DEFORMATIONS OF LIE ALGEBRAS AND GROUPS AND THEIR APPLICATIONS*

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ABSTRACT: The concept of deformations of Lie algebras has had applications in both physics and representation theory. Despite this, little is known about the general classification of the possible deformations of a given Lie algebra and its representations. We present here a general survey of what is known, along with its application to representation theory and physics. The paper is essentially divided into two parts. In the first part we discuss two prominent physical examples of deformations namely the deformation of the Galilei algebra to the Poincaré algebra and the deformation of the Poincaré algebra to the de Sitter algebra. In the second part, we concentrate on what are called first order deformations, by applying a well-known algorithm to various inhomogenizations of semi-simple Lie algebras. A discussion is given indicating which representations can be deformed to which algebras. We also discuss the corre-

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sponding deformations of the group and their connection with multiplier representations and the Iwasawa decomposition. Tables are presented giving a classification of the main results.

INTRODUCTION

It is the purpose of this article to give a review of deformations of Lie algebras, and provide somewhat of a synthesis between the physics and mathematics literature in a way easily accessible to physicists. This paper is then arranged in essentially two parts. First, after stating exactly what a deformation is, we discuss two important physical examples of deformations. In the second part we concentrate on what are called first order deformations, giving an idea of what is more or less the range of validity of such deformations, and then discussing the important connection of these deformations on the group level with multiplier representations and the Iwasawa decomposition. There is no attempt to be rigorous, although some points concerning rigor are discussed. For the details we refer the reader to the literature. Also in sections 3 and 4 some material is presented for the first time especially that concerning the deformations $i^k so(n) \oplus so(k) \Longrightarrow so(n,k)$ which has not appeared in completed form.

The procedure of deforming a Lie algebra (intuitively in many cases the inverse of contraction[1]) has had several applications to physics. Its most natural setting is when one is given an invariance algebra, which is valid in some limited domain, and one wants to consider possible generalizations giving a new invariance or symmetry algebra, which yields in some parametric limit the original invariance. Two prominent examples come readily to mind: passing from the nonrelativistic invariance algebra (Galilei) to the special relativistic algebra (Poincaré); passing from the special relativistic algebra to a possible general relativistic algebra, e.g., the de Sitter algebra. Indeed the latter is exactly how deformations first appeared in the physics literature[2]. There have been, however, other applications, for example, it has been used to obtain relativistic position operators[3] and has provided a means for building non-compact generators[4] to obtain spectrum generating algebras.

On the other hand in the mathematics literature, deformations were first discussed in the context of cohomology groups on Lie algebras[5]. The hope here being that one could possibly classify all nonequivalent deformations, by placing them in a one-to-one correspondence with the members of certain cohomology groups. Some results have been obtained[6,7], however, it turns out that in general the above correspondence is not valid[6,7]. Recently, progress has been made in finding certain first order deformations[8,9,10] of various inhomogeneous Lie algebras. While a complete classification is still lacking, we have a pretty good idea of which representations of which algebras can give rise to first order deformations and which can be obtained through first order deformations. These techniques have various applications to the representation theory of Lie algebras and the corresponding groups[11]. In essence it allows one to describe the properties of the deformed algebra or group in terms of the inhomogeneous ones. This fact is particularly important where multiplicity problems arise since it is easier to handle them in the case of the latter.

1. DEFORMATIONS OF REPRESENTATIONS

We are interested in the general question: When can a given representation of a Lie algebra be deformed into another representation of a (in general) different Lie algebra? Since we are interested in representations, we want our definition of deformation to be representation dependent.

Definitions: An ***expansion*** of a representation ϕ of a Lie algebra $\mathcal{G}$ is a mapping $\phi \to \phi_\lambda$, such that the ϕ_λ form a representation of another Lie algebra $\mathcal{G}'$. If the conditions

$$\phi_\lambda \xrightarrow{\lambda \to 0} \phi \qquad\qquad \phi_\lambda = \phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \dots$$

are satisfied the expansion is called a ***deformation***. If for some subalgebra $\mathcal{K} \subset \mathcal{G}$, the deformation is trivial, i.e. $\phi_\lambda(\mathcal{K}) = \phi(\mathcal{K})$, the deformation is called *relative to* $\mathcal{K}$ and the subalgebra $\mathcal{K}$ is said to be ***stable***. It is emphasized that although ϕ_λ is a representation of $\mathcal{G}'$, ϕ_0 is the original representation of $\mathcal{G}$, since $\phi_\lambda \xrightarrow{\lambda \to 0} \phi_0$, i.e. $\phi_0 = \phi$. A deformation with $\phi_2 = \phi_3 = \dots = 0$ is called a *first-order* deformation.

Before proceeding to some physical examples we give a brief result from cohomology theory[6,7]. Rigidity Theorem:

If $\mathcal{G}$ is semisimple and ϕ acts in a finite dimensional vector space, all deformations of $\mathcal{G}$ are trivial.

This follows essentially from Whitehead's Lemma[12] and agrees with the intuitive picture of deformation as inverses of contractions. The problem is that this result is ***not true*** when the representation is infinite dimensional and an example will be the first order deformation we discuss next.

2. PHYSICAL EXAMPLES

In this section we discuss the two examples of deformations mentioned in the introduction, namely Poincaré $\Longrightarrow$ de Sitter and Galilei $\Longrightarrow$ Poincaré.

Poincaré $\Longrightarrow$ de Sitter

This represents the prototype of the first order deformations, which will be considered more thoroughly in section 3. Physically, of course, it represents the deformation of a flat space-time manifold to one with a constant but nonzero curvature. Consider then the Poincaré Lie algebra spanned by the generators of the homogeneous Lorentz group $M_{\mu\nu}$ and translations P_μ

$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho} + g_{\mu\sigma}M_{\nu\rho}$$

$$[M_{\mu\nu}, P_\rho] = g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu \tag{2.1}$$

$$[P_\mu, P_\nu] = 0 .$$

With $\mu, \nu, \sigma, \rho = 0, 1, 2, 3$ and the metric $g_{0i} = 0$, $g_{00} = 1 = -g_{11} = -g_{22} = -g_{33}$. Now it turns out that one can perform the desired nontrivial deformation for all representations[13] such that $P^2 = P_\mu P^\mu \neq 0$. If we now construct

$$N_\mu = \tfrac{1}{2}[\Phi, P_\mu] + \tau P_\mu \tag{2.2}$$

where $\Phi = M_{\mu\nu}M^{\mu\nu}$, the second order Casimir operator of the homogeneous Lorentz algebra $o(3, 1)$, we find that the N's and M's form a representation of one of the de Sitter algebras, viz.

$$[N_\mu, N_\nu] = P^2 M_{\mu\nu} \tag{2.3}$$

One notices from (2.2) that all the information of the de Sitter algebras is contained in the inhomogeneous Poincaré algebra.

Before discussing further this construction a brief historical digression is made about Eq. (2.2). It seems that this equation has been propagated in the literature as "the Gell-Mann formula"; however, it first appeared in articles by Melvin[2] and was later discovered independently by Sankaranarayanan[2] and Dothan, Gell-Mann, and Ne'emann[4]. I think it therefore, should be called the Melvin, Sankaranarayanan, Dothan, Gell-Mann, Ne'emann (MSDGN) relation.

Now it can be seen from (2.3) that there are three distinct cases:

i) $P^2 > 0$ $\qquad iso(3,1) \Longrightarrow so(4,1)$

ii) $P^2 < 0$ $\qquad iso(3,1) \Longrightarrow so(3,2)$

iii) $P^2 = 0$ $\qquad$ deformation is trivial

To make the deformation and contraction procedures manifest, we first construct the properly normalized generators[14] $(P^2)^{-\frac{1}{2}} N_\mu$ and then multiply by a parameter ϵ, i.e. consider

$$N_\mu^\epsilon = \epsilon(P^2)^{-\frac{1}{2}} N_\mu = \tfrac{1}{2}\epsilon(P^2)^{-\frac{1}{2}}[\Phi, P_\mu] + u(P^2)^{-\frac{1}{2}} P_\mu \qquad (2.4)$$

where $u = \epsilon\tau$. Then the Inönü-Wigner contraction[1] limit is given by $\epsilon \to 0$, $\tau \to \infty$ such that $\epsilon\tau = u$ a constant. We see that we can deform a representation $(u, s)(P^2 = u^2 > 0)$ of the Poincaré algebra into a representation (τ, s) of the de Sitter algebra $so(4,1)$. We notice a few things about this deformation: 1) it is relative to the homogeneous Lorentz algebra $o(3,1)$, i.e. $o(3,1)$ remains stable; 2) it is a first order deformation; 3) the spin label s remains the same for both $iso(3,1)$ and $so(4,1)$. This last point reflects the fact that on the group level for both $ISO_0(3,1)$ and $SO_0(4,1)$, the spin label s is induced by the "little" group $SO(3)$, and this algebra remains stable through the deformation. Indeed, it will be seen later (heuristically at least) that there is a close connection between these first order deformations of the algebras and Mackey's theory of induced representations for the corresponding groups.

In the preceeding discussion we have "swept under the rug" a very important point, namely that a Lie algebra representation is more than just a set of formal operators, it is an algebra of operators together with the vector space upon which these operators act. This is all the more important since not only is $iso(3, 1)$ noncompact but also the stable subalgebra $so(3, 1)$ is noncompact. In fact in order to obtain an integrable Lie algebra[15] for $so(4, 1)$, we must start from the reducible representation $(u, s, P_0 > 0) \oplus (u, s, P_0 < 0)$, $u > 0$ of the Poincaré algebra $iso(3, 1)$, i.e. we consider both particles and anti-particles together. We will obtain in de Sitter space an energy operator which is *not* positive definite[16].

Before proceeding to the next example we mention that by adding to Eq. (2.2) a term of the form ρP_μ we can pass from a representation (τ, s) to an inequivalent representation $(\tau + \rho, s)$ thus providing a counterexample to the rigidity theorem for semisimple Lie algebras mentioned in the previous section.

Galilei $\Longrightarrow$ *Poincaré*[17]

Owing to the somewhat greater complexity of the Galilei algebra[18] this deformation is more complicated than the previous case. Physically the deformation corresponds to passing from a nonrelativistic domain to a relativistic domain, i.e. from inertial systems with an absolute time, to inertial systems with relative time. The composition of this deformation with the previous one then allows us to pass from a flat space, absolute time manifold to a curved space-time manifold. There are, of course, other possible kinematics as considered in ref. (19), e.g. one could start with the usual nonrelativist Galilei algebra and deform it to one of the Newton[19] algebras representing the automorphisms of a curved space with an absolute time. We will not consider such cases here.

The algebra $\mathcal{G}$ of the Galilei group is spanned by the generators of rotation M_i, space translation P_i, time translation H, and inertial transformations K_i,

$$[M_i, M_j] = \epsilon_{ijk} M_k \qquad [M_i, P_j] = \epsilon_{ijk} P_k$$

$$[M_i, K_j] = \epsilon_{ijk} K_k \qquad [P_i, P_j] = [K_i, K_j] = 0$$

$$[K_i, H] = P_i \qquad [P_i, H] = [M_i, H] = [K_i, P_j] = 0 \tag{2.5}$$

It is known[20], that none of the true representations of the Galilei group admit a reasonable physical interpretation; they are not localizable[21]. In order to obtain localizable representations one must deal with projective representations, or what amounts to the same thing true representations of a nontrivial central extension[20]. We can make such an extension of $\mathcal{G}$ by the replacement

$$[K_i , P_j] = 0 \longrightarrow [K_i , P_j] = m\,\delta_{ij} \tag{2.6}$$

Indeed the connection between this extension and quantum mechanics is apparent. Eq. (2.5) with this replacement yields the extended Galilei algebra $\tilde{\mathcal{G}}$. We now consider the representation[18] of a free nonrelativistic spinning particle labeled by $(m, U = 0\,,\, s)$ with nonrelativistic mass m, internal energy $U = H - P^2/2m$, and spin s. Without loss of generality we can decompose the angular momentum into orbital and spin parts,

$$M_{ij} = \epsilon_{ijk}\, M_k = m^{-1}\,(K_i P_j - K_j P_i) + \epsilon_{ijk}\, S_k \tag{2.7}$$

where the S_k's satisfy and $so\,(3)$ algebra and commute with everything else. If we construct, following Inönü and Wigner[1],

$$P_0 = [\,u^2 + (P^2/\epsilon^2)]^{\frac{1}{2}}\ , \qquad P_0' = P_0 - u \tag{2.8a}$$

by making a scale transformation $P_i \to \epsilon^{-1} P_i \equiv P_i'$ and $K_i \to \epsilon K_i$ which has no effect on $\tilde{\mathcal{G}}$ and build[17]

$$N_i = m^{-1} P_0\, \epsilon K_i - \epsilon_{ijk}\, S_j P_k\, \epsilon^{-1}/(P_0 + u) \tag{2.8b}$$

we find that P_0, P_i', $N_i \equiv M_{i0}$, M_{ij} close on the algebra of the Poincaré group, (2.1).

If we now consider the contraction limit, we will find that it is P_0' not P_0 which has a finite contraction limit. Thus the deformation is a deformation "up to a factor" in analogy with the Inönü-Wigner contraction[1] up to a factor. In other words we replace P_0 with P_0' obtaining a trivial extension of the Poincaré algebra $\mathcal{P}$ which upon contraction yields a nontrivial extension

$\tilde{\mathcal{G}}$ of the Galilei algebra. The contraction is detailed as follows: take the limit $\epsilon \to 0$, $u \to \infty$ such that $u\epsilon^2 = m$ of the operators

$$\epsilon P_i' \longrightarrow P_i$$
$$\epsilon N_i \longrightarrow K_i \tag{2.9}$$

which imposes $P_0' \longrightarrow P^2/2m = H$ performed explicitly by considering the power series expansion of P_0',

$$P_0' = (P^2/2m) - u^{-1}(P^4/8m^2) + \mathcal{O}(1/u^2) \ . \tag{2.10}$$

Hence, as $u \to \infty$, we obtain the energy of a free nonrelativistic spinning particle. In the appropriate units $u \to \infty$ implies the speed of light $c \to \infty$ which of course corresponds to the physical picture. It is also seen readily from (2.10) that the deformation is of infinite order. Comparing this infinite order deformation with the first order deformation (2.4), the virtue of the latter can be seen. One can not simply truncate the series (2.10) and retain the desired invariance properties. It is also seen that the Euclidian subalgebra M_i, P_i' remains stable under the deformation (2.8); however, given the algebra P_μ', N_i, M_i the Inönü–Wigner contraction (2.8) is performed relative to the generators M_i, P_0'. Apparently this deformation and the Inönü–Wigner contraction are not strictly inverse operations. The difference appears to be the rescaling of P_i. One starts with the extended Galilei algebra $\tilde{\mathcal{G}}$ rescales the momentum P_i and inertial transformations K_i and then deforms $\tilde{\mathcal{G}}$ to $\mathcal{P}$ which upon contraction yields the original unrescaled algebra.

The question of whether such deformations are unique has not yet been answered; however, one can find inequivalent expansions of $\tilde{\mathcal{G}}$ to $\mathcal{P}$. These expansions look very similar to the above deformation and not at all like the first order deformation of the previous sections. Indeed it seems improbable that one could use an algorithm of the type (2.2) to go from the Galilei algebra to the Poincaré algebra. Although these infinite order deformations are somewhat complicated, it is certainly interesting that one can start with quite general *physical* representations of the Galilei algebra and obtain upon deformation *physical* representations of the Poincaré algebra.

3. FIRST ORDER DEFORMATIONS

In this section we want to focus our attention on first order deformations of the type given by the MSDGN relation (2.2), and try to provide an answer as to which algebras it can be applied to yield a deformation. We will not attack here the more general problem of the classification of all deformations of a given Lie algebra or even the classification of all first order deformations. We formulate the problem as follows: let $\mathcal{K}$ be a semisimple Lie algebra, $\mathcal{A}$ an abelian Lie algebra and consider the semidirect sum $\mathcal{K} \oplus\!\!\!\!\!\supset \mathcal{A}$. We construct the following set of operators from the universal enveloping algebra of $\mathcal{K} \oplus\!\!\!\!\!\supset \mathcal{A}$,

$$\mathcal{N} = \tfrac{1}{2}\,[\Phi, \mathcal{A}] + \tau\mathcal{A} \tag{3.1}$$

where Φ is the second order Casimir operator of $\mathcal{K}$. First notice that

$$[\mathcal{K}, \mathcal{N}] \subset \mathcal{N} \tag{3.2}$$

which follows immediately from the semidirect sum structure of the original algebra. The question is under what conditions do the $\mathcal{N}$'s close to form some Lie algebra $\mathcal{G}$? A partial answer to this question has been given by many[2,8] and summarized by Gilmore[9]. Gilmore showed that if one starts with a semisimple algebra $\mathcal{G}$ and one makes a Cartan decomposition i.e. $\mathcal{G} = \mathcal{K} + \mathcal{N}$ where $\mathcal{K}$ is compact, Eq. (3.2) is satisfied and

$$[\mathcal{N}, \mathcal{N}] \subset \mathcal{K} \tag{3.3}$$

then $\mathcal{N}$ can be written in the form (3.1) if the Riemannian symmetric space[22] $\exp(\mathcal{N})$ is of rank[23] 1. The key to the proof is that these symmetric spaces are spaces of constant sectional curvature[22]. Actually the fact that $\mathcal{K}$ is compact is not really essential, the procedure can be carried through for all symmetric decompositions (i.e. $\mathcal{K}$ is a maximal subalgebra of $\mathcal{G} = \mathcal{K} + \mathcal{N}$ with Eqs. (3.2) and (3.3) satisfied) such that $\exp(\mathcal{N})$ is of rank 1. Now in general it turns out that (3.1) is not quite a deformation; however, by taking $\mathcal{K}$ and hence $\mathcal{G}$ slightly larger so that $[\mathcal{N}, \mathcal{N}]$ spans $\mathcal{K}$ one does get a defomation. This change allows us to separate expansions of the type (3.1) from much more radical expansions which have very little to do with the process of contraction. One additional point here is that all realizations of Eq. (3.1) yield hermitian representations of ***noncompact*** algebras. If one considers, however, $\mathfrak{G}^* = \mathfrak{K} + \mathfrak{N}^*$, where $\mathfrak{N}^* = i\mathfrak{N}$, then $\mathfrak{G}^*$ is compact and $\exp(\mathfrak{N}^*)$ is a compact symmetric space, yet no explicit realizations of this type have been given despite the fact that the contraction procedure is very similar to the noncompact case. Examples of rank 1 deformations are given in table 1.

Thus Gilmore's result provides a rather large class of algebras for which the first order deformation procedure given by (3.1) holds. This class, however, is not a necessary condition. One can find deformations given essentially by (3.1) such that the rank of $\exp(\mathfrak{N})$ is greater than one. Such defomations, however, may only be valid for certain representations of $\mathfrak{K} \supset\!\!\!\!+ \mathfrak{A}$. There are two new distinct classes of first order defomations of this type.

For example, let $\mathfrak{A}$ be the Lie algebra of abelian second rank mixed symmetric tensors with respect to the compact[24] algebra $\mathfrak{K}$. Then the deformation[4, 10, 25] given by Eq. (3.1) can be carried through when one represents $\mathfrak{A}$ as a product of commuting vectors $x_i x_j$, and $\mathfrak{K}$ only operates on this vector space, i.e. one cannot build an additional vector space over this space. More explicitly, let $\mathfrak{K} = so(n)$ and $\mathfrak{A}$ be a commuting set of second rank symmetric tensors with respect to $so(n)$, then one represents $so(n)$, by

$$M_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \tag{3.4}$$

and $\mathfrak{A}$ by $x_i x_j / x^2$ and finds that

$$N_{ij} = \tfrac{1}{2}[\Phi, x_i x_j / x^2] + \tau x_i x_j / x^2 \tag{3.5}$$

satisfies

$$[N_{ij}, N_{kl}] = \delta_{jk} M_{il} + \delta_{ik} M_{jk} + \delta_{jk} M_{ik} + \delta_{il} M_{jk} \;. \tag{3.6}$$

Thus for such representations we have the deformation $i_2 so(n) \Longrightarrow sl(n, R)$. If we multiply N_{ij} by ϵ and take the limit $\epsilon \to 0$, $\tau \to \infty$, we obtain the original representation of $i_2 so(n)$. We notice here the absence of the "spin" contribution to M_{ij}. This is not at all surprising since the principal series of $sl(n, R)$ has no spin labels induced by a compact subgroup. This will become clearer in the group theoretic context in the next section. But we can see emerging what appears to be a general maxim for our first order deformations

TABLE 1: TYPE 1 DEFORMATIONS – RANK 1: COMPACT FORMS (i.e. K compact)

x_i, z_i, and q_i denote real, complex, and quaternionic variables respectively.

Inhomogeneous Group $K \rtimes A_n$	Stability Subgroup $K_0 \rtimes A_n$	Coset Space $K \rtimes A_n / K_0 \rtimes A_n \approx G/H$	Deformed Group $G \approx K A N$	Stability Subgroup $H \approx K_0 \rtimes AN$	Representations of G remarks
$SO(n) \rtimes A_n$	$SO(n-1) \rtimes A_n$	$x_1^2 + \ldots + x_n^2 = 1$	$SO_0(n,1)$	$SO(n-1) \rtimes AN$	General; A - 1 dim $SO(n-1)$ is the Centralizer of A
$U(1) \otimes [U(n) \rtimes A_n]$	$U(1) \otimes [U(n-1) \rtimes A_n]$	$\lvert z_1 \rvert^2 + \ldots + \lvert z_n \rvert^2 = 1$	$U(n,1)$	$U(1) \otimes [U(n-1) \rtimes AN]$	General; A - 1 dim $U(1) \otimes U(n-1)$ centralizer of A
$Sp(1) \otimes [Sp(n) \rtimes A_n]$	$Sp(1) \otimes [Sp(n-1) \rtimes A_n]$	$\lvert q_1 \rvert^2 + \ldots + \lvert q_n \rvert^2 = 1$	$Sp(n,1)$	$Sp(1) \otimes [Sp(n-1) \rtimes AN]$	A - 1 dim $Sp(1) \otimes Sp(n-1)$ centralizer of A

TABLE 1: (Continued) NONCOMPACT FORMS (i.e. K noncompact)

Inhomogeneous Group $K \mathbin{\circledS} A$	Stability Subgroup $K_0 \mathbin{\circledS} A$	Coset Space $K \mathbin{\circledS} A / K_0 \mathbin{\circledS} A \approx X$	Deformed Group G	Representations of G
$SO(p,q) \mathbin{\circledS} A_{p+q}$	$SO(p,q-1) \supset A_{p+q}$	$x_1^2 + \ldots + x_p^2 - \ldots - x_{p+q}^2 = \pm 1$	$SO(p+1,q)$	general
	$SO(p-1,q) \supset A_{p+q}$		$SO(p,q+1)$	
$U(1) \otimes [U(p,q) \mathbin{\circledS} A_{p+q}]$	$U(1) \otimes [U(p,q-1) \supset A_{p+q}]$	$\lvert z_1\rvert^2 + \ldots + \lvert z_p\rvert^2 - \ldots - \lvert z_{p+q}\rvert^2 = \pm 1$	$U(p+1,q)$	general
	$U(1) \otimes [U(p-1,q) \supset A_{p+q}]$		$U(p,q+1)$	
$Sp(1) \otimes [Sp(p,q) \mathbin{\circledS} A_{p+q}]$	$Sp(1) \otimes [Sp(p,q-1) \supset A_{p+q}]$	$\lvert q_1\rvert^2 + \ldots + \lvert q_p\rvert^2 - \ldots - \lvert q_{p+q}\rvert^2 = \pm 1$	$Sp(p+1,q)$	general
	$Sp(1) \otimes [Sp(p-1,q) \supset A_{p+q}]$		$Sp(p,q+1)$	
$F = R$		$x_1' x_{n+1} + \ldots + x_n' \wedge x_{2n} = \text{Const.}$	$Sp(n+1,R)$	
$Sp(1) \otimes [Sp(n,F) \mathbin{\circledS} A_n]$	$Sp(1) \otimes [Sp(n-1,F) \supset A_n]$			general
$F = C$		$z_1' \wedge z_{n+1} + \ldots + z_n' \wedge z_{2n} = \text{Const.}$	$Sp(n+1,C)$	
$SO(n,C) \mathbin{\circledS} A_n$	$SO(n-1,C) \mathbin{\circledS} A_n$	$z_1^2 + \ldots + z_n^2 = 1$	$SO(n+1,C)$	general
$SO(n,Q) \mathbin{\circledS} A_n$	$SO(n-1,Q) \mathbin{\circledS} A_n$	$q_1^2 + \ldots + q_n^2 = 1$	$SO(n+1,Q)$	general

and that is that our deformations can contain only *one* continuous label for the representations, that is we can deform only along one continuous path at a time. That the spin labels play an innocuous role will become more apparent in the group context. In table 2 we have compiled a list of the first order deformations of this type along with some properties concerning homogeneous spaces and the allowed representations.

In the last type of deformation[26] to be discussed, the above discussion becomes even more apparent. As an example of this type of deformation we take $\mathcal{K}$ to be $so(n) \oplus so(k)$, $1 \leqslant k \leqslant n$ and $\mathcal{A}$ to be k direct sums of abelian n-vector operators, i. e. $\mathcal{A} = \mathcal{A}_n^{(1)} \oplus \ldots \oplus \mathcal{A}_n^{(k)}$ where $\mathcal{A}_n^{(\,)}$ transform as n-vectors. There are two ways to construct deformations both of which are representation-dependent. We choose the identity representation of $so(k)$, an arbitrary representation of $so(n)$ and take $\mathcal{A}_n^{(\,)}$ to be a set of k orthonormal spheres (one has n-orthonormal spheres available from the group manifold $SO(n)$). It is then found that the operators

$$N_i^{(\alpha)} = \tfrac{1}{2}\,[\Phi\,, x_i^{(\alpha)}] + \tau x_i^{(\alpha)} \tag{3.7}$$

$$M_{\alpha\beta} = \tfrac{1}{2}\,\{\,[\Phi,\, x_i^{(\alpha)}]\,,\, x^{(\beta)}\}\ ;\ x_i^{(\alpha)} x_i^{(\beta)} = \delta_{\alpha\beta} \tag{3.8}$$

with $i = 1, \ldots, n$; $\alpha, \beta = 1, \ldots, k$ close along with M_{ij} to form a representation of the Lie algebra of $so(n, k)$; hence we have the deformation $\overset{k}{i}\ so(n) \oplus so(k) \Longrightarrow so(n, k)$. By multiplying $N_i^{(\alpha)}$ and $M_{\alpha\beta}$ by ϵ and taking the limit $\epsilon \to 0$, $\tau \to \infty$, we arrive back at the original representation of the original algebra. Now if we had started with M_{ij} as a representation of $so(n)$ obtained as the infinitesimal generators of the group $SO(n)$ acting on itself say from the right, then the generators (3.8) would be nothing more than the generators of the $SO(k)$ subgroup of $SO(n)$ acting from the left. Starting from this prescription one can see that the k n-vectors of $\mathcal{A}$ also transform under $so(k)$ as n k-vectors (not all independent of course). In this case[27] we obtain via (3.7) the same representations of $so(n, k)$. The contraction is performed by only multiplying $N_i^{(\alpha)}$ by ϵ. Again the general maxim is that we can deform only along *one* continuous direction, i. e. τ must be a constant independent of i and α in order that the algebra close. Thus we find in general only certain degenerate representations. For example $so(n, k)$ has rank[28] $[(n+k)/2]$ of which k parameters are continuous for the principal series; hence for $so(n, 1)$ we get all the representations of the principal series as before, however, for

TABLE 2: TYPE 2 DEFORMATIONS: COMPACT FORMS

x_i, z_i, and q_i denote real, complex, and quatemionic variables respectively.

Inhomogeneous Group $K \rtimes A_{2,n}$	Stability Subgroup $K_0 \rtimes A_{2,n}$	Coset Space $K \rtimes A_{2,n} / K_0 \rtimes A_{2,n} \approx G/H$	Deformed Group $G \approx K\ AN$	Stability Subgroup $H \approx K_0 \rtimes AN$	Representations of G remarks
$SO(n) \rtimes A_{2,n}$	$SO(n-1) \rtimes A_{2,n}$	$x_1^2 + \ldots + x_n^2 = 1$	$SL(n, R)$	$SO(n-1) \rtimes AN$	Most degenerate 1 continuous label A - $(n-1)$ dim The centralizer of A is a discrete subgroup of K
$U(n) \rtimes A_{2,n}$	$U(n-1) \rtimes A_{2,n}$	$\lvert z_1 \rvert^2 + \ldots + \lvert z_n \rvert^2 = 1$	$SL(n, C) \otimes U(1)$	$U(n-1) \rtimes AN$	degenerate 1 continuous label A - $(n-1)$ dim (n times) $U(1) \otimes \ldots U(1)$ centralizer of A
$Sp(1) \otimes [Sp(n) \rtimes A_{2,n}]$	$Sp(1) \otimes [Sp(n-1) \rtimes A_{2,n}]$	$\lvert q_1 \rvert^2 + \ldots + \lvert q_n \rvert^2 = 1$	$SL(n, Q) \otimes Sp(1)$	$Sp(1) \otimes Sp(n-1) \rtimes AN$	degenerate 1 continuous label A - $(n-1)$ dim ($n-1$ times) $Sp(1) \otimes U(1) \otimes \ldots U(1)$ centralizer of A

TABLE 2: (continued) NONCOMPACT FORMS (i.e. K noncompact)

Inhomogeneous Group $K \rtimes A_2$	Stability Subgroup $K_0 \rtimes A_2$	Coset Space $K \rtimes A_2 / K_0 \rtimes A_2 \simeq X$	Deformed Group G	Representations of G
$SO(p,q) \rtimes A_{2,p+q}$	$SO(p,q-1) \supset A_{2,p+q}$ $SO(p-1,q) \rtimes A_{2,p+q}$	$x_1^2 + \ldots + x_p^2 - x_{p-1}^2 - \ldots - x_{p+q}^2 = \pm 1$	$SL(p+q,R)$	Most Degenerate
$U(p,q) \rtimes A_{2,p+q}$	$U(p,q-1) \rtimes A_{2,p+q}$ $U(p-1,q) \rtimes A_{2,p+q}$	$\|z_1\|^2 + \ldots + \|z_p\|^2 - \ldots - \|z_{p+q}\|^2 = \pm 1$	$SL(p+q,C) \otimes U(1)$	Degenerate
$Sp(1) \otimes [Sp(p,q) \rtimes A_{2,p+q}]$	$Sp(1) \otimes [Sp(p,q-1) \rtimes A_{2,p+q}]$ $Sp(1) \otimes [Sp(p-1,q) \rtimes A_{2,p+q}]$	$\|q_1\|^2 + \ldots + \|q_p\|^2 - \ldots - \|q_{p+q}\|^2 = \pm 1$	$SL(p+q,Q) \otimes Sp(1)$	Degenerate
$SO(n,C) \rtimes A_{2,n}$	$SO(n-1,C) \rtimes A_{2,n}$	$z_1^2 + \ldots + z_n^2 = 1$	$SL(n,C)$	Most Degenerate
$SO(n,Q) \rtimes A_{2,n}$	$SO(n-1,Q) \rtimes A_{2,n}$	$q_1^2 + \ldots + q_n^2 = 1$	$SL(n,Q)$	Most Degenerate

$so(n,k)$ $1 < k < n$ we get those labeled by $[(n-k)/2]$ discrete parameters and one k-fold degenerate continuous label. Thus for $k = n-1$ or n, we get a "most degenerate" representation labeled by the continuous parameter τ. Notice the number of discrete labels is just the rank of the algebra of the little group $SO(n-k)$. In table 3, we give the first order deformations of this type with compact stable subalgebras along with some properties of the homogeneous spaces and representations.

4. GROUP DEFORMATIONS

Deformations of representations of Lie groups have been discussed by Hermann [17], by considering cohomology theory over groups and establishing a relation with multiplier representations, although he has not given explicit forms or discussed the connection with the Iwasawa decomposition. Indeed, it is an interesting and important property of the previous first order deformations, that these generators are exactly those obtained from the group by considering certain multiplier representations which are closely related to the Iwasawa decomposition. In this section we want to explore this connection.

The deformation process of algebras has a natural generalization to the deformation of the group. However, explicit but general realizations analogous to Eq. (3.1) on the group level are hard to obtain by the straightforward integration of (3.1). One obstacle is that expressions which appear very similar on the Lie algebra level, like (3.1) for $so(n,1)$ and $sl(n,R)$ have very different actions on the group level. Even so, all of these actions exhibit the common feature of having a strong connection with the Iwasawa decomposition. The deformation process on the group can be roughly stated as follows: consider a representation of the inhomogeneous group $K \circledS A$, K compact [24], A abelian where K acts as a transformation group over a homogeneous space X which is closely related to the space $\phi(\mathfrak{a})$ of hermitian representations [29] of the algebra $\mathfrak{a}$, and the action of A is not effective

$$T(A)\, f(X) = \exp\left[i\phi(\mathfrak{a})\right] f(X) \tag{4.1}$$

A deformation of this representation of the group $K \circledS A$ corresponding to the deformation of the algebra is a map

$$T(A) \longrightarrow T^{\tau}(g) \tag{4.2}$$

TABLE 3: TYPE 3 DEFORMATIONS: COMPACT FORMS

x_i, z_i, and q_i denote real, complex, and quaternionic variables respectively

Inhomogeneous Group $K \circledS A_n$	Stability Subgroup $K_0 \circledS A_n$	Coset Space $K \circledS A_n / K_0 \circledS A_n \approx G/H$	Deformed Group $G \approx K\ AN$	Stability Subgroup $H \approx K_0 \circledS AN$	Representations of G remarks
$[SO(n) \otimes SO(k)]$ $\circledS [A_n^{(1)} \otimes \ldots A_n^{(k)}]$	$[SO(n-k) \otimes \widehat{SO}(k)]$ $\supset [A_n^{(1)} \otimes \ldots A_n^{(k)}]$	$x_i^{(\alpha)} x_i^{(\beta)} = \delta_{\alpha\beta}$	$SO_0(n,k)$	$[SO(n-k) \otimes \widehat{SO}(k)] \circledS AN$	degenerate 1 continuous label $A-k$ dim $SO(n-k)$ is Centralizer of A.
$[U(n) \otimes U(k)]$ $\circledS [A_n^{(1)} \ldots A_n^{(k)}]$	$[U(n-k) \otimes \widehat{U(k)}]$ $\supset [A_n^{(1)} \otimes \ldots A_n^{(k)}]$	$z_i^{(\alpha)} \overline{z}_i^{(\beta)} = \delta_{\alpha\beta}$	$U(n,k)$	$[U(n-k) \otimes \widehat{U(k)}] \circledS AN$	degenerate 1 continuous label $A-k$ dim (k times) $U(n-k) \otimes U(1) \otimes \ldots U(1)$ Centralizer of A.
$[Sp(n) \otimes Sp(k)]$ $\circledS [A_n^{(1)} \ldots A_n^{(k)}]$	$[Sp(n-k) \otimes \widehat{Sp(k)}]$ $\supset [A_n^{(1)} \otimes \ldots A_n^{(k)}]$	$q_i^{(\alpha)} \overline{q}_i^{(\beta)} = \delta_{\alpha\beta}$	$Sp(n,k)$	$[Sp(n-k) \otimes \widehat{Sp}(k)] \supset AN$	degenerate 1 continuous label $A-k$ dim (k-1) times $Sp(n-k) \otimes Sp(1) \otimes U(1) \otimes \ldots U(1)$ Centralizer of A.

where $g = \exp(i\mathfrak{N})$ such that

$$T^{\tau}(g)\, f(X) = \mu^{\tau}(g, X')\, f(X') \tag{4.3}$$

where the group action $X \xrightarrow{g^{-1}} X'$ and the multiplier $\mu^{\tau}(g, X)$ must satisfy certain limit conditions, that is, they must give back the original representation of the original group. This contraction limit (Lim) is given by the Inönü-Wigner[1] procedure of taking a sequence of representations labeled by an index τ (in our case continuous) and letting $\tau \to \infty$ while at the same time taking a sequence of neighborhoods of the identity of the group $\{g_t\}$ and letting $t \to 0$ in such a way that $t\tau \to \xi$ a constant. Thus we demand

$$\mathrm{Lim}\; T^{\tau}(g_t)\, f(X) = \mathrm{Lim}\; \mu^{\tau}(g, X)\, f(X_t') = \exp\,[i\phi(\mathfrak{A})\,\xi]\, f(X) \tag{4.4}$$

Accordingly it follows that

$$\mathrm{Lim}\; \mu^{\tau}(g_t, X') = \exp\,[i\phi(\mathfrak{A})\,\xi] \tag{4.5a}$$

$$\mathrm{Lim}\; X_t' = X \tag{4.5b}$$

It is understood here that the subgroup K remains stable under the deformation. We take for the functions f infinitely differentiable functions over X, where in general we allow f to be vector-valued and we have suppressed transformations in this vector space in the foregoing discussion. In the three first order deformations discussed, the space X turned out to be a sphere or a product of spheres. Had the stable subgroup been non-compact, as in the case of the deformation of the Poincaré group and algebra, the homogeneous space X would have been hyperboloids and we would take infinitely differentiable functions. of compact support.

The outstanding question now is how does one ascertain the action of the group over X and the multiplier $\mu^{\tau}(g, X)$? The problem is that as of yet no nice prescription analogous to Eq. (3.1) has been found. Although one can express the infinitesimal generators (3.1) in terms of the derivatives of the action and the multiplier, the inverse procedure of integration has not been found in general. We can exhibit some properties possessed by the deformations of the group (4.2). In the deformations of the algebra the space X

was compact and homogeneous; thus there exists a subgroup K_0 of K such that

$$K \otimes A/K_0 \otimes A \approx K/K_0 \approx X \ .$$

Furthermore, this space does not change under the deformation although the measure on X is "deformed" (i.e. is not G-invariant), so one has a deformation of the $K \otimes A \Longrightarrow G$ and of its stability subgroup $K_0 \otimes A \Longrightarrow H$ such that $G/H \approx K/K_0 \approx X$. This homogeneous space can be illustrated via the Iwasawa decomposition[30] $G \approx KAN$, $H \approx K_0 AN$. We can check that the subgroup H does indeed exist. It is, however, not unimodular and thus the measure over X is not G-invariant; hence, one needs multipliers to get unitary representations of G over X.

Finally, we give explicit forms for the three types of first order deformations presented in the last section. Again for simplicity we give details only for the case of real groups. The complex and quaternionic cases are similar and some of their properties appear in the tables. In each case it can be checked that the infinitesimal generators of the corresponding representation are exactly those obtained through the algebra deformation procedure in the last section.

1) $G = SO_0(n,1) \qquad X = \{x_i : x_i\, x_i = 1 \ \ i = 1, \ldots, n\}$

$$x_i' = (g_{ij}^{-1} x_j + g_{i0}^{-1})/(g_{0j}^{-1} x_j + g_{00}^{-1}) \tag{4.6}$$

The action is just the projective transformation of the sphere X onto itself. The stability subgroup of the point $x_n = 1$, $x_1 = \ldots = x_{n-1} = 0$ is the subgroup composed of those transformations which satisfy

$$g_{0n}^{-1} + g_{00}^{-1} = g_{nn}^{-1} + g_{n0}^{-1} \qquad\qquad g_{in}^{-1} + g_{i0}^{-1} = 0 \quad i \neq n \tag{4.7}$$

It can be shown that such transformations are just those of the subgroup $SO(n-1)\,AN$ of $SO_0(n,1)$. The multiplier is given by

$$\mu^{\tau}(g, x') = [g_{0j}^{-1} x_j + g_{00}^{-1}]^{\sigma} \ , \tag{4.8}$$

where $\sigma = -(n-1)/2 + \tau$, which will give rise to unitary representations when τ is pure imaginary. Now the subgroup $SO(n-1)$ is the centralizer of A (one-parameter subgroup generated by one of the boosts) in $SO(n)$, thus the representations of H are direct products of the representations of $SO(n-1)$ and the characters of A. The representation of $SO_0(n,1)$ described by (4.8) yields the principal nondegenerate series induced by the representations of H. The multiplier (4.8) is just enough to offset the transformation of the measure under G, so that for τ pure imaginary the representation is unitary.

2) $G = SL(n,R)$

X is the same as in 1). The action of G on X is

$$x_i' = \frac{r}{r'} g_{ij}^{-1} x_j$$

$$\frac{r}{r'} = [x_i g_{ji}^{-1} g_{jk}^{-1} x_k]^{\frac{1}{2}} . \tag{4.9}$$

The stability subgroup is again given by those transformations which satisfy

$$g_{nn}^{-1} = (g_{jn}^{-1} g_{jn}^{-1})^{\frac{1}{2}} , \quad g_{in}^{-1} = 0 \quad i \neq n . \tag{4.10}$$

Actually the second of these equations implies the first. Again it can be shown that this subgroup is just $SO(n-1)\,AN$. Of course the subgroups A and N are here quite different than in 1). A is the subgroup of diagonal matrices, whereas N is the subgroup of lower triangular matrices. In this case $SO(n-1)$ is *not* the centralizer of A. In fact the centralizer of A in K is discrete. As a consequence only a most degenerate series of representations of H over X can be induced to representations of $SL(n,R)$. This is just labeled by one of the characters of A. The multiplier in this case is

$$\mu^\tau(g,x) = (r'/r)^\sigma \tag{4.11}$$

where $\sigma = -(n-1)/2 + \tau$, τ pure imaginary for unitary representations. Again the multiplier just cancels the change in the measure under transformations in G.

3) $G = SO_0(n,k)\ \{x_i : x_i^{(\alpha)} x_i^{(\beta)} = \delta_{\alpha\beta}\ i = 1, \ldots, n;\ \alpha, \beta = 1, \ldots, k\}$

The work presented for this case is not yet in completed form[26]. This situation is somewhat more complicated; the orthonormal spheres $x_i^{(\alpha)}$ can be represented as $n \times k$ rectangular matrices and the action is a generalizatioı of the projective transformation (4.6).

$$x_i'^{(\alpha)} = (x_k^{(\alpha)} g_{k,n+\beta} + g_{n+\alpha,n+\beta})^{-1} (x_j^{(\beta)} g_{ji} + g_{n+\beta,i}) \tag{4.12}$$

The $n \times n$ submatrices $SO(n)$ of $SO_0(n,k)$ parameterized by g_{ij} act as rigid transformations from the right whereas the $k \times k$ submatrices $SO(k)$ of $SO_0(n,k)$ are written as $g_{n+\alpha,n+\beta}$ and act as rigid transformations from the left. The stability subgroup of the point $x_i^{(\alpha)} = \delta_{\alpha i}$, $i = 1, \ldots, k$, $x_i = 0$ for $i > k$, is the subgroup of those transformations which satisfy.

$$\delta_{\alpha i} = g^{-1}_{\alpha,n+\beta} g_{\beta i} + g^{-1}_{\alpha,n+\beta} g_{n+\beta,i} + g^{-1}_{n+\alpha,n+\beta} g_{\beta i} + g^{-1}_{n+\alpha,n+\beta} g_{n+\beta,i}$$

It can be shown that this subgroup is just $H = [SO(n-k) \otimes \widehat{SO}(k)]\, A\, N$, where $\widehat{SO}(k)$ indicates that this is the subgroup $SO^r(k) \otimes SO^l(k)$ of matrices of $SO(n) \otimes SO(k)$, r and l designate action from the right and left respectively. The homogeneous space G/H is then isomorphic with the product of spheres $S^{(n-1)} \otimes \ldots \otimes S^{(n-k)}$ which is isomorphic with X. The centralizer of A in $SO(n) \otimes SO(k)$ is $SO(n-k)$ and representations of this subgroup along with a degenerate continuous representation (character) of A can be induced to representations of $SO_0(n,k)$ in correspondence with the discussion given at the end of the last section.

In conclusion again we emphasize the close connection between the deformations of Lie algebras and groups presented here and the theory of multiplier representations developed originally by Bargmann[32], and Gel'fand and colaborators[33]. and elaborated into the theory of induced representations by Mackey[31]. Many of the multiplier representations can be constructed by considering the space of homogeneous functions over a manifold of higher dimension realized by removing the subgroup A or part of it from the stability subgroup H. See, for example, Gel'fand and Graev[33]. It remains to be seen whether all multiplier representations can be obtained from the deformation of appropriate representations of some inhomogeneous group.

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23. The rank of a symmetric space $e^{\mathfrak{n}}$ is equal to the dimension of $\mathcal{H} \cap \mathfrak{n}$ where $\mathcal{H}$ is a Cartan subalgebra of $\mathcal{G}$.
24. Actually this construction goes through when the subalgebra $\mathcal{K}$ is non-compact, but we choose it compact for simplicity as well as facilitating the discussion on the group level with the Iwasawa decomposition.
25. This type of deformation was first considered by Y. Dothan, M. Gell-Mann, and Y. Ne'eman and appears implicitly in their paper in Ref. 4.
26. C. P. Boyer and K. B. Wolf (to appear). This type of deformation was first suggested by Professor Y. Ne'eman.
27. These two deformations are inequivalent since they have different stable subalgebras; however, they yield equivalent representations of $SO_0(n,k)$. In the first, however, the representation of the original algebra is not faithful.
28. Here $[x]$ means the greatest integer less than or equal to x.
29. K acts as a group of automorphisms over the algebra $\mathcal{A}$. For type 1) deformations the space X is just an orbit of K in $\mathcal{A}$. i. e. spheres. For type 2) X is again an orbit of K, but only for certain degenerate representations of $\mathcal{A}$ does one get spheres X. Again for type 3) one has to specify that the subalgebras $\mathcal{A}_n^{(\,)}$ be orthogonal.
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RESUMEN

El concepto de deformaciones de álgebras de Lie ha tenido aplicaciones tanto en física como en la teoría de representaciones. A pesar de esto, se sabe poco sobre la clasificación general de las posibles deformaciones de un álgebra de Lie dada y sus representaciones. Presentamos aquí un estudio general de lo que se sabe, junto con sus aplicaciones a teoría de representaciones y a física. El artículo se divide esencialmente en dos partes. En la primera, discutimos dos prominentes ejemplos físicos de deformaciones, la deformación del álgebra de Galileo al álgebra de Poincaré y la deformación del álgebra de Poincaré al álgebra de de Sitter. En la segunda parte, nos concentramos en lo que se llaman deformaciones de primer orden, aplicando un algoritmo bien conocido a varias inhomogenizaciones de álgebras de Lie semi-simples. Se discute e indica qué representaciones se pueden deformar a qué álgebras. También se discute la deformación correspondiente del grupo y su conexión con representaciones multiplicativas y la descomposición de Iwasawa. Se presentan tablas dando una clasificación de los principales resultados.