely to
the results of the algebraic method (which after all can be carried out in a
number and angular momentum conserving approximation), by treating x, $y_{\pm}$
as generator coordinates after the method of Hill and Wheeler. This program
has not yet been started. The technique based on (23), however, has proved
extremely useful for studying, albeit in a more approximate manner than we
have hitherto countenanced, the properties of the ground state band up to
high angular momentum. For example, by considering a direct product
$R(5) \times R(5)$, we have obtained models of the phase transition which occurs
in the ground state band at high angular momentum. This has been interpre-
ted as the Coriolis coupling between a rotating core (frame) and a set of
superfluid particles, eventually undoing the pairing correlations of the latter.
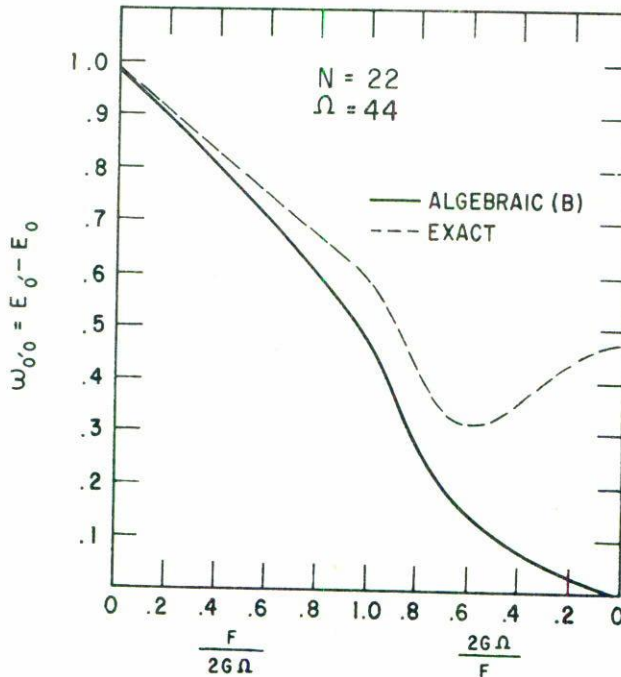This work is still in progress.

Fig. 7. Calculation analogous to that of Fig. 6 for the state O', the second state of angular momentum zero.

(e) The single-j shell model. The study of this model, utilizing a conventional pairing plus quadrupole-quadrupole interaction Hamiltonian is the closest we have come to real physics. An exact diagonalization program succeeded in producing exact results for systems of maximal size [10], $j^n = (21/2)^6$ which proved extremely useful in assessing approximations. Most of this work preceded the work on $R(5)$, an unfortunate temporal sequence from the point of view of efficiency of development of ideas. The algebraic work has produced two ideas which we are confident will stand up upon further development. We have both argued and showed by numerical comparison [11] that there is a valid concept which we call the collective subalgebra: Although fermion pairs constructed from individual j's of large magnitude can couple to very large resultant angular momentum, collective effects appear dominantly in the low multipoles. We have shown, convincingly but perhaps not yet cogently, how to incorporate this idea into the algebraic approach. Secondly [11], we have produced a more accurate alternative to the conventional theory of nuclear shapes based on Hartree-Bogoliubov theory. In Fig. 10, the pairing corre-
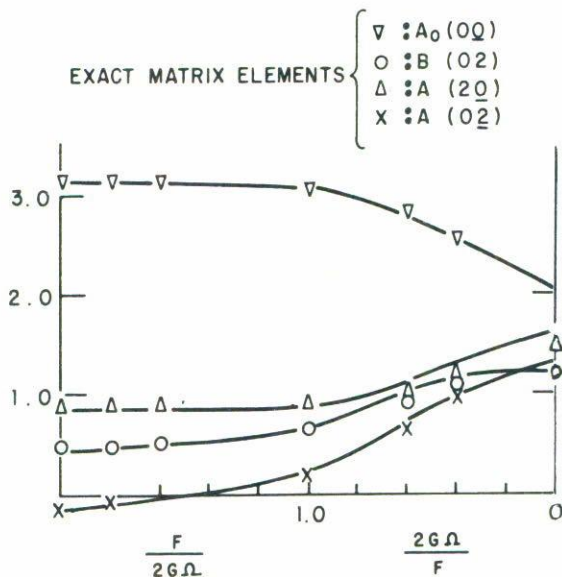
Fig. 8.   Some matrix elements for the ground state and first excited state. The continuous line shows the solutions for the four matrix elements as provided by the algebraic procedure, whereas the contiguous triangles, etc., identified in the legend, are the exact values.
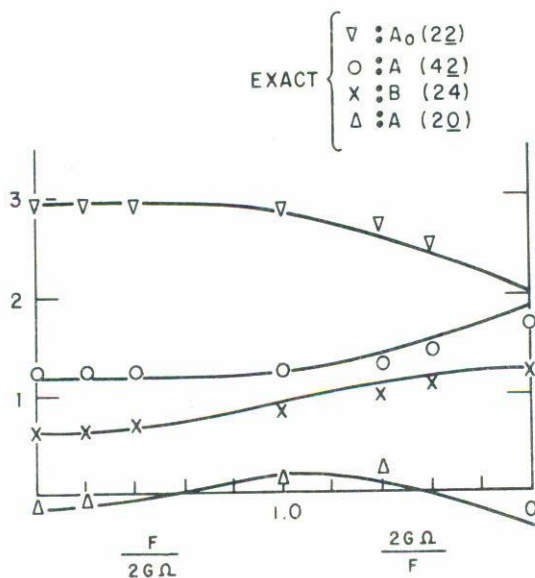

Fig. 9.   The same comparison as in Fig. 8 for a selected set of matrix elements connecting the first excited state and the second layer of excited states.

lation and intrinsic quadrupole moment are compared with the exact answers for $j^n = (21/2)^6$ and with the corresponding results of the conventional theory. More work remains to be done on this model.
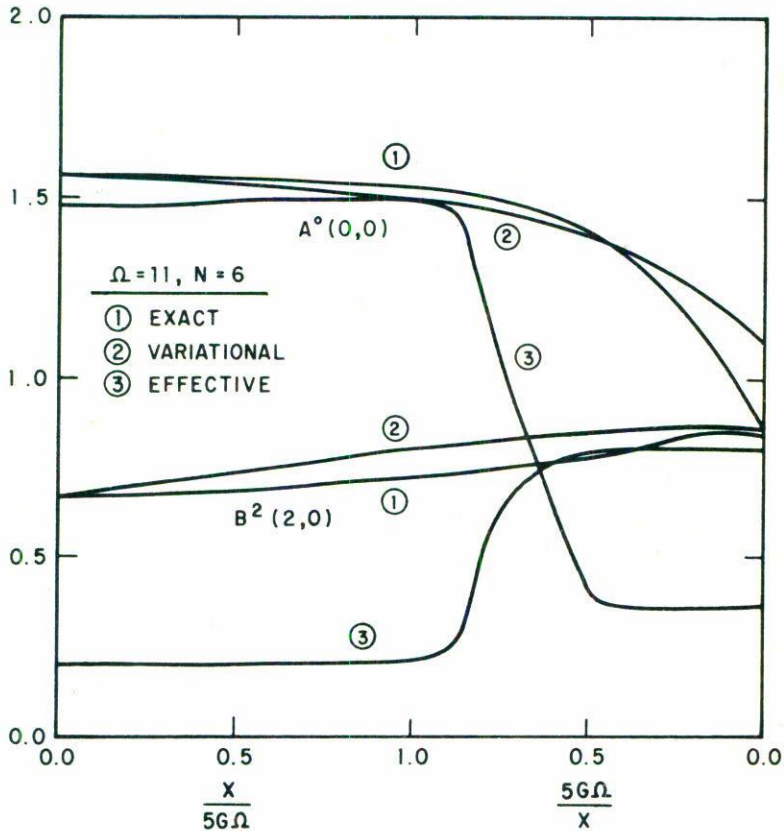

Fig. 10.   Comparison between exact, algebraic, and BCS approximations to the matrix elements $A^0(0,0)$ and $B^2(2,0)$. The configuration is $(21/2)^6$ of the single j model with rotational invariance in three dimensions.

(f) Ground state correlations at closed shells. We merely allude to related work on the theory of long-range (random phase approximation type) correlations in the ground state wave functions of closed shell nuclei[12]. Some advances in formalism can be claimed, but no applications have been attempted.

4. OUTLOOK

Work continues for the time being on models which have rotational symmetry about a fixed axis, including $R(5)$ as described, $R(5) \times R(5)$, and still more involved algebras which contain these as subalgebras. The aim is to obtain a deeper understanding of the concept of collective subalgebra which proved useful for the single-j model. There, the choice of this collective subalgebra was kinematically obvious. In the new examples, the subalgebra will be dynamically determined: the lessons to be learned should then be applicable to the case of complete rotational invariance. Technically, the experience gained in solving sizeable sets of non-linear algebraic equations should prove useful in improving the previous studies described in our introductory remarks. Finally contact has been made with other methods in the theory of collective motion, such as boson expansions and generator coordinates.

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RESUMEN

Se define un modelo de capas con solución exacta, prescribiendo como Hamiltoniano un polinomio en los generadores del álgebra de Lie y escogiendo un álgebra suficientemente simple para que la diagonalización, llevada a cabo por computadora, se pueda interpretar como la solución de un sistema grande de fermiones. Hasta ahora los modelos basados en $SU(2)$, $SU(2) \times SU(2) \times \ldots$, y $R(5)$ se han estudiado exhaustivamente. El principal interés de estos estudios es la forma excepcionalmente pura en la que tales sistemas manifiestan grados de libertad colectivos, interpretables en casos límites como vibraciones y rotaciones. Consecuentemente, el objeto de los trabajos en estos modelos es la comparación de varias de las más usuales teorías aproximadas de movimiento colectivo, con las soluciones exactas. Además de lo anterior, nuestro mayor interés ha sido el desarrollo de un esquema diferente que tiene su origen tanto en los métodos de ecuaciones de movimiento de la teoría moderna de muchos cuerpos, como en el enfoque del "álgebra de corrientes" de la teoría de partículas elementales. En este método, tanto los comutadores cinemáticos como los dinámicos y otras relaciones algebráicas se convierten en reglas de suma para observables, donde el total constituye un conjunto cerrado de ecuaciones inhomogéneas, no lineales que definen las propiedades de un subconjunto colectivo de estados para el sistema estudiado. Se muestra su significado físico, exactitud y relación con otros métodos