NAGEL-MOSHINSKY OPERATORS FOR DISCRETE UNITARY REPRESENTATIONS OF $U(p, q)$*

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ABSTRACT: It is shown that the operators of Nagel and Moshinsky, which raise and lower the representations of $U(n-1)$ contained in a representation of $U(n)$ can also be used, after a relabelling, to raise and lower the representations of $U(p, q-1)$ contained in a discrete unitary representation of $U(p, q)$ which is bounded above or below. An algorithm is given for deriving similar operators for the unbounded discrete series of $U(p, q)$ and the details worked out for the simplest non-trivial case, that of the conformal group, $U(2, 2)$.

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1. INTRODUCTION

Nagel and Moshinsky[1] have constructed explicit operators which raise and lower IR's of $U(n-1)$ contained in an IR of $U(n)$; recently one of us[2] has shown that identical operators serve to raise and lower IR's of $U(p)$ contained in a discrete unitary irreducible representation (DUIR) of $U(p,1)$.

In the present paper we consider the problem of constructing operators which raise and lower IR's of $U(p,q-1)$ contained in a DUIR of $U(p,q)$, $p \geqslant q \geqslant 2$. The raising and lowering operators are to be polynomial functions of the generators of the group. The utility of such operators is apparent. Starting from the extreme (or other) state of a basis they enable us to obtain any other particular basis state.

In Sec. 2 we show that operators essentially identical to those of Nagel and Moshinsky, apart from a relabelling, serve to raise and lower $U(p,q-1)$ IR's contained in a bounded above or bounded below DUIR of $U(p,q)$. In Sec. 3 an algorithm is presented for the construction of raising and lowering operators for $U(p,q-1)$ IR's contained in unbounded DUIR's of $U(p,q)$ and is applied to obtain explicit raising and lowering operators for the simplest non-trivial case, the unbounded DUIR's of the conformal group $U(2,2)$. The remainder of this section contains a discussion of the generators of generators of $U(p,q)$ and the Gel'fand patterns which provide a basis for its DUIR's.

The group $U(p,q)$, $p \geqslant q$, has n^2 generators E_{jk}, $1 \leqslant j,k \leqslant n$ with commutation rules

$$[E_{hj}, E_{kl}] = \delta_{jk} E_{hl} - \delta_{hl} E_{kj} \tag{1}$$

(here and elsewhere we use the notation $n \equiv p+q$). The generators have the following hermiticity properties:

$$\begin{aligned} E_{jj}^{\dagger} &= E_{jj}, \quad && \text{all } j, \\ E_{jk}^{\dagger} &= E_{kj}, \quad && 1 \leqslant j,k \leqslant p \text{ or } p+1 \leqslant j,k \leqslant n, \\ E_{jk}^{\dagger} &= -E_{jk}, \quad && 1 \leqslant j \leqslant p < k \leqslant n. \end{aligned} \tag{2}$$

A finite unitary transformation may be written exp iG where G is a real linear combination of the hermitian generators E_{jj}, $E_{jk} + E_{jk}^{\dagger}$, $(E_{jk} - E_{jk}^{\dagger})/i$ (here $j < k$ to avoid duplication).

As basis states for DUIR's of $U(p, q)$ we use the patterns of Gel'fand and Zetlin[3] and Gel'fand and Graev[4]:

$$|m\rangle = |m_{jk}\rangle , \qquad 1 \leqslant j \leqslant k \leqslant n . \tag{3}$$

They provide IR's for the subgroup chain $U(p, q) \supset U(p, q-1) \supset \ldots .$ $\supset U(p) \supset U(p-1) \supset \ldots \supset U(1)$. The integers m_{jk} are the representation labels of the k'th group from the right in the above chain.

For the compact group $U(n)$ the m's satisfy the betweenness inequalities

$$m_{j,k+1} \geqslant m_{jk} \geqslant m_{j+1,k+1} . \tag{4}$$

Fig. 1(a) shows the general pattern for $U(4)$; the inequalities (4) imply that each m lies between the two m's immediately above it.

Gel'fand and Graev[4] showed that with each finite-dimensional IR of the compact group $U(n)$ there is associated a set of $p+1$ infinite dimensional DUIR's of the non-compact group $U(p, q)$ which we distinguish by the label i, $0 \leqslant i \leqslant p$. For the i'th series the betweenness relations are as follows

For $\quad 1 \leqslant k \leqslant p-1 , \quad 1 \leqslant j \leqslant k ,$

$$m_{j,k+1} \geqslant m_{jk} \geqslant m_{j+1,k+1} . \tag{5a}$$

For $\quad p \leqslant k \leqslant n-1 , \quad 1 \leqslant j \leqslant i ,$

$$m_{j-1,k+1} \geqslant m_{jk} - 1 \geqslant m_{j,k+1} \quad (m_{0,k+1} \equiv \infty) . \tag{5b}$$

$$\begin{array}{ccccccc}
m_{14} & & m_{24} & & m_{34} & & m_{44} \\
& m_{13} & & m_{23} & & m_{33} & \\
& & m_{12} & & m_{22} & & \\
& & & m_{11} & & &
\end{array}$$

(a) $U(4)$

$$\begin{array}{ccccccccc}
& & m_{14} & & m_{24} & & m_{34} & & m_{44} \\
& m_{13} & & m_{23} & & & & m_{33} & \\
m_{12} & & m_{22} & & & & & & \\
& m_{11} & & & & & & &
\end{array}$$

(b) $i = 0$ series of $U(2, 2)$

$$\begin{array}{ccccccccccc}
& & m_{14} & & m_{24} & & m_{34} & & m_{44} & & \\
& m_{13} & & & & m_{23} & & & & m_{33} & \\
m_{12} & & & & & & & & & & m_{22} \\
& & & & & m_{11} & & & & &
\end{array}$$

(c) $i = 1$ series of $U(2, 2)$

$$\begin{array}{ccccccccc}
m_{14} & & m_{24} & & m_{34} & & m_{44} & & \\
& m_{13} & & & & m_{23} & & m_{33} & \\
& & & & & & m_{12} & & m22 \\
& & & & & & & m_{11} &
\end{array}$$

(d) $i = 2$ series of $U(2, 2)$

Fig. 1 Gel'fand patterns for IR's of $U(4)$ and for the $i = 0, 1, 2$ series of DUIR's of $U(2, 2)$. The form of each pattern suggests the inequalities (4, 5) satisfied by the m's.

For $\quad p+1 \leqslant k \leqslant n-1\ ,\quad i+1 \leqslant j \leqslant i+k-p\ ,$

$$m_{j,k+1} \geqslant m_{jk} \geqslant m_{j+1,k+1}\ . \tag{5c}$$

For $\quad p \leqslant k \leqslant n-1\ ,\quad i+k-p+1 \leqslant j \leqslant k\ ,$

$$m_{j+1,k+1} > m_{jk} + 1 \geqslant m_{j+2,k+1} \qquad (m_{k+2,k+1} \equiv -\infty)\ . \tag{5d}$$

Roughly, the inequalities (5) state that in the rows $p \leqslant k \leqslant n-1$ each of the first i m's is moved left past one m of the row above, and each of the last $p-i$ m's is moved right past one m from the position it would occupy in a pattern of the compact group $U(n)$. Fig. 1(b, c, d) shows the general pattern for the $i = 0, 1, 2$ series respectively of $U(2, 2)$. We refer to the $i = 0$ series as bounded above, $i = p$ as bounded below and all other series $1 \leqslant i < p - 1$ as unbounded.

The patterns are eigenstates of the diagonal generators:

$$E_{kk} \,|\, m > \; = w_k \,|\, m > \, , \tag{6}$$

where the n weights w_k are given by

$$w_k = \sum_{j=1}^{k} m_{jk} - \sum_{j=1}^{k-1} m_{j\,k-1} \, , \quad 2 < k < n \, , \tag{7a}$$

$$w_1 = m_{11} \, . \tag{7b}$$

The non-zero matrix elements of the generators $E_{k,\,k+1}$, $E_{k+1,\,k}$ are

$$< m_{jk} + 1 \,|\, E_{k,\,k+1} \,|\, m > \; = \; < m \,|\, E_{k+1,\,k} \,|\, m_{jk} + 1 >$$

$$= i^N \left| - \frac{\prod_{l=1}^{k+1} (m_{l,\,k+1} - m_{jk} - l + j) \prod_{l=1}^{k-1} (m_{l,\,k-1} - m_{jk} - l + j - 1)}{\prod_{l=1,\, l \neq j}^{k} (m_{lk} - m_{jk} - l + j)(m_{lk} - m_{jk} - l + j - 1)} \right|^{\frac{1}{2}} \, , \tag{8}$$

where $|m>$ is an arbitrary pattern of the IR and $|m_{jk} + 1>$ is the same pattern with the label m_{jk} increased by unity. In the compact case $U(n)$, the matrix elements are all positive ($N = 0$); for DUIR's of the non-compact group $U(p, q)$ the phase of the matrix element is i^N where N is the number, 0, 1 or 2, of factors in the numerator of (8) whose signs are reversed compared to the $U(n)$ case. This choice of matrix elements implies the commutation rules (1), and is consistent with the betweenness relations (5). Matrix elements of other non-diagonal generators can be found from (8) with the help of (1).

2. BOUNDED DUIR's OF $U(p,q)$

We briefly review Nagel and Moshinsky's derivation[1] of operators that raise or lower the IR's of $U(n-1)$ contained in a fixed IR of $U(n)$.

They define an extreme state of $U(n-1)$ as a pattern $|m>_E$ which satisfies

$$m_{jk} = m_{j,n-1}, \qquad 1 \leqslant j \leqslant k \leqslant n-1; \tag{9}$$

it is the highest weight state of a $U(n-1)$ IR contained in the IR of $U(n)$. A $U(n-1)$ raising generator annihilates an extreme state:

$$E_{k,k+1} |m>_E = 0, \qquad 1 \leqslant k \leqslant n-2. \tag{10}$$

The Nagel–Moshinsky raising and lowering operators R_{jn} and L_{nj} are defined by the properties

$$R_{jn} |m>_E = (N_{jn})^{-1} |m_{j,n-1} + 1>_E, \tag{11a}$$

$$L_{nj} |m>_E = (N_{nj})^{-1} |m_{j,n-1} - 1>_E, \tag{11b}$$

where $|m_{j,n-1} \pm 1>_E$ is the extreme state of $U(n-1)$ obtained from $|m>_E$ by increasing/decreasing $m_{j,n-1}$ by unity. $(N_{jn})^{-1}$, $(N_{nj})^{-1}$ are non-zero proportionality constants.

Because of Eqs. (10, 11) it follows that R_{jn}, L_{nj} satisfy the commutation rules

$$[E_{k,k+1}, R_{jn}] |m>_E = [E_{k,k+1}, L_{jn}] |m>_E = 0, \quad 1 \leqslant k \leqslant n-2. \tag{12}$$

Nagel and Moshinsky exploit the commutation rules (12) to derive explicit R_{jn} and L_{nj}:

$$R_{jn} = \Big(\sum_{r=0}^{j-1} \sum_{\mu_r > \mu_{r-1} > \ldots > \mu_2 > \mu_1 > 1}^{j-1} E_{j\mu_r} E_{\mu_r \mu_{r-1}} \ldots E_{\mu_2 \mu_1} E_{\mu_1 n} \times$$

$$\times \prod_{l=1}^{r} \xi_{j\mu l}^{-1} \Big) \prod_{\mu=1}^{j-1} \xi_{j\mu} , \quad 1 \leqslant j \leqslant n-1 , \tag{13a}$$

$$L_{nj} = \Big(\sum_{n=0}^{n-j-1} \sum_{\mu_r > \mu_{r-1} > \ldots > \mu_2 > \mu_1 = j+1}^{n-1} E_{\mu_1 j} E_{\mu_2 \mu_1} \ldots E_{\mu_r \mu_{r-1}} E_{n\mu_r} \times$$

$$\times \prod_{l=1}^{r} \xi_{j\mu l}^{-1} \Big) \prod_{\mu=j+1}^{n-1} \xi_{j\mu} , \quad 1 \leqslant j \leqslant n-1 , \tag{13b}$$

where

$$\xi_{jk} \equiv E_{jj} - E_{kk} + k - j \ . \tag{14}$$

See Nagel and Moshinsky [1] for alternative formulas, discussion and particular examples of R_{jn}, L_{nj}.

They find the proportionality constants of Eq. (11) to be

$$N_{jn} = (-1)^{j-1} \left| \frac{\prod_{l=j+1}^{n-1} (m_{l,n-1} - m_{j,n-1} - l + j - 1)}{\prod_{l=1}^{n} (m_{l,n} - m_{j,n-1} - l + j) \prod_{l=1}^{j-1} (m_{l,n-1} - m_{j,n-1} - l + j)} \right|^{\frac{1}{2}} , \tag{15a}$$

$$N_{nj} = \left| \frac{\prod_{l=1}^{j-1} (m_{l,n-1} - m_{j,n-1} - l + j + 1)}{\prod_{l=1}^{n} (m_{ln} - m_{j,n-1} - l + j + 1) \prod_{l=j+1}^{n-1} (m_{l,n-1} - m_{j,n-1} - l + j)} \right|^{\frac{1}{2}} \tag{15b}$$

We now seek to modify the above results so they will apply to the bounded above ($i = 0$) series of DUIR's of $U(p, q)$. Define an extreme state of the subgroup $U(p, q-1)$ as a pattern $|m>_E$ which satisfies

$$m_{jk} = \begin{cases} m_{j,n-1}, & p+1 \leqslant k \leqslant n-1,\ 1 \leqslant j \leqslant k-p, \\ m_{n+j-k-1,n-1} + k - n + 1, & p \leqslant k \leqslant n-1,\ k-p+1 \leqslant j \leqslant k, \\ m_{q+j-1,n-1} - q + 1, & 1 \leqslant k \leqslant p,\ 1 \leqslant j \leqslant k; \end{cases} \tag{16}$$

it is the highest weight state of a $U(p, q-1)$ IR.

Equations (10, 11, 12) are still valid; it follows that the raising and lowering operators given by (13) are still the ones we require, apart from a relabelling. We find

$$R^0_{jn} = \begin{cases} R_{p+j,n}, & 1 \leqslant j \leqslant q-1, \\ R_{j-q+1,n}, & q \leqslant j \leqslant n-1. \end{cases} \tag{17a}$$

where the superscript represents the value of $i\,(=0)$. That (17a) is the correct identification is confirmed by comparison of the change in weight induced by the operators on the two sides. The lowering operators must be relabelled in the same way.

$$L^0_{nj} = \begin{cases} L_{n,p+j} & 1 \leqslant j \leqslant q-1 \\ L_{n,j-q+1} & q \leqslant j \leqslant n-1. \end{cases} \tag{17b}$$

The normalization coefficients N^0_{nj}, N^0_{jn} are found by evaluating the matrix elements of R^0_{nj}, L^0_{jn} between the appropriate extreme states, using the similar calculation of Nagel and Moshinsky[1] as a model. For $1 \leqslant j \leqslant q-1$ we find

$$N_{jn}^{0} = (-1)^{j-1} \left| \frac{\prod\limits_{l=j+1}^{q-1} (m_{l,n-1} - m_{j,n-1} - l + j - 1)}{\prod\limits_{l=1}^{n} (m_{l,n} - m_{j,n-1} - l + j) \prod\limits_{l=1}^{j-1} (m_{l,n-1} - m_{j,n-1} - l + j)} \right|^{\frac{1}{2}} \times$$

$$\times \left| \prod_{l=q}^{n-1} (m_{l,n-1} - m_{j,n-1} - l + j) \right|^{-\frac{1}{2}} , \qquad (18a)$$

$$N_{nj}^{0} = \left| \frac{\prod\limits_{l=1}^{j-1} (m_{l,n-1} - m_{j,n-1} - l + j + 1) \prod\limits_{l=q}^{n-1} (m_{l,n-1} - m_{j,n-1} - l + j + 1)}{\prod\limits_{l=1}^{n} (m_{l,n} - m_{j,n-1} - l + j + 1) \prod\limits_{l=j+1}^{q-1} (m_{l,n-1} - m_{j,n-1} - l + j)} \right|^{\frac{1}{2}} ,$$

(18b)

and for $q \leqslant j \leqslant n-1$,

$$N_{jn}^{0} = (-1)^{n} \left| \frac{\prod\limits_{l=1}^{q-1} (m_{l,n-1} - m_{j,n-1} - l + j - 1) \prod\limits_{l=j+1}^{n-1} (m_{l,n-1} - m_{j,n-1} - l + j - 1)}{\prod\limits_{l=1}^{n} (m_{l,n} - m_{j,n-1} - l + j) \prod\limits_{l=q}^{j-1} (m_{l,n-1} - m_{j,n-1} - l + j)} \right|^{\frac{1}{2}} ,$$

(18c)

$$N_{nj}^{0} = (-1)^{n-j-1} \left| \frac{\prod\limits_{l=q}^{j-1} (m_{l,n-1} - m_{j,n-1} - l + j + 1)}{\prod\limits_{l=1}^{n} (m_{l,n} - m_{j,n-1} - l + j + 1) \prod\limits_{l=1}^{q-1} (m_{l,n-1} - m_{j,n-1} - l + j)} \right|^{\frac{1}{2}} \times$$

(18d)

$$\times \left| \prod_{l=j+1}^{n-1} (m_{l,n-1} - m_{j,n-1} - l + j) \right|^{-\frac{1}{2}} .$$

(18d)

The results we have just derived for the bounded above ($i = 0$) series are easily converted to apply to the bounded below ($i = p$) series of DUIR's of

$U(p, q)$. We take the extreme state of $U(p, q-1)$ as the *lowest* weight state of a $U(p, q-1)$ IR. Then the raising (lowering) operators for $i = p$ are the hermitian conjugates of the lowering (raising) operators for $i = 0$, with an appropriate relabelling.

Specifically, an extreme state of $U(p, q-1)$ is now a pattern $|m>_E$ for which

$$m_{jk} = \begin{cases} m_{j,n-1} + n - k - 1, & p \leqslant k \leqslant n-1, \quad 1 \leqslant j \leqslant p, \\ m_{j,n-1}, & p+1 \leqslant k \leqslant n-1, \quad p+1 \leqslant j \leqslant k, \\ m_{p+j-k,n-1} + q - 1, & 1 \leqslant k \leqslant p, \quad 1 \leqslant j \leqslant k. \end{cases} \tag{19}$$

$$R^p_{jn} = (L^0_{n,n-j})^\dagger , \tag{20a}$$

$$L^p_{nj} = (R^0_{n-j,n})^\dagger . \tag{20b}$$

The normalization coefficients are the same as for $i = 0$, Eq. (18), apart from the relabelling and a phase factor:

For $1 \leqslant j \leqslant p$,

$$N^p_{jn} = (-1)^{p-j-1} N^0_{n,n-j} , \tag{21a}$$

$$N^p_{nj} = (-1)^{p-j-1} N^0_{n-j,n} ; \tag{21b}$$

and for $p+1 \leqslant j \leqslant n-1$,

$$N^p_{jn} = (-1)^{j-q-1} N^0_{n,n-j} , \tag{21c}$$

$$N^p_{nj} = (-1)^{j-q-1} N^0_{n-j,n} . \tag{21d}$$

3. UNBOUNDED DUIR's OF $U(p,q)$

For unbounded $(1 \leqslant i \leqslant p-1)$ series of $U(p,q)$ DUIR's, define an extreme state of the subgroup $U(p,q-1)$ as a pattern $|m>_E$ which satisfies

$$m_{jk} = \begin{cases} m_{j,n-1} + n - k - 1, & p \leqslant k \leqslant n-1, \quad 1 \leqslant j \leqslant i, \\ m_{j,n-1}, & p+1 \leqslant k \leqslant n-1, \quad i+1 \leqslant j \leqslant i+k-p, \\ m_{j+n-1-k,n-1} - n + k + 1, & p \leqslant k \leqslant n-1, \quad i+k-p+1 \leqslant j \leqslant k, \\ m_{j,n-1} + q - 1, & i \leqslant k \leqslant p, \quad 1 \leqslant j \leqslant i, \\ m_{j+q-1,n-1} - q + 1, & i+1 \leqslant k \leqslant p, \quad i+1 \leqslant j \leqslant k, \\ m_{j,n-1} + q - 1, & 1 \leqslant k \leqslant i, \quad 1 \leqslant j \leqslant k. \end{cases} \tag{22}$$

The first i elements of the k'th row, $p \leqslant k \leqslant n-2$, are at the extreme right of their ranges (as small as possible); consistently with this, all other elements are at the left of their ranges (as large as possible).

Raising operators R^i_{jn} for the ranges $i+1 \leqslant j \leqslant i+q-1$ and $i+q \leqslant j \leqslant n-1$ are readily constructed

$$R^i_{jn} = \xi_{n-1,p+j-i} \, R^i_{j,n-1} E_{n-1,n}, \quad i+1 \leqslant j \leqslant i+q-2, \tag{23a}$$

$$R^i_{jn} = (\xi_{n-1,j-q+1} + i) \, R^i_{j-1,n-1} E_{n-1,n}, \quad i+q \leqslant j \leqslant n-1 \,. \tag{23b}$$

To construct $R^i_{i+q-1,n}$ we first define operators $\Omega^i_{n,n-1}$ for certain j which are polynomials in the diagonal generators; they satisfy

$$\Omega^{i}_{n,n-1} \left| m_{j,n-1} + 1 \right>_{E'} = \begin{cases} L^{i}_{n-1,j} R^{i}_{j,n-1} \left| m_{j,n-1} + 1 \right>_{E'} , \\ \qquad i + 1 \leq j \leq i + q - 2 , \\ L^{i}_{n-1,j-1} R^{i}_{j-1,n-1} \left| m_{j,n-1} + 1 \right>_{E'} , \\ \qquad i + q \leq j \leq n - 1 . \end{cases} \tag{24}$$

Here $\left| m_{j,n-1} + 1 \right>_{E'}$ is the pattern obtained from the extreme pattern $\left| m \right>_E$ by increasing the single element $m_{j,n-1}$ by unity; it is not itself an extreme pattern. When $L^{i}_{n-1,j} R^{i}_{j,n-1}$, $i + 1 \leq j \leq i + q - 2$ or $L^{i}_{n-1,j-1} R^{i}_{j-1,n-1}$, $i + q \leq j \leq n - 1$, acts on such a pattern, the pattern is multiplied by a certain polynomial in the m's. The operator $\Omega^{i}_{j,n-1}$ is obtained by replacing each m in the polynomial by the linear function of a diagonal generator whose eigenvalue it is for the pattern in question. That the operators Ω defined by Eqs. (24, 26, 29, 31) are in fact polynomials is insured by the presence in Eqs. (23, 26, 28, 31) of the diagonal factors involving ξ; the proof is by induction. Then

$$R^{i}_{i+q-1,n} = \left(\Pi_j \Omega^{i}_{j,n-1}\right) \Big[1 - \sum_{j=i+1}^{i+q-2} \left(L^{i}_{n-1,j} R^{i}_{j,n-1} / \Omega^{i}_{j,n-1}\right) - \\ - \sum_{j=i+q}^{n-1} \left(L^{i}_{n-1,j-1} R^{i}_{j-1,n-1} / \Omega^{i}_{j,n-1}\right) \Big] E_{n-1,n} ; \tag{25}$$

the product in (25) is over the same ranges as the two sums in the middle factor.

To construct R^{i}_{jn}, $1 \leq j \leq i$, we define further polynomials $\Omega^{i}_{j,j',n-1}$ in the diagonal operators which satisfy

$$\Omega^{i}_{j,j',n-1} \left| m_{j',n-1} + 1 \right\rangle_{jE''}$$

$$= \begin{cases} L^{i}_{n-1,j'} (\xi_{j,p+j'-i} - q - i + 2)\, R^{i}_{j',n-1} \left| m_{j',n-1} + 1 \right\rangle_{jE''} , \\ \qquad\qquad i + 1 \leqslant j' \leqslant i + q - 2 , \\ L^{i}_{n-1,j'-1} (\xi_{j,j'-q+1} - q + 2)\, R^{i}_{j'-1,n-1} \left| m_{j',n-1} + 1 \right\rangle_{jE''} , \\ \qquad\qquad i + q \leqslant j' \leqslant n - 1 , \end{cases} \tag{26a}$$

and

$$\Omega^{i}_{j,i+q-1,n-1} \left| m_{i+q-1,n-1} + 1 \right\rangle_{jE''} = R^{i}_{j,n-1} L^{i}_{n-1,j} \left| m_{i+q-1,n-1} + 1 \right\rangle_{jE''} ; \tag{26b}$$

$| m_{j',n-1} + 1 \rangle_{jE''}$ denotes the pattern obtained from $| m \rangle_E$ by increasing by unity $m_{j',n-1}$ and also the "tail" of $m_{j,n-1}$, i.e., m_{jk} for all k in the range $j \leqslant k \leqslant n-2$. The factors $\xi_{j,p+j'-1} - q - i + 2$ and $\xi_{j,j'-q+1} - q + 2$ in Eqs. (26a, 27) are not required and should be omitted for $q = 2$. The raising operators for $1 \leqslant j \leqslant i$ are

$$R^{i}_{jn} = (\Omega^{i}_{j,i+q-1,n-1} - R^{i}_{j,n-1} L^{i}_{n-1,j}) (\prod_{j'} \Omega^{i}_{j,j',n-1}) \times$$

$$\times \left\{ 1 - \sum_{j'=i+1}^{i+q-2} [L^{i}_{n-1,j'} (\xi_{j,p+j'-1} - q - i + 2)\, R^{i}_{j',n-1} / \Omega^{i}_{j,j',n-1}] - \right.$$

$$- \sum_{j'=i+q}^{n-1} [L^{i}_{n-1,j'-1} (\xi_{j,j'-q+1} - q + 2)\, R^{i}_{j'-1,n-1} / \Omega^{i}_{j,j',n-1}] \times$$

$$\left. \times E_{n-1,n} R^{i}_{j,n-1} \right\} . \tag{27}$$

The product in (27) is over the same j' ranges as the two sums in the factor which follows it.

Lowering operators for $1 \leqslant j \leqslant i$ are

$$L^{i}_{nj} = (\xi_{n-1,j} + i + q - 2)\, L^{i}_{n-1,j}\, E_{n,n-1}\,, \qquad 1 \leqslant j \leqslant i\ . \tag{28}$$

To construct $L^{i}_{n,i+q-1}$ we define polynomials $\Omega^{i}_{n-1,j}$ in the diagonal generators satisfying

$$\Omega^{i}_{n-1,j}\, |m_{j,n-1} - 1 >_{E'} = R^{i}_{j,n-1}\, L^{i}_{n-1,j}\, |m_{j,n-1} - 1 >_{E'}\,, \quad 1 \leqslant j \leqslant i\ , \tag{29}$$

where $|m_{j,n-1} - 1 >_{E'}$ is the pattern obtained from $|m>_E$ by decreasing $m_{j,n-1}$ by unity. Then

$$L^{i}_{n,i+q-1} = \Big(\prod_{j=1}^{i} \Omega^{i}_{n-1,j}\Big)\Big(1 - \sum_{j=1}^{i} [R^{i}_{j,n-1}\, L^{i}_{n-1,j}/\Omega^{i}_{n-1,j}]\Big)\, E_{n,n-1}\ . \tag{30}$$

For the construction of L^{i}_{nj}, $i + 1 \leqslant j \leqslant i + q - 2$ or $i + q \leqslant j \leqslant n - 1$ we need polynomials $\Omega^{i}_{n-1,j,j'}$ in the diagonal generators for $1 \leqslant j' \leqslant i$. For $i + 1 \leqslant j \leqslant i + q - 2$ they satisfy

$$\Omega^{i}_{n-1,j,j'}|m_{j',n-1} - 1 >_{jE''} = R^{i}_{j',n-1}(\xi_{j+p-i,j'} + i + q - 2)\, L^{i}_{n-1,j'}|m_{j,n-1} - 1 >_{jE''} \tag{31a}$$

and for $i + q \leqslant j \leqslant n - 1$,

$$\Omega^{i}_{n-1,j,j'}\, |m_{j',n-1} - 1 >_{jE''}$$

$$= R^{i}_{j',n-1}(\xi_{j-q+1,j'} + q - 2)\, L^{i}_{n-1,j'}\, |m_{j',n-1} - 1 >_{jE''}\ . \tag{31b}$$

$m_{j',n-1} - 1 >_{jE''}$ denotes the pattern obtained from $|m>_E$ by reducing by unity $m_{j',n-1}$ and also the "tail" of $m_{j,n-1}$, i.e., all m_{jk} for which k is in the range $p + j - i < k < n - 2$ in case $i + 1 \leqslant j \leqslant i + q - 2$ or all $m_{j+k-n+1,k}$ for which k is in the range $j - q + 1 \leqslant k \leqslant n - 2$ in case $i + q \leqslant j \leqslant n - 1$. The factors $\xi_{j+p-i,j'} + i + q - 2$ and $\xi_{j-q+1,j'} + q - 2$ should be omitted from Eqns. (31, 32) in the case $q = 2$. Further,

$$\Omega^i_{n-1,j,i+q-1} | m_{i+q-1} - 1 \rangle_{jE''} = \begin{cases} L^i_{n-1,j} R^i_{j,n-1} | m_{i+q-1,n-1} - 1 \rangle_{jE''} , \\ \qquad i + 1 \leqslant j \leqslant q - 2 , \\ L^i_{n-1,j-1} R^i_{j-1,n-1} | m_{i+q-1,n-1} - 1 \rangle_{jE''} , \\ \qquad i + q \leqslant j \leqslant n - 1 . \end{cases} \tag{31c}$$

Then, for $i + 1 < j < i + q - 2$,

$$L^i_{nj} \quad (\Omega^i_{n-1,j,i+q-1} - L^i_{n-1,j} R^i_{j,n-1}) (\prod_{j'=1}^{i} \Omega^i_{n-1,j,j'}) \times$$

$$\times (1 - \sum_{j'=1}^{i} [R^i_{j',n-1} (\xi_{j+p-i,j'} + i + q - 2) L^i_{n-1,j'} / \Omega^i_{n-1,j,j'}]) \times$$

$$\times E_{n,n-1} L^i_{n-1,j} , \tag{32a}$$

while for $i + q \leqslant j < n - 1$

$$L^i_{nj} = (\Omega^i_{n-1,j,i+q-1} - L^i_{n-1,j-1} R^i_{n-1,j-1}) (\prod_{j'=1}^{i} \Omega^i_{n-1,j,j'}) \times$$

$$\times (1 - \sum_{j'=1}^{i} [R^i_{j',n-1} (\xi_{j-q+1,j'} + q - 2) L^i_{n-1,j'} / \Omega^i_{n-1,j,j'}]) \times$$

$$\times E_{n,n-1} L^i_{n-1,j-1} . \tag{32b}$$

We have succeeded in constructing raising and lowering operators for unbounded series, $1 \leqslant i \leqslant p-1$, of DUIR's of $U(p,q)$, $q \geqslant 2$ in terms of those for $U(p, q-1)$ and the elementary generators $E_{n-1,n}$ and $E_{n,n-1}$. When raising and lowering in the construction of $U(p, 1)$ are required in the construction of those for $U(p, 2)$ it is understood that those of Ref. 2 are to be used.

It is a straightforward matter in any particular case to calculate the matrix elements of the raising and lowering operators between the states they are designed to connect. The terms in the expressions for R^i_{jn} or L^i_{nj} involving products $R^i_{n-1,j'}$ $L^i_{j',n-1}$ or $L^i_{j',n-1}$ $R^i_{n-1,j'}$ may be omitted for this purpose; their role is simply to cancel unwanted final states when R^i_{jn} or L^i_{nj} acts on an initial extreme state. The normalization coefficients N^i_{jn}, N^i_{nj} are just the reciprocals of the matrix elements of R^i_{jn}, L^i_{nj}.

We conclude this section by writing down explicitly the three raising and lowering operators for the simplest unbounded series of $U(p, q)$ DUIR's namely the $i = 1$ series for the conformal group $U(2, 2)$:

$$R^1_{14} = -\{(\xi_{13}-3)(\xi_{12}-1) + R_{13}L_{31}\} \times$$

$$\times \{(\xi_{12}-2)(\xi_{32}+1) - L_{32}R_{23}\} E_{34} R_{13}, \tag{33a}$$

$$R^1_{24} = \{(\xi_{12}-1)\,\xi_{32} - L_{32}R_{23}\} E_{34}, \tag{33b}$$

$$R^1_{34} = (\xi_{32}+1)\, R_{23} E_{34}, \tag{33c}$$

$$L^1_{41} = (\xi_{31}+1)\, L_{31} E_{43}, \tag{34a}$$

$$L^1_{42} = -\{(\xi_{13}-3)(\xi_{12}-1) + R_{13}L_{31}\} E_{43}, \tag{34b}$$

$$L^1_{43} = -\{\xi_{13}\xi_{32} - L_{32}R_{23}\}\{(\xi_{13}-2)(\xi_{12}-2) + R_{13}L_{31}\} E_{43} L_{32}. \tag{34c}$$

The normalizing coefficients for the raising and lowering operators operating on the pattern $|m>_E$ are as follows

$$N^1_{14} = i\,[(m_{13}-m_{14})(m_{13}-m_{24}+1)(m_{13}-m_{34}+2)(m_{13}-m_{44}+3)\times$$
$$\times(m_{13}-m_{23}+1)(m_{13}-m_{33}+2)]^{-\frac{1}{2}}(m_{13}-m_{33}+3)^{-\frac{3}{2}}\times$$
$$\times(m_{13}-m_{33}+4)^{\frac{1}{2}}(m_{23}-m_{33}+1)^{-1}, \tag{35a}$$

$$N^1_{24} = [(m_{14}-m_{23}+1)(m_{24}-m_{23})(m_{23}-m_{34}+1)(m_{23}-m_{44}+2)\times$$
$$\times(m_{23}-m_{33}+1)]^{-\frac{1}{2}}(m_{13}-m_{23})^{\frac{1}{2}}(m_{13}-m_{33}+2)^{-1}, \tag{35b}$$

$$N^1_{34} = -i\,[(m_{14}-m_{33}+2)(m_{24}-m_{33}+1)(m_{34}-m_{33})(m_{44}-m_{33}-1)\times$$
$$\times(m_{13}-m_{33}+2)(m_{13}-m_{33}+3)(m_{23}-m_{33})]^{-\frac{1}{2}}(m_{13}-m_{33}+1)^{\frac{1}{2}}, \tag{35c}$$

$$N^1_{41} = -i\,[(m_{13}-m_{14}-1)(m_{13}-m_{24})(m_{13}-m_{34}+1)(m_{13}-m_{44}+2)\times$$
$$\times(m_{13}-m_{33}+3)]^{-\frac{1}{2}}[(m_{13}-m_{23})(m_{13}-m_{33}+1)]^{\frac{1}{2}}(m_{13}-m_{23}+1)^{-1}, \tag{36a}$$

$$N^1_{42} = -[(m_{14}-m_{23}+2)(m_{24}-m_{23}+1)(m_{23}-m_{34})(m_{23}-m_{44}+1)\times$$
$$\times(m_{13}-m_{23}+1)]^{-\frac{1}{2}}(m_{23}-m_{33})^{\frac{1}{2}}(m_{13}-m_{33}+2)^{-1}, \tag{36b}$$

$$N^1_{43} = -i\,[(m_{14}-m_{33}+3)(m_{24}-m_{33}+2)(m_{34}-m_{44}+1)(m_{44}-m_{33})\times$$
$$\times(m_{13}-m_{33}+2)(m_{23}-m_{33}+1)]^{-\frac{1}{2}}[(m_{12}-m_{23}+1)(m_{13}-m_{33}+3)]^{-1}. \tag{36c}$$

It is probably possible to generalize the raising and lowering operators of other compact groups, to apply to the corresponding non-compact groups, by methods similar to those used here.

REFERENCES

1. J.G. Nagel and M. Moshinsky, J. Math. Phys. 6 (1965) 682.
2. J. Patera, J. Math. Phys. 14 (1973) 279.
3. I.M. Gel'fand and M.L. Zetlin, Dokl. Akad. Nauk SSSR 71 (1950) 825.
4. I.M. Gel'fand and M.I. Graev, Izv. Akad. Nauk SSSR Ser. Math. 29 (1965) 1329.

RESUMEN

Se muestra que los operadores de Nagel y Moshinsky que suben y bajan las representaciones de $U(n-1)$ contenidas en una representación de $U(n)$, se pueden usar también, después de cambiarles índices, para subir y bajar las representaciones de $U(p, q-1)$ contenidas en una representación unitaria y discreta de $U(p,q)$, que está acotada por arriba o por abajo. Se da un algoritmo para deducir operadores similares para las series discretas no acotadas de $U(p,q)$ y se calcula para el caso no trivial más simple, el del grupo conforme, $U(2,2)$.