DILATATION INVARIANT BILOCAL QUANTUM THEORY FOR MASSIVE PARTICLES

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ABSTRACT: Within the framework of a previously developed algebraic approach to relativistic dilatation physics, bilocal wave equations for arbitrary spin are set up. The principal result of this study is that, despite dilatation invariance, particles can have nonzero restmass. A canonical formalism is developed and various currents are studied.

I. INTRODUCTION

At the Philadelphia conference in 1972, we suggested a new relativistic dynamical group[1] which contains dilatations in a natural manner. The best way to visualize the genesis of our group is to think of hadrons as excitable blobs of matter and to imagine that, in the average, the excitation is more or less localizable within the blob. Thus, to characterize the hadron[2], we need its c.m. coordinates x^{μ} and the coordinates ξ^{μ} of the "exciton", relative to the c.m., as illustrated in Fig. 1. Since x^{μ} and ξ^{μ} are kinematically independent, we have the relativistic Poisson brackets[3]

$$\{x_\mu, x_\nu\}_P = \{\xi_\mu, \xi_\nu\}_P = 0\,, \quad \{x_\mu, \xi_\nu\}_P = g_{\mu\nu} \tag{1.1}$$

where the bracket is defined in the usual way,

$$\{A, B\}_P \equiv \partial_\rho A 6^\rho B - 6_\rho A \partial^\rho B\ . \tag{1.1a}$$

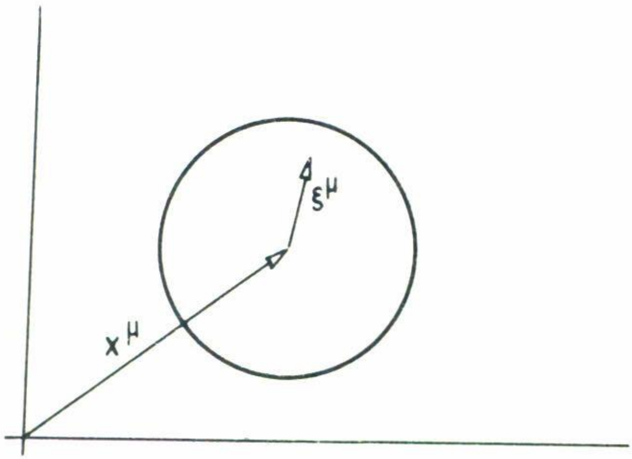

FIG. 1

HADRON MODEL

Here, and in the following,

$$\partial_\rho = \partial/\partial x^\rho\,, \qquad 6_\rho = \partial/\partial \xi^\rho\ . \tag{1.1b}$$

Thus, we are led to a quasi-phase space structure $M(x, \xi)$ and it is natural to ask about a nontrivial group of "canonical transformations" that leaves (1.1) invariant. We find that the simplest nontrivial set of canonical transformations is the following 17-parameter group $\mathcal{H}_5$:

$$J_\mu : \begin{cases} x_\mu \to \Lambda_{\mu\rho} x^\rho\,, \\ \xi_\mu \to \Lambda_{\mu\rho} \xi^\rho\,, \end{cases} \qquad (\Lambda : \text{Lorentz matrix}) \tag{1.2a}$$

$$P_\mu : \begin{cases} x_\mu \to x_\mu + a_\mu , \\ \xi_\mu \to \xi_\mu , \end{cases} \qquad (a_\mu \text{ real}) \qquad (1.2b)$$

$$\Pi_\mu : \begin{cases} x_\mu \to x_\mu , \\ \xi_\mu \to \xi_\mu + b_\mu , \end{cases} \qquad (b_\mu \text{ real}) \qquad (1.2c)$$

$$S : \begin{cases} x_\mu \to x_\mu - \sigma\xi_\mu , \\ \xi_\mu \to \xi_\mu , \end{cases} \qquad (-\infty < \sigma < +\infty) \qquad (1.2d)$$

$$C : \begin{cases} x_\mu \to x_\mu , \\ \xi_\mu \to \xi_\mu + \alpha x_\mu , \end{cases} \qquad (-\infty < \alpha < +\infty) \qquad (1.2e)$$

$$D : \begin{cases} x_\mu \to e^{\lambda} x_\mu , \\ \xi_\mu \to e^{-\lambda} \xi_\mu . \end{cases} \qquad (-\infty < \lambda < +\infty) \qquad (1.2f)$$

The P and Π transformations represent independent translations in the external and internal space and their generators can be interpreted as *total (c. m.) momentum* and *exciton (relative)* momentum, respectively. S and C are "dynamical transformations" inasmuch as the change in the external coordinates depends on the internal coordinates and vice versa. Now, it is important to observe that after S and C have been introduced, we do not have a group *unless we adjoin the dilatations D*. Hence the latter arise in a very natural way.

The Lie algebra is found to be as follows:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\mu\sigma}J_{\rho\nu} + g_{\nu\sigma}J_{\rho\mu}) \; ; \qquad (1.3a)$$

$$[J_{\rho\sigma}, P_\mu] = i(g_{\mu\sigma}P_\rho - g_{\mu\rho}P_\sigma) , \qquad [J_{\rho\sigma}, \Pi_\mu] = i(g_{\mu\sigma}\Pi_\rho - g_{\mu\rho}\Pi_\sigma) ,$$

$$[P_\mu, P_\nu] = [\Pi_\mu, \Pi_\nu] = [P_\mu, \Pi_\nu] = 0 \; ; \qquad (1.3b)$$

$$[S, P_\mu] = 0 , \quad [S, \Pi_\mu] = iP_\mu , \quad [C, P_\mu] = -i\Pi_\mu , \quad [C, \Pi_\mu] = 0 ; \tag{1.3c}$$

$$[D, P_\mu] = -iP_\mu , \quad [D, \Pi_\mu] = i\Pi_\mu ; \tag{1.3d}$$

$$[S, C] = iD , \quad [S, D] = 2iS , \quad [D, C] = 2iC ; \tag{1.3e}$$

$$[J_{\mu\nu}, S] = [J_{\mu\nu}, C] = [J_{\mu\nu}, D] = 0 . \tag{1.3f}$$

Equation (1.3e) shows that we have an $SU(1, 1)$ subgroup, generated by

$$I_1 = \tfrac{1}{2}D , \quad I_2 = \tfrac{1}{2}(C + S) , \quad I_3 = \tfrac{1}{2}(C - S) , \tag{1.4}$$

with I_2 being the compact generator. Thus, the physically so important dilatation occurs in our group not in isolation, but rather as a *member of a semisimple subgroup*. The whole group structure can now be read off[4]:

$$\mathcal{H}_5 = \{ SL(2, C)^J \times SU(1, 1)^I \} \otimes \{ T_4^P \times T_4^\Pi \} . \tag{1.5}$$

Since the dynamical $SU(1, 1)^I$ commutes with the kinematical $SL(2, C)^J$, we are able to set up a *classification scheme*. If we consider finite dimensional (non-unitary) representations of the $SU(1, 1)^I$ dynamical subgroup, then observables will form tensor or spinor operators relative to this classification. The usual *scale quantum number d* of X will be determined by $[D, X] = idX$. Thus, a two-component $SU(1, 1)^I$ spinor operator has a $d = +1$ and a $d = -1$ component and so on. Similarly, sets of fields will be classified according to their $SU(1, 1)^I$ transformation behavior. This will permit us to consider *wave equations for dilatation covariant multicomponent fields*.

The two Casimir invariants of $\mathcal{H}_5$ are found to be

$$C_1 = (P\Pi)^2 - P^2\Pi^2 , \tag{1.6}$$

$$C_2 = \tfrac{1}{2} J^{\mu\nu} R_{\mu\nu} + S\Pi^2 + CP^2 - DP\Pi \ , \tag{1.7}$$

where

$$R_{\mu\nu} \equiv P_\mu \Pi_\nu - P_\nu \Pi_\mu \ . \tag{1.8}$$

Finally, we succeeded in classifying all unitary irreducible representations of $\aleph_5$. For these results and for other details, cf. Reference 1.

In our subsequent work, M. Lorente, P. L. Huddleston, and I endeavored to construct a simple (first quantized) field theory, which is invariant under $\aleph_5$. The rest of this report sketches our findings.

II. GENERALITIES ABOUT THE WAVE EQUATIONS

Since we have two sets (x_μ, ξ_μ) of coordinates, we clearly shall have a *bilocal theory*. Naturally, we need a non-vanishing orbit-equation. This forces us to select one particular class of representations which is distinguished from the others by the fact that $C_1 \neq 0$. However, for this class the little group is just the identity, so that we have no natural "internal labels" for the $SU(1,1)^I$ characterization, nor even for "intrinsic spin". In addition, it later turned out that for this class $C_2 = 0$. It is therefore necessary to shift our interest from the representations of the global group to those of its *Lie algebra*. There is another reason for this, too. One readily sees that finite transformations generated by S and C can destroy causal order. Thus, it is necessary to confine oneself to the corresponding infinitesimal transformations; i.e., to consider realizations of the Lie algebra rather than those of the group.

If we take a space of sufficiently well-behaved functions $\phi(x, \xi)$ over $M(x, \xi)$, we easily find the following realization of the Lie algebra (1.3a – 1.3f):

$$P_\mu = i\partial_\mu \ , \qquad \Pi_\mu = i6_\mu \ ; \tag{2.1a}$$

$$J_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu + \xi_\mu 6_\nu - \xi_\nu 6_\mu) + I_{\mu\nu} \ ; \tag{2.1b}$$

$$S = -i\xi^\mu \partial_\mu + \Sigma \tag{2.1c}$$

$$C = i x^{\mu} \delta_{\mu} + \Gamma \ ; \tag{2.1d}$$

$$D = i\,(x_{\mu} \partial^{\mu} - \xi_{\mu} \delta^{\mu}) + \Delta \ . \tag{2.1e}$$

Here the $I_{\mu\nu}$ are the familiar spin matrices, and similarly, Σ, Γ, Δ are (coordinate independent) matrices obeying the same commutator algebra as S, C, D. That is,

$$[\Sigma, \Gamma] = i\Delta \ , \qquad [\Sigma, \Delta] = 2i\Sigma \ , \qquad [\Delta, \Gamma] = 2i\Gamma \ . \tag{2.2}$$

Using this realization, the equation for the second Casimir operator (1.7) becomes[5]

$$C_2 \phi(x, \xi) = \{ \tfrac{1}{2} I^{\mu\nu} (\partial_{\nu} \delta_{\mu} - \partial_{\mu} \delta_{\nu}) - \Sigma \Box_{\xi} - \Gamma \Box_{x} + \Delta \partial_{\mu} \delta^{\mu} \} \, \phi(x, \xi) \ , \tag{2.3}$$

and for the first Casimir operator

$$C_1 \phi(x, \xi) = \{ \partial_{\mu} \delta^{\mu} \partial_{\nu} \delta^{\nu} - \Box_{x} \Box_{\xi} \} \, \phi(x, \xi) \ . \tag{2.4}$$

To start with, we concentrate on the fundamental two-dimensional non-unitary spinor representation of the internal $SU(1,1)^{I}$ algebra. Then ϕ is a two-component spinor

$$\phi(x, \xi) = \begin{pmatrix} \phi_1(x, \xi) \\ \phi_2(x, \xi) \end{pmatrix} \ ,$$

and an explicit representation of Σ, Γ, Δ is

$$\Gamma = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} \ , \quad \Sigma = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \ , \quad \Delta = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \ . \tag{2.5}$$

Apart from (2.2), in this representation we also have the additional properties:

$$\Gamma^2 = 0 \ , \quad \Sigma^2 = 0 \ , \quad \Delta^2 = -\underset{\sim}{1} \ , \tag{2.6a}$$

$$\Gamma\Delta + \Delta\Gamma = 0 \ , \quad \Sigma\Delta + \Delta\Sigma = 0 \ , \quad \Gamma\Sigma + \Sigma\Gamma = \underset{\sim}{1} \ ; \tag{2.6b}$$

$$\Gamma^\dagger = \Sigma \ , \quad \Sigma^\dagger = \Gamma \ , \quad \Delta^\dagger = -\Delta \ . \tag{2.6c}$$

The dilatation transformations are represented by the matrix

$$T^D(\lambda) = e^{i\lambda\Delta} = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix} \tag{2.7}$$

so that the two $SU(1,1)^I$ basis spinors

$$\phi_+ \equiv \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix} \quad \text{and} \quad \phi_- \equiv \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix} \ , \tag{2.8}$$

(which are eigenstates of $i\Delta$) *are dilatation eigenstates with scale quantum number $d = +1$ and $d = -1$, respectively.*

Higher representations of $SU(1,1)^I$ will be briefly considered in Section V.

III. SPINLESS PARTICLES

We first study the case of spin zero, when $I_{\mu\nu} = 0$ in Equation (2.3). We then find that the condition of solvability of (2.3) is

$$C_2^2 = -C_1 \ . \tag{3.1}$$

We select a representation with $C_1 > 0$. Then (3.1) tells us that C_2 is pure imaginary. For convenience we set

$$C_2 = -iK \ , \quad K \text{ real.} \tag{3.2}$$

Then (2.3) assumes the form

$$(\Gamma\Box_x + \Sigma\Box_\xi - \Delta\partial_\mu 6^\mu - iK)\,\phi(x,\xi) = 0 \,. \tag{3.3}$$

If we operate on this with $(\Gamma\Box_x + \Sigma\Box_\xi - \Delta\partial_\mu 6^\mu + iK)$, we obtain the equation for the first Casimir operator, cf. (2.4), with

$$C_1 = K^2 \ . \tag{3.4}$$

Thus, *our basic wave equation is not the orbit equation, but rather the equation for the second Casimir operator*, given by (3.3).

In momentum space the wave equation (3.3) becomes

$$(\Gamma p^2 + \Sigma\pi^2 - \Delta p\pi + iK)\,\phi(p,\pi) = 0 \ . \tag{3.5}$$

Taking the basis state ϕ_+ (cf. 2.8) which has $d = +1$, we find that

$$p^2\phi_+ = 0 \ , \qquad p\pi\phi_+ = -K\phi_+ \ . \tag{3.6}$$

Consequently, this state has zero mass, $M_+^2 = 0$. However, taking the basis state ϕ_- which has $d = -1$, we obtain

$$\pi^2\phi_- = 0 \ , \qquad p\pi\phi_- = K\phi_- \ . \tag{3.7}$$

Thus, in this case p^2 need not vanish even though $\pi^2 = 0$. In fact, writing $M_-^2 = p^2$ and going to the rest frame $\underset{\sim}{p} = 0$, the second relation in (3.7) gives

$$M_-^2 = K^2/(\pi_0^R)^2 = C_1/(\pi_0^R)^2 > 0 \tag{3.8}$$

for the mass. Here π_0^R stands for the energy of the exciton in the restframe of the extended particle.

General states (which are not dilatation eigenstates) will not be mass-eigenstates. However, writing for an arbitrary state

$$|\psi> = a|\phi_+> + b|\phi_->\ , \quad |a|^2 + |b|^2 = 1\ ,$$

we easily compute the *expectation value*

$$\overline{M^2} \equiv <\psi|p^2|\psi> = |b|^2 M_-^2 + |a|^2 M_+^2 = |b|^2 K^2/(\pi_0^R)^2\ . \qquad (3.9)$$

Thus, the admixture of the $d = -1$ component provides a *non-vanishing mass*. We have a *dilatation invariant* (in fact, $\mathcal{H}_5$ invariant) *theory for massive particles*. The mass is determined by the energy of the exciton; i.e., by the force to which it is subject. Our model, naturally does not provide this force, so that the mass is not fixed, even if a fixed representation is used.

We now return to the general discussion of the wave equation. The hermitean conjugate of (3.3) is

$$\Box_x \phi^\dagger \Sigma + \Box_\xi \phi^\dagger \Gamma + \partial_\mu \sigma^\mu \phi^\dagger \Delta + iK\phi^\dagger = 0\ .$$

Both (3.3) and this conjugate equation can be derived from the *Lagrangian*

$$\mathcal{L} = i\partial_\mu \bar{\phi} \Gamma \partial^\mu \phi + i\sigma_\mu \bar{\phi} \Sigma \sigma^\mu \phi - \tfrac{1}{2} i\partial_\mu \bar{\phi} \Delta \sigma^\mu \phi - \tfrac{1}{2} i\sigma_\mu \bar{\phi} \Delta \partial^\mu \phi - K\bar{\phi}\phi\ , \qquad (3.10)$$

where the adjoint wave function $\bar{\phi}(x, \xi)$ is defined as

$$\bar{\phi}(x, \xi) \equiv \phi^\dagger(x, -\xi)\,\theta\ , \qquad (3.11a)$$

with

$$\theta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\ . \qquad (3.11b)$$

The Lagrangian density (3.10) is not hermitean – in fact, it can be shown that there does not exist any hermitean $\mathcal{L}$. However, "averaged" over the ξ-space[6],

$$<\mathcal{L}> \equiv \int d^4\xi\,\mathcal{L}$$

is hermitean, and so is the action

$$W = \int d^4x <\mathcal{L}> \quad .$$

We are now prepared to set the canonical formalism in action. If an infinitesimal symmetry transformation

$$x \to x + \delta x \ , \quad \xi \to \xi + \delta\xi \ , \quad \phi \to \phi + \delta\phi , \quad \overline{\phi} \to \overline{\phi} + \delta\overline{\phi} \tag{3.12a}$$

is performed, we get

$$\delta W = \int d^4x\, d^4\xi\ [\,\partial_\mu j^\mu + \eth_\mu k^\mu] \quad , \tag{3.12b}$$

with

$$j_\mu(x,\xi) = [\mathcal{L} g_{\mu\nu} - \partial_\nu\phi_a(\partial\mathcal{L}/\partial\partial^\mu\phi_a)]\ \delta x^\nu -$$

$$- \eth_\nu\phi_a(\partial\mathcal{L}/\partial\partial^\mu\phi_a)\ \delta\xi^\nu + (\partial\mathcal{L}/\partial\partial^\mu\phi_a)\ \delta\phi_a \ , \tag{3.13a}$$

and

$$k_\mu(x,\xi) = -\ \partial_\nu\phi_a(\partial\mathcal{L}/\partial\eth^\mu\phi_a)\ \delta x^\nu +$$

$$+\ [\mathcal{L} g_{\mu\nu} - \eth_\nu\phi_a(\partial/\partial\eth^\mu\phi_a)]\ \delta\xi^\nu + (\partial\mathcal{L}/\partial\eth^\mu\phi_a)\ \delta\phi_a \ . \tag{3.13b}$$

As is standard practice in bilocal theories, we define the physical x-dependent currents by "averaging" over the internal ξ- space,

$$<j_\mu(x)> \equiv \int d^4\xi\, j_\mu(x,\xi) \ .$$

Since $\int d^4\xi\,(6_\mu k^\mu) = \int d\sigma_\mu(\xi)\, k^\mu$, Equation (3.12b) yields

$$\delta W = \int d^4x\, \partial_\mu <j^\mu(x)> + \int d^4x \int d\sigma_\mu(\xi)\, k^\mu(x,\xi) \ .$$

However, in consequence of the assumed suitably good behavior of ϕ at the boundary of the ξ-space, the surface integral vanishes and we have

$$\delta W = \int d^4x\, \partial_\mu <j^\mu(x)> \ . \tag{3.14}$$

Specializing to various symmetry transformations, we then get the following currents:

a) Electromagnetic current (from phase transformations):

$$<J_\mu(x)> = <\bar{\phi}\Gamma\overleftrightarrow{\partial}_\mu\phi - \tfrac{1}{2}\bar{\phi}\Delta\overleftrightarrow{6}_\mu\phi> \ . \tag{3.15a}$$

b) C.m. energy momentum tensor (from c.m. translations):

$$<T^P_{\nu\mu}(x)> = <i\partial_\nu\bar{\phi}\Gamma\partial_\mu\phi + i\partial_\mu\bar{\phi}\Gamma\partial_\nu\phi - \tfrac{1}{4}i[\partial_\nu\bar{\phi}\Delta 6_\mu\phi + 6_\mu\bar{\phi}\Delta\partial_\nu\phi +$$

$$+ \partial_\mu\bar{\phi}\Delta 6_\nu\phi + 6_\nu\bar{\phi}\Delta\partial_\mu\phi] - g_{\mu\nu}\mathcal{L}> \ . \tag{3.15b}$$

c) Exciton energy momentum tensor (from relative translations):

$$<T^\Pi_{\nu\mu}(x)> = <i6_\nu\bar{\phi}\Gamma\partial_\mu\phi + i\partial_\mu\bar{\phi}\Gamma 6_\nu\phi - \tfrac{1}{2}i\,6_\nu\bar{\phi}\Delta 6_\mu\phi - \tfrac{1}{2}i\,6_\mu\bar{\phi}\Delta 6_\nu\phi> \cdot \tag{3.15c}$$

d) Covariant angular momentum tensor (from Lorentz transformations):

$$<M_{\nu\rho\mu}(x)> = <x_\nu T^P_{\rho\mu} - x_\rho T^P_{\nu\mu} + \xi_\nu T^\Pi_{\rho\mu} - \xi_\rho T^\Pi_{\nu\mu}> \quad . \tag{3.15d}$$

e) S-current (from S-transformation):

$$<S_\mu(x)> = <i\xi^\nu \theta^P_{\nu\mu}> \quad . \tag{3.15e}$$

f) C-current (from C-transformation):

$$<C_\mu(x)> = <ix^\nu \theta^\Pi_{\nu\mu}> \quad . \tag{3.15f}$$

g) Dilatation current (from D-transformation):

$$<D_\mu(x)> = <x^\nu \theta^P_{\nu\mu} - \xi^\nu \theta^\Pi_{\nu\mu}> \quad . \tag{3.15g}$$

In the last three formulae,

$$\theta^P_{\nu\mu} = T^P_{\nu\mu} + t^P_{\nu\mu} \ , \qquad \theta^\Pi_{\nu\mu} = T^\Pi_{\nu\mu} + t^\Pi_{\nu\mu} \tag{3.16}$$

are "improved stress tensors". Without quoting the detailed structure of t^P and t^Π we only mention that

$$\partial^\mu < t^P_{\nu\mu} > = 0 \ , \qquad \partial^\mu < t^\Pi_{\nu\mu} > = 0 \ ,$$

$$\int d\sigma^\mu(x) < t^P_{\nu\mu} > = 0 \ , \qquad \int d\sigma^\mu(x) < t^\Pi_{\nu\mu} > = 0 \ ,$$

so that the redefinitions (3.16) are permissible.

Noteworthy features of these results are the following:

i) All currents are hermitean (except $< T^\Pi_{\nu\mu} >$ which is antihermitean). Thus, formal unitary representations in terms of the integrated currents can be written down (disregarding questions of domain).

ii) The electromagnetic current has scale quantum number $q = 3$. As is well known, this leads to "scaling" in deep inelastic scattering.

iii) $T^{P}_{\nu\mu}$ is symmetric and has scale quantum number $d = 4$, as in a canonical local theory.

iv) The angular momentum tensor, the S, C, and D currents are (combinations of) moments of the energy momentum tensors.

v) Translation invariance implies S-conservation.

vi) C-conservation imposes the trace condition $\mathrm{Tr}\, \theta^{\Pi} = 0$.

vii) D-conservation imposes the trace condition $\mathrm{Tr}\, \theta^{P} = 0$.

Interestingly, dilatation invariance does not imply S and/or C-conservation, in contrast to the usual local conformal theory where dilatation-invariance (together with translation invariance) implies conformal invariance. On the other hand, S and C conservation together imply dilatation invariance, as can be inferred from Equation (1.3e).

IV. PARTICLES WITH SPIN 1/2

If we have spin ½, then in Equation (2.3) we must put $I^{\mu\nu} = \frac{1}{4} i\, [\gamma^{\mu}, \gamma^{\nu}]$. Since $SL(2, C)^{J}$ and $SU(1,1)^{I}$ commute, the wave function will be the direct product of an $SU(1,1)^{I}$ and an $SL(2, C)^{J}$ part. Using the representation

$$\gamma^{0} = \begin{pmatrix} \underset{\sim}{1} & 0 \\ 0 & -\underset{\sim}{1} \end{pmatrix}, \qquad \gamma^{k} = \begin{pmatrix} 0 & \underset{\sim}{\sigma}^{k} \\ -\underset{\sim}{\sigma}^{k} & 0 \end{pmatrix}$$

of the Dirac matrices, we can write for the 8-component wave function

$$\psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \times \begin{pmatrix} \chi_A \\ \chi_B \end{pmatrix} \equiv \begin{pmatrix} \psi_1^{A} \\ \psi_1^{B} \\ \psi_2^{A} \\ \psi_2^{B} \end{pmatrix} \tag{4.1}$$

where χ_A and χ_B are two-component spinors corresponding to the subdivision of the Dirac matrices. The first factor (ϕ) is the now familiar two-component $SU(1,1)$ spinor. We used the notation $\psi_1^A = \phi_1 \chi_A$ etc. in the last identity.

If we introduce the notations

$$(R_{23}, R_{31}, R_{12}) = \underset{\sim}{R} \equiv \underset{\sim}{p} \times \underset{\sim}{\pi} \quad , \tag{4.2a}$$

$$(R_{01}, R_{02}, R_{03}) = \underset{\sim}{Q} \equiv p_0 \underset{\sim}{\pi} - \pi_0 \underset{\sim}{p} \quad , \tag{4.2b}$$

then the wave equation in the momentum space that arises from (2.3) can be conveniently written in the split form

$$\begin{pmatrix} \tfrac{1}{2}\underset{\sim}{\sigma}\underset{\sim}{R} + ip\pi - C_2 & \tfrac{1}{2}i\underset{\sim}{\sigma}\underset{\sim}{Q} \\ \tfrac{1}{2}i\underset{\sim}{\sigma}\underset{\sim}{Q} & \tfrac{1}{2}\underset{\sim}{\sigma}\underset{\sim}{R} + ip\pi - C_2 \end{pmatrix} \begin{pmatrix} \psi_1^A \\ \psi_1^B \end{pmatrix} + \begin{pmatrix} i\pi^2 & 0 \\ 0 & i\pi^2 \end{pmatrix} \begin{pmatrix} \psi_2^A \\ \psi_2^B \end{pmatrix} = 0 \ , \tag{4.3a}$$

$$\begin{pmatrix} \tfrac{1}{2}\underset{\sim}{\sigma}\underset{\sim}{R} - ip\pi - C_2 & \tfrac{1}{2}i\underset{\sim}{\sigma}\underset{\sim}{Q} \\ \tfrac{1}{2}i\underset{\sim}{\sigma}\underset{\sim}{Q} & \tfrac{1}{2}\underset{\sim}{\sigma}\underset{\sim}{R} + ip\pi - C_2 \end{pmatrix} \begin{pmatrix} \psi_2^A \\ \psi_2^B \end{pmatrix} - \begin{pmatrix} ip^2 & 0 \\ 0 & ip^2 \end{pmatrix} \begin{pmatrix} \psi_1^A \\ \psi_1^B \end{pmatrix} = 0 \ . \tag{4.3b}$$

We find that the condition for solvability for this homogeneous linear set of equations is

$$C_2^2 = -k^2 C_1 \ , \qquad \text{where} \quad k^2 = \tfrac{1}{4} \text{ or } \tfrac{9}{4} \quad . \tag{4.4}$$

Thus, we have now a reducible representation. As in the spinless case, we select a representation with $C_1 > 0$, so that then C_2 must be pure imaginary. We write

$$C_2 = -iK \ , \quad K \text{ real} \ . \tag{4.5}$$

A compatible set of labeling operators is given as follows:

a) C_2^2 (generalization of the familiar "large and small component" distinction),

b) $$i\delta = i\Delta \times \begin{pmatrix} \underset{\sim}{1} & 0 \\ 0 & \underset{\sim}{1} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{1} & & & \\ & \underset{\sim}{1} & & \\ & & -\underset{\sim}{1} & \\ & & & -\underset{\sim}{1} \end{pmatrix}$$ (dilatation operator),

c) $$N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \frac{1}{|\underset{\sim}{R}|} \begin{pmatrix} \underset{\sim}{\sigma}\underset{\sim}{R} & 0 \\ 0 & -\underset{\sim}{\sigma}\underset{\sim}{R} \end{pmatrix} = \frac{1}{|\underset{\sim}{R}|} \begin{pmatrix} \underset{\sim}{\sigma}\underset{\sim}{R} & & & \\ & -\underset{\sim}{\sigma}\underset{\sim}{R} & & \\ & & \underset{\sim}{\sigma}\underset{\sim}{R} & \\ & & & -\underset{\sim}{\sigma}\underset{\sim}{R} \end{pmatrix}$$

(generalized helicity operator).

In summary, a basis set of solutions of the wave equation can be labeled as $\psi_d^{k^2 n}$ and we have

$$(C_2^2/C_1)\,\psi_d^{k^2 n} = -k^2 \psi_d^{k^2 n}\ , \quad k^2 = \tfrac{1}{4} \text{ or } \tfrac{9}{4} \quad ; \tag{4.6a}$$

$$i\delta\psi_d^{k^2 n} = d\psi_d^{k^2 n}\ , \quad d = +1 \text{ or } -1\ ; \tag{4.6b}$$

$$N\psi_d^{k^2 n} = n\psi_d^{k^2 n}\ , \quad n = +1 \text{ or } -1\ . \tag{4.6c}$$

If we now substitute a $d = +1$ basis state ψ_+ into the wave equation (4.3b), we immediately see that it is an eigenstate of p^2 belonging to eigenvalue zero. Thus, such states are massless, $M_+^2 = 0$. However, taking a basis state ψ_- which has $d = -1$, Equation (4.3a) tells us that now π^2 has zero eigenvalue, but p^2 need not vanish. In fact, writing $M_-^2 = p^2$ and going to the restframe $\vec{p} = 0$, we get from (4.3b) with (4.2a), (4.2b), (4.5) and with some calculation concerning the condition of solvability,

$$M_-^2 = m^2 K^2/(\pi_0^R)^2 \quad , \quad \text{where} \quad m^2 = \begin{cases} 4 & \text{if } k^2 = \frac{1}{4} \quad , \\ \frac{4}{9} & \text{if } k^2 = \frac{9}{4} \quad . \end{cases} \tag{4.7}$$

As in the spinless case, we see that $d = -1$ basis states can have *non-vanishing mass*, but now we also have *two distinct mass values* which correspond to the two possible values of C_2^2. But, as expected, the mass does not depend on the "helicity" n. General states, once again, are not mass eigenstates, but they will have a nonvanishing mass expectation value.

As in the spinless case, we can find a slightly modified Lagrangian and work out a canonical formalism. Since, however, no new features emerge, we omit here the details.

V. ARBITRARY $SU(1, 1)^I$ AND $SL(2, C)^J$ SPIN

We now wish to indicate the generalization of our major results when, instead of the $n = 2$ dimensional defining representation of $SU(1, 1)$ we allow for an arbitrary $n = 2j + 1$ ($j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$) dimensional representation, and when we consider arbitrarily high ordinary spin s.

The matrix representations of Δ, Σ, and Γ for arbitrary finite n can be calculated from Bargmann's paper[7]. We find that

$$\Delta = i \begin{pmatrix} -2j & & & & \\ & -2j+2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & +2j \end{pmatrix} ,$$

$$\Sigma = i \begin{pmatrix} 0 & A_{j,j-1} & & & \\ & 0 & A_{j-1,j-2} & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot & A_{-j+1,-j} \\ & & & & & 0 \end{pmatrix} ,$$

$$\Gamma = i \begin{pmatrix} 0 & & & & & \\ -A_{j-1,j} & 0 & & & & \\ & -A_{j-2,j-1} & 0 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & -A_{-j,-j+1} & 0 \end{pmatrix},$$

where

$$A_{ab} = [1/(a-b)!]\,[\{(j+a)!(j-b)!\}/\{(j+b)!(j-a)!\}]^{\frac{1}{2}}$$

Concerning higher $SL(2, C)$ spin, we found it convenient to use the Bargmann–Wigner formalism[8], where, for spin $s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ we have

$$I^{\mu\nu} = \tfrac{1}{2} i \sum_{r=1}^{2s} \gamma^{\mu}(r)\,\gamma^{\nu}(r) \quad ,$$

with

$$\gamma^{\mu}(r) = 1 \times 1 \times \ldots \times \gamma^{\mu} \times 1 \times \ldots \times 1 \quad ,$$

the Dirac γ^{μ} standing in the r–th place.

When we now write down the wave equation in momentum space, a lengthy but elementary calculation shows that the condition of solvability is

$$C_2^2 \equiv -K^2 = -k^2 C_1 \quad , \tag{5.1a}$$

where

$$k = 2j + s, \quad 2j + s - 1, \ldots, -2j - s \quad . \tag{5.1b}$$

To find the possible mass values, we substitute different basis states into the wave equation and obtain the following results:

i) Basis states with dilatation quantum number $d = j$ have $p^2 = 0$, $\pi^2 = 0$.

ii) Basis states with $d = j - 1, j - 2, \ldots, -j + 1$ have both $p^2 = 0$ and $\pi^2 = 0$.

iii) The only basis states that can have non-vanishing mass are those which have $d = -j$. Then $\pi^2 = 0$ but $p^2 \neq 0$, and going to the restframe we find that

$$M_-^2 = m^2 K^2 / (\pi_0^R)^2 \quad , \tag{5.2a}$$

where now

$$m^2 = (2j - s)^{-2}, \quad (2j - s + 1)^{-2}, \ldots, (2j + s)^{-2} \quad . \tag{5.2b}$$

Thus, we have a finite, discrete mass-spectrum[9] for any fixed value of π_0^R.

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REFERENCES AND FOOTNOTES

1. For a detailed account, see P. Roman, J.J. Aghassi, and P.L. Huddleston, Jour. Math. Phys. 13 (1972) 1852.
2. A similar picture of hadronic matter was first suggested by Japanese physicists, cf. S. Ishida, Prog. Theor. Phys. 46 (1971) 1570, ibid. p. 1905, and subsequent papers in the same journal. An algebraic model

with bilocal coordinates has been used recently by F. Gürsey and S. Orfanidis, Nuovo Cim. 11A (1972) 225. An interesting connection between bilocal wave equations and the chain model has been studied, for example, by T. Takabayashi, Prog. Theor. Phys. 48 (1972) 1718.

3. Our metric is $g_{00} = 1$, $g_{kk} = -1$, $g_{\mu\nu} = 0$ for $\mu \neq \nu$.
4. We denote direct products by $\times$ and semidirect products by $\otimes$.
5. Here and in the following, $\Box_x = \partial_\mu \partial^\mu$ and $\Box_\xi = \partial_\mu \partial^\mu$.
6. We shall use the symbol < . . . > throughout to indicate integration over the entire ξ-space.
7. V. Bargmann, Ann. of Math. 48 (1947) 568. See especially pp. 629-630.
8. V. Bargmann and E. P. Wigner, Proc. Nat. Acad. Sci. 34 (1948) 211.
9. Incidentally, it may happen that we have a value $k^2 = K^2 = 0$.
   For such a state the mass is undetermined.
   A closer inspection reveals that the emergence of *several* distinct mass values (as stated after Eqs. (4.7) and (5.2b)) is only fictitious.
   Cf. P. Roman, M. Lorente and P. L. Huddleston, Nuovo Cimento A, 1974, in press.

RESUMEN

Se establecen ecuaciones de onda bilocales para espin arbitrario, dentro del marco algebráico desarrollado previamente para física relativista de dilatación. El principal resultado de este estudio es que, a pesar de la invariancia de dilatación, las partículas pueden tener masa no-nula. Se desarrolla un formalismo canónico y se estudian varias corrientes.