ON THE PROPERTIES OF COHERENT STATES

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ABSTRACT: Coherent states are obtained by applying a dynamical unitary transformation to an extremal state in an invariant subspace of a quantum mechanical hamiltonian. The properties of coherent states are completely characterized mathematically. In addition, we prove the following very useful theorem: A physical system initially in a coherent state, or in particular in its ground state, will evolve into a coherent state. We give various examples of the utility of coherent states.

I. INTRODUCTION

A large number of quantum mechanical models have the following properties:

1) The gross energy level structure is defined by a static hamiltonian;
2) Perturbations can be written as a linear superposition of shift operators;
3) The static hamiltonian and the shift operators close under commutation and form a finite dimensional Lie algebra.

We define coherent states with respect to a Lie group G, a stability subgroup H, and an irreducible representation $\Gamma^{\lambda}(G)$, as the state obtained by applying the operator $\Gamma^{\lambda}(G/H)$ to an extremal state (e.g., the ground state of the unperturbed hamiltonian) in the invariant subspace of G characterized by the quantum numbers λ.

Such coherent states have all the usual properties of the field coherent states. Baker-Campbell-Hausdorff formulas, depending only on G and not on Γ^{λ}, can be constructed and used to simplify calculations. The coherent states themselves are non-orthogonal and over-complete within any invariant subspace. Under an arbitrary perturbation, a system which is initially in a coherent state, or in particular in its ground state, will evolve into a coherent state.

These statements are valid whenever the dynamical transformation group G is compact, or if G is non-compact, whenever Γ^{λ} is semi-bounded.

In §II we describe the forces motivating the search for generalization of the coherent state concept. This is directly related to the extreme usefulness and the widespread applicability of the field coherent states. The properties of these states are reviewed in §III. These mathematical mechanisms are applied, in §IV, to the construction of the atomic coherent states for an ensemble of 2-level atoms. The extreme similarity between the field coherent states described in §III, and the atomic coherent states described in §IV, is made manifest by a group contraction procedure in §V. In this process the Bloch sphere (describing atomic coherent states) is contracted to the phase plane of the harmonic oscillator (describing field coherent states) In §VI we illustrate the utility of the atomic coherent states by indicating how they have been used to solve non-trivial problems.

In §VII we return to a general discussion of the properties of coherent states, and in particular we prove the theorem stated in the abstract. Finally, we apply this formalism in §VIII to obtain a swift solution to a particular model of a superfluid system.

II. BACKGROUND AND MOTIVATION

What are now called the field coherent states were first discussed by Schrödinger[1] in connection with the semiclassical limit of the quantum mechanical harmonic oscillator. They were later used by Bloch and Nordsieck[2] to treat the "infrared catastrophe." The properties of these states were then formalized by Schwinger.[3] Finally, Glauber[4,5] introduced these states under the name "coherent states," into Quantum Optics.

Because of the intimate relationship between coherent states on the one hand, and the output of a laser cavity on the other, field coherent states have maintained a central position in the development of Quantum Optics since their introduction by Glauber in 1963.[6]

Quantum Optics involves the description of the interaction between N atoms and an electromagnetic field confined to a cavity of finite volume. A suitable model hamiltonian for such a system is

$$\mathcal{H} = \sum_{\boldsymbol{k}} \hbar\omega_{\boldsymbol{k}} a_{\boldsymbol{k}}^{\dagger} a_{\boldsymbol{k}} + \Delta E \sum_{j=1} S_{3(j)} +$$

$$+ \sum_{\boldsymbol{k}} \sum_{j} g_{\boldsymbol{k}} a_{\boldsymbol{k}} S_{+(j)} \exp(i\boldsymbol{k}\cdot\mathbf{x}_j) + g_{\boldsymbol{k}}^{*} a_{\boldsymbol{k}}^{\dagger} S_{-(j)} \exp(-i\boldsymbol{k}\cdot\mathbf{x}_j) \ . \qquad (2.1)$$

In this expression, $a_{\boldsymbol{k}}^{\dagger}$ and $a_{\boldsymbol{k}}$ are the Bose creation and annihilation operators for photons in the field mode $\boldsymbol{k}$, and $S_{3(j)}$, $S_{\pm(j)}$ are the angular momentum operators describing the atom located at position $\mathbf{x}_j$ as a 2-level system.

Equation (2.1) has not yet been solved in general. In particular, the operators appearing in this equation do not close under commutation, and as a result do not form a finite-dimensional Lie algebra. As a result the procedures described in the introduction are not directly applicable to this hamiltonian.

If the "atomic part" of the system described in (2.1) behaves classically, so that the operators $S_{3(j)}$, $S_{\pm(j)}$ can be replaced by c-number driving fields, then the resulting hamiltonian can be solved explicitly and exactly.[4,5] If the system is originally in a vacuum state of the electromagnetic field, then it will evolve into a field coherent state. We conclude from the quantum-classical hamiltonian (2.1) that a classical current, when applied to a vacuum state of the electromagnetic field, will produce a coherent state of the electromagnetic field, and that such a coherent state is in some sense the closest possible quantum analog of a classical electromagnetic field.

It is instructive to ask whether these results can be dualized. That is: is it possible to replace the electromagnetic field operators appearing in (2.1) by c-number driving fields and then solve the resulting hamiltonian? The resulting hamiltonian then describes the interaction between a classical electromagnetic field and an ensemble of N identical 2-level atoms. This hamiltonian can be solved exactly in three cases of extreme physical interest :

i) point laser (cavity length $\ll\lambda$);

ii) single mode traveling wave laser;

iii) traveling electromagnetic wave in an aplifying or absorbing medium.

In any of these cases, if the atomic system is originally in its ground state, it will evolve into a coherent atomic state.

Although the hamiltonian (2.1) cannot be solved exactly, the semi-classical hamiltonians arising from (2.1) can be solved exactly. The semi-classical hamiltonians are obtained by assuming either that the atomic system is classical and the field system is quantum mechanical, or that the field system is classical and the atomic system is quantized. In either case, if the quantum mechanical system is originally in a coherent state, or in particular in its ground state, then it will evolve into a coherent state. In both instances the coherent state is the closest possible quantum analog of the corresponding classical state. These remarks are summarized in Fig. 1.

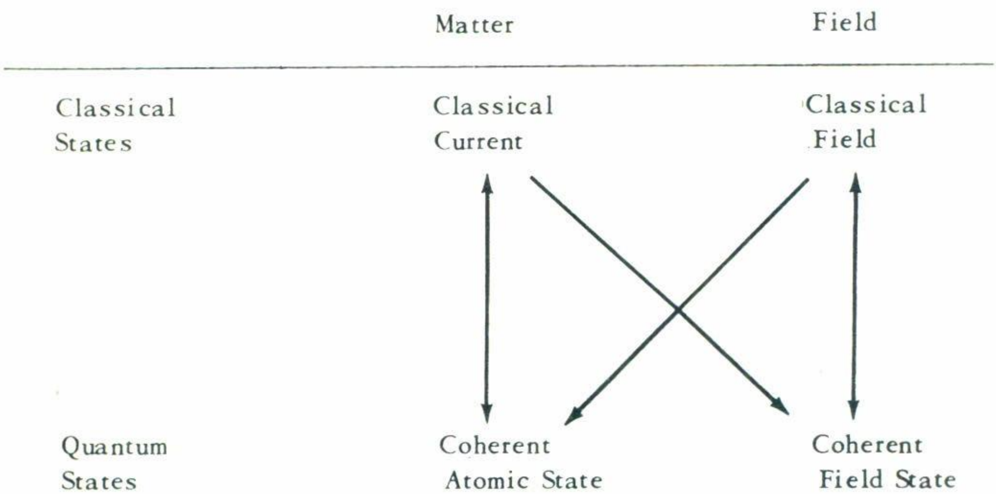

Fig. 1. Interacting atomic and field systems may be considered as dual to each other. If either quantum system is driven by its dual classical counterpart, a coherent state results (diagonal arrows). The coherent state is the closest possible quantum analog of the corresponding classical state (vertical arrows).

III. REVIEW OF FIELD COHERENT STATES

We summarize here the properties of coherent states for a single mode of the electromagnetic field.[4-7]

1. Model Hamiltonian: a model hamiltonian describing the interaction of a classical current with the electromagnetic field is

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_0 + \mathcal{H}_{\text{pert}} \\ \mathcal{H}_0 &= \hbar\omega a^\dagger a \\ \mathcal{H}_{\text{pert}} &= \lambda(t)\, a^\dagger + \lambda^*(t)\, a \ . \end{aligned} \tag{3.1}$$

Here $a^\dagger a$ is the single mode photon number operator, and $a^\dagger$ and a are the photon creation and annihilation operators for a single mode, respectively.

2. Commutation Relations: the hamiltonian described in (3.1) is a linear superposition of operators that close under commutation. These operators obey the commutation relations

$$\begin{aligned} [n, a^\dagger] &= +a^\dagger & \qquad [n, I] &= 0 \\ [n, a] &= -a & \qquad [a^\dagger, I] &= 0 \\ [a, a^\dagger] &= +I & \qquad [a, I] &= 0 \ . \end{aligned} \tag{3.2}$$

The four operators $n = a^\dagger a$, $a^\dagger$, a, and I span the Lie algebra h_4, called the harmonic oscillator algebra.

3. Diagonal States: the eigenstates of $\mathcal{H}_0$ contain a fixed number of photons in each field mode

$$\mathcal{H}_0 \,|\, n > \, = \hbar\omega n \,|\, n > \ . \tag{3.3a}$$

The normalized eigenstates can be obtained by applying the creation operator to the ground state $|\,0>$ n successive times:

$$|\, n > \, = (a^\dagger)^n (n!)^{-\frac{1}{2}} \,|\, 0 > \ . \tag{3.3b}$$

These diagonal field states are called "Fock" states.[8]

4. Ground State: the ground state is defined uniquely, up to a complex phase factor of modulus unity, as the eigenstate of $\mathcal{H}_0$ with lowest energy eigenvalue:

$$\mathcal{H}_0 \, |0\rangle = E_{min} \, |0\rangle \; ,$$

$$E_{min} = 0 \; . \tag{3.4a}$$

It can equivalently be defined as the state annihilated by the shift-down operator a:

$$a|0\rangle = 0 \quad \text{or} \quad \exp a|0\rangle = |0\rangle \tag{3.4b}$$

5. Unitary Translation Operator: under the influence of a classical driving current, the ground state $|0\rangle$ will evolve under a unitary operator $U(\alpha)$:

$$U(\alpha) = \exp(\alpha a^{\dagger} - \alpha^{*} a)$$

$$U(\alpha)\,|0\rangle = |\alpha\rangle \; . \tag{3.5}$$

In general, $\alpha(t)$ is a time-dependent complex number, and $\alpha(t)$ is related to $\lambda(t)$ through the equations of motion which are derivable from (3.1).

The transformation $U(\alpha)$ is a unitary representation ($\infty \times \infty$ matrix) of the coset representatives[9] of $H_4/U(1) \otimes U(1)$, which is isomorphic with the phase plane of the harmonic oscillator. The states $|\alpha\rangle$ are called "coherent" states and for the particular case of the electromagnetic field they are called "Glauber" states.[4,5,7,9]

6. Coherent State Eigenvalue Equation: the coherent states obey an eigenvalue equation easily derivable from (3.4b):

$$\{U(\alpha)\, a\, U^{-1}(\alpha)\}\, U(\alpha)\,|0\rangle = (a - \alpha)\,|\alpha\rangle = 0 \; . \tag{3.6}$$

7. Baker-Campbell-Hausdorff Formulas: these formulas allow for rearrangements in the ordering of exponential operator products. They are extremely useful for dealing with the properties of coherent states. A useful

BCH formula for the Lie group H_4 is

$$\exp(\alpha a^\dagger + \beta a) = \exp(-\tfrac{1}{2}\alpha\beta)\exp(\beta a)\exp(\alpha a^\dagger)$$

$$= \exp(\tfrac{1}{2}\alpha\beta)\exp(\alpha a^\dagger)\exp(\beta a)\,. \tag{3.7}$$

8. Expansion of Coherent States: the coherent states can be expanded in terms of the eigenstates (3.3b) of $\mathcal{H}_0$, since these form a complete set of orthonormal states. This expansion is facilitated by the BCH relation (3.7):

$$|\alpha\rangle = U(\alpha)|0\rangle = \exp(-\tfrac{1}{2}\alpha^*\alpha)\exp(\alpha a^\dagger)\exp(-\alpha^* a)|0\rangle$$

$$= \exp(-\tfrac{1}{2}\alpha^*\alpha)\sum_0^\infty(\alpha a^\dagger)^n(n!)^{-1}|0\rangle$$

$$= \exp(-\tfrac{1}{2}\alpha^*\alpha)\sum_0^\infty(\alpha)^n(n!)^{-\frac{1}{2}}|n\rangle\,. \tag{3.8}$$

9. Non-orthogonality: the field coherent states are non-orthogonal:

$$\langle\alpha|\beta\rangle = \langle 0|U^\dagger(\alpha)\,U(\beta)|0\rangle$$

$$= \exp\left(\alpha^*\beta - \tfrac{1}{2}(\alpha^*\alpha + \beta^*\beta)\right) \tag{3.9a}$$

$$|\langle\alpha|\beta\rangle|^2 = \exp\left(-|\alpha-\beta|^2\right)\,. \tag{3.9b}$$

10. Over-completeness: the coherent states are overcomplete. The resolution of the identity operator in terms of coherent states is not unique. A useful resolution is

$$\int|\alpha\rangle\,(d^2\alpha/\pi)\,\langle\alpha| = I = \sum_0^\infty|n\rangle\langle n|\,. \tag{3.10}$$

11. Uncertainty Relations: the creation and annihilation operators are not hermitian, but their "real" and "imaginary" parts are:

$$q = (a + a^{\dagger})/\sqrt{2}$$

$$[p, q] = -i$$

$$p = (a - a^{\dagger})/i\sqrt{2} \tag{3.11a}$$

The non-commuting hermitian operators p, q have minimum uncertainty within a coherent state:

$$(\Delta p)^2 \, (\Delta q)^2 = (\tfrac{1}{2})^2$$

$$(\Delta q)^2 = \langle \alpha | (q - \langle q \rangle)^2 | \alpha \rangle$$

$$\langle q \rangle = \langle \alpha | q | \alpha \rangle \; . \tag{3.11b}$$

12. Generating Functions: in correlation experiments it is often necessary to compare correlation data with matrix elements of the form:

normal form	$\langle \alpha | (a^{\dagger})^m (a)^n | \alpha \rangle$
anti-normal form	$\langle \alpha | (a)^n (a^{\dagger})^m | \alpha \rangle$
symmetrized form	$\langle \alpha | S\{(a)^n (a^{\dagger})^m\} | \alpha \rangle$. (3.12a)

Such matrix elements are most simply obtained from a generating function:

$$\langle \alpha | (a)^n (a^{\dagger})^m | \alpha \rangle = (\partial/\partial\gamma)^n (\partial/\partial\delta)^m \langle \alpha | \exp(\gamma a) \exp(\delta a^{\dagger}) | \alpha \rangle \Big|_{\gamma = \delta = 0} \tag{3.12b}$$

The generating function is simple to compute:

$$\langle\alpha|\exp(\gamma a)\exp(\delta a^\dagger)|\alpha\rangle$$

$$= \exp(-\alpha^*\alpha)\langle 0|\exp(\alpha^* a)\exp(\gamma a)\exp(\delta a^\dagger)\exp(\alpha a^\dagger)|0\rangle$$

$$= \exp(-\alpha^*\alpha)\langle 0|\exp\big((\alpha+\delta)a^\dagger\big)\exp\big((\alpha^*+\gamma)(\alpha+\delta)\big)\exp\big((\alpha^*+\gamma)a\big)|0\rangle$$

$$= \exp(-\alpha^*\alpha)\exp\big((\alpha+\delta)(\alpha^*+\gamma)\big) \;. \tag{3.12c}$$

Other generating functions can be obtained as simply.

IV. ATOMIC COHERENT STATES

We now "dualize" the treatment given in the preceeding section.

1. Model Hamiltonian: if the electromagnetic field operators appearing in (2.1) are replaced by their (macroscopic) classical average values using the analog of a mean-field approximation scheme, the hamiltonian simplifies greatly. In the case of a single mode traveling wave laser, it is

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{pert}}$$

$$\mathcal{H}_0 = \Delta E \sum_{j=1}^{N} S_{3(j)}$$

$$\mathcal{H}_{\text{pert}} = \gamma(t)\sum_{j=1}^{N} S_{+(j)}\exp(i\mathbf{k}\cdot\mathbf{x}_j) + \gamma^*(t)\sum_{j=1}^{N} S_{-(j)}\exp(-i\mathbf{k}\cdot\mathbf{x}_j) \;. \tag{4.1}$$

2. Commutation Relations: the single-atom operators are kinematically independent and obey the usual $SU(2)$ commutation relations:

$$[S_{3(j)}, S_{+(j')}] = S_{+(j)}\delta_{jj'} \;,\; \text{etc.}$$

The many-atom operators

$$J_3 = \sum_{j=1}^{N} S_{3(j)}$$

$$J_+ = \sum_{j=1}^{N} S_{+(j)} \exp(ik \cdot \mathbf{x}_j)$$

$$J_- = \sum_{j=1}^{N} S_{-(j)} \exp(-ik \cdot \mathbf{x}_j) = J_+^{\dagger}$$

obey the usual $SU(2)$ commutation relations

$$\begin{aligned} [J_3, J_+] &= J_+ \qquad & [J_3, J_0] &= 0 \\ [J_3, J_-] &= -J_- \qquad & [J_+, J_0] &= 0 \\ [J_-, J_+] &= -2J_3 \qquad & [J_-, J_0] &= 0 \quad . \end{aligned} \tag{4.2}$$

The operator J_0 is a multiple of the identity within any irreducible representation.

3. Diagonal States: the eigenstates of $\mathcal{H}_0$ are essentially angular momentum eigenstates

$$\mathcal{H}_0 \left| {j \atop m} \right\rangle = \Delta E \, m \left| {j \atop m} \right\rangle \quad . \tag{4.3a}$$

The normalized eigenstates can be obtained by applying the shift-up operator to the ground state $\left| {j \atop -j} \right\rangle$ $(j+m)$ times

$$\left| {j \atop m} \right\rangle = \begin{pmatrix} 2j \\ j \pm m \end{pmatrix}^{-\frac{1}{2}} \frac{(J_+)^{j+m}}{(j+m)!} \left| {j \atop -j} \right\rangle \quad . \tag{4.3b}$$

These diagonal states are called "Dicke" states.[7, 10]

4. Ground State: the ground state is defined uniquely, up to a complex phase factor of modulus unity, as the eigenstate of $\mathcal{H}_0$ with lowest energy eigenvalue:

$$\mathcal{H}_0 \left| {j \atop -j} \right\rangle = E_{\min} \left| {j \atop -j} \right\rangle \tag{4.4a}$$

$$E_{\min} = -j\Delta E$$

It can equivalently be defined as the state annihilated by the shift-down operator J_-:

$$J_- \left| {j \atop -j} \right\rangle = 0 \quad \text{or} \quad \exp(J_-) \left| {j \atop -j} \right\rangle = \left| {j \atop -j} \right\rangle \tag{4.4b}$$

5. Unitary Transformation Operator: under the influence of a classical driving field, the ground state $\left| {j \atop -j} \right\rangle$ will evolve under a unitary operator[7,9] $U(\theta\phi)$:

$$U(\theta\phi) = \exp(\zeta J_+ - \zeta^* J_-)$$

$$\zeta = \tfrac{1}{2}\theta \exp(-i\phi)$$

$$U(\theta\phi) \left| {j \atop -j} \right\rangle = \left| {j \atop \theta\phi} \right\rangle \tag{4.5}$$

In general $\zeta(t)$ is a time-dependent complex number, and $\zeta(t)$ is related to $\gamma(t)$ through the equations of motion which are derivable from (4.1).

The transformation $U(\theta\phi)$ is a unitary representation ($2j+1 \times 2j+1$ matrix) of the coset representatives of $U(2)/U(1) \otimes U(1)$, which is isomorphic[7,9] with the sphere S^2. This sphere is often called[7,9] the "Bloch sphere" since it was introduced by Bloch[11] for the discussion of the nuclear induction experiment.[12] The states $\left| {j \atop \theta\phi} \right\rangle$ are called "coherent" states, and for the particular case of two-level atoms, they are called "Bloch" states.[7,9,11]

6. Coherent State Eigenvalue Equation: the coherent atomic states obey several "eigenvalue equations" easily derivable from the eigenvalue equations defining the ground state $\left| {}^{j}_{-j} \right\rangle$.

$$\{U(\theta\phi)\, J^2\, U^{-1}(\theta\phi)\}\, U(\theta\phi) \left| {}^{j}_{-j} \right\rangle = j(j+1) \left| {}^{j}_{\theta\phi} \right\rangle$$

$$\{U(\theta\phi)\, J_3\, U^{-1}(\theta\phi)\}\, U(\theta\phi) \left| {}^{j}_{-j} \right\rangle = -j \left| {}^{j}_{\theta\phi} \right\rangle$$

$$\{U(\theta\phi)\, J_-\, U^{-1}(\theta\phi)\}\, U(\theta\phi) \left| {}^{j}_{-j} \right\rangle = 0 \quad . \tag{4.6}$$

These equations do not have the classic structure of eigenvalue equations since the operator $\{U(\theta\phi)\, \mathbb{Q}\, U^{-1}(\theta\phi)\}$ on the left hand side of each equation depends explicitly on the parameters $(\theta\phi)$ serving to label the coherent (eigen) states.

7. Baker-Campbell-Hausdorff Formulas: a large number of BCH formulas can be derived for the Lie group $SU(2)$. These have been treated in detail elsewhere.[7, 13] Some particularly useful BCH fomulas for current purposes are given in (4.7):

$$\exp(\zeta J_+ - \zeta^* J_-) = \exp(\tau J_+)\exp\left(\ln(1+\tau^*\tau)\, J_3\right)\exp(-\tau^* J_-)$$

$$= \exp(-\tau^* J_-)\exp\left(-\ln(1+\tau^*\tau)\, J_3\right)\exp(\tau J_+) \tag{4.7}$$

where

$$\zeta = \exp(-i\phi)\, \tfrac{1}{2}\theta$$

$$\tau = \exp(-i\phi)\tan \tfrac{1}{2}\theta \quad .$$

8. Expansion of Coherent States: the coherent states can be expanded in terms of the eigenstates (4.3b) of $\mathcal{H}_0$, since these form a complete set of orthonormal states. This expansion is facilitated by the BCH relation (4.7):

$$\left|\begin{matrix} j \\ \theta\phi \end{matrix}\right\rangle = U(\theta\phi)\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle = \exp(\tau J_+)\exp\left(\ln(1+\tau^*\tau)J_3\right)\exp(-\tau^* J_-)\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle$$

$$= (1+\tau^*\tau)^{-j}\sum_{n=0}^{\infty}(\tau J_+)^n (n!)^{-1}\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle$$

$$= \sum_{m=-j}^{+j}\left|\begin{matrix} j \\ m \end{matrix}\right\rangle \binom{2j}{j\pm m}^{\frac{1}{2}}(\cos\tfrac{1}{2}\theta)^{j-m}\left(\exp(-i\phi)\sin\tfrac{1}{2}\theta\right)^{j+m} \qquad (4.8)$$

9. Non-orthogonality: the atomic coherent states are non-orthogonal:

$$\left\langle\begin{matrix} j' \\ \theta'\ \phi' \end{matrix}\right|\left.\begin{matrix} j \\ \theta\ \phi \end{matrix}\right\rangle$$

$$= [\cos\tfrac{1}{2}\theta'\cos\tfrac{1}{2}\theta + \exp\left(i(\phi'-\phi)\right)\sin\tfrac{1}{2}\theta'\sin\tfrac{1}{2}\theta]^{2j}\delta_{jj'} \qquad (4.9a)$$

$$\left|\left\langle\begin{matrix} j' \\ \theta'\ \phi' \end{matrix}\right|\left.\begin{matrix} j \\ \theta\phi \end{matrix}\right\rangle\right|^2 = \left\{\frac{1+\hat{n}(\Omega')\cdot\hat{n}(\Omega)}{2}\right\}^j \delta_{j'j} \ . \qquad (4.9b)$$

In the later expression, $\hat{n}(\Omega)$ is the unit vector from the center to the point $(\theta\phi)$ on the surface of the Bloch sphere.

10. Over-completeness: within any $SU(2)$-invariant subspace the identity operator may be resolved with respect to either the diagonal or the coherent states. The resolution of the identity operator in terms of coherent states is not unique, since they are over-complete. A useful resolution is

$$\int\left|\begin{matrix} j \\ \theta\phi \end{matrix}\right\rangle \frac{2j+1}{4\pi}d\Omega \left\langle\begin{matrix} j \\ \theta\phi \end{matrix}\right| = I_{2j+1} = \sum_{m=-j}^{+j}\left|\begin{matrix} j \\ m \end{matrix}\right\rangle\left\langle\begin{matrix} j \\ m \end{matrix}\right| \qquad (4.10)$$

11. Uncertainty Relations: the canonical uncertainty relations

$$\Delta J_x^2 \Delta J_y^2 \geqslant (\tfrac{1}{2})^2 \Delta J_z^2$$

become, after the unitary transformation by $U(\theta\phi)$:

$$U(\theta\phi)(J_x, J_y, J_z)\, U^{-1}(\theta\phi) = (J_\xi, J_\eta, J_\zeta) \tag{4.11a}$$

$$\Delta J_\xi^2\, \Delta J_\eta^2 \geqslant (\tfrac{1}{2})^2 \Delta J_\zeta^2 \quad . \tag{4.11b}$$

Within a coherent state, this uncertainty relation assumes the minimum allowed value.

12. Generating Functions: these generating functions play the same role in atomic physics that the functions (3.12) play for the electromagnetic field. They are derived in substantially the same way. For example

$$\left\langle {}^{j}_{\phi\theta} \right| \exp(a_- J_-) \exp(a_3 J_3) \exp(a_+ J_+) \left| {}^{j}_{\theta\phi} \right\rangle$$

$$= (\cos^2 \tfrac{1}{2}\theta)^{2j} \left\langle {}^{\;j}_{-j} \right| \exp\left((\tau^* + a_-) J_-\right) \exp(a_3 J_3) \exp\left((\tau + a_+) J_+\right) \left| {}^{\;j}_{-j} \right\rangle$$

$$= (\cos^2 \tfrac{1}{2}\theta)^{2j} \left\langle {}^{\;j}_{-j} \right| \exp\left((\;)' J_+\right) \exp(a_3' J_3) \exp\left((\;)' J_-\right) \left| {}^{\;j}_{-j} \right\rangle$$

$$= (\cos^2 \tfrac{1}{2}\theta)^{2j} \left\{\exp(-\tfrac{1}{2} a_3) + \exp(\tfrac{1}{2} a_3)(a_- + \tau^*)(a_+ + \tau)\right\}^{2j} \quad . \tag{4.12}$$

All such generating functions can be expressed succinctly as the $2j$-th power of the trace of a product of two matrices, one of which depends only on the parameters $\theta\phi$, the other only on the parameters a_i.[13]

V. RELATION BETWEEN ATOMIC AND FIELD STATES

The field states described in §III and the atomic states described in §IV have an extremely close formal resemblance. This resemblance is emphasized in Fig. 2, which indicates how the Dicke and Fock states are related to the Bloch and Glauber states.

This resemblance is not solely formal. Nor does it come about because the treatments followed in §III and §IV follow the same basic pattern. Rather, it comes about for the following reasons:

	Field System	Atomic System
Diagonal States	Fock	Dicke
Coherent States	Glauber	Bloch

Fig. 2. A Rosetta stone for the terminology of Quantum Optics

i) The dynamical group for the single field mode, which is H_4, and the dynamical group for an ensemble of 2-level atoms, which is $U(2)$, are both 4-parameter Lie groups.

ii) Field coherent states exist in 1-1 correspondence with coset representatives $H_4/U(1) \otimes U(1)$, which is essentially the phase plane of the harmonic oscillator. Atomic coherent states exist in 1-1 correspondence with coset representatives $U(2)/U(1) \otimes U(1)$, which is essentially the Bloch sphere.

iii) The groups $U(2)$ and H_4 and their cosets, the Bloch sphere and the oscillator phase plane, are related to each other by a group contraction process.[14, 15]

When the non-singular transformation ($c \neq 0$ in 5.1) is performed on the generators J_μ of the group $U(2)$, the new basis vectors b_μ obey the commutation relations given in (5.2).

$$\begin{bmatrix} b_+ \\ b_- \\ b_3 \\ b_0 \end{bmatrix} = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 1/2c^2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} J_+ \\ J_- \\ J_3 \\ J_0 \end{bmatrix} \tag{5.1}$$

$$\begin{aligned} [b_3, b_+] &= +b_+ & [b_3, b_0] &= 0 \\ [b_3, b_-] &= -b_- & [b_+, b_0] &= 0 \\ [b_-, b_+] &= b_0 - 2c^2 b_3 & [b_-, b_0] &= 0 \end{aligned} \tag{5.2}$$

Although the transformation (5.1) becomes singular in the limit $c \to 0$, the commutation relations (5.2) remain well defined. In fact, in the limit $c \to 0$, the commutation relations for the Lie algebra $\boldsymbol{u}(2)$ become the commutation relations for the Lie algebra $\boldsymbol{h}_4$.

It is useful at this point to define the following limits:

$$\begin{aligned} &\gamma(t)/c \to \lambda(t) \\ &&\Delta E \to \hbar\omega \quad . \qquad (5.3) \\ &\tfrac{1}{2}\exp(-i\phi)\theta/c \to \alpha \end{aligned}$$

In the limit $c \to 0$ the hamiltonian (4.1) becomes equal to the hamiltonian (3.1) up to a constant additive term:

$$\begin{aligned} &\Delta E J_3 + \gamma(t)\, J_+ + \gamma^*(t)\, J_- \\ &= \Delta E\,\left(b_3 - (b_0/2c^2)\right) + \left(\gamma(t)/c\right)(cJ_+) + \left(\gamma^*(t)/c\right)(cJ_-) \\ &\xrightarrow{c \to 0} \hbar\omega b_3 + \lambda(t)\, b_+ + \lambda^*(t)\, b_- - (\Delta E/2c^2)\, b_0 \quad . \qquad (5.4) \end{aligned}$$

In addition, the BCH formulas valid for $U(2)$ (4.7) can be contracted to the corresponding BCH formulas, valid for H_4 (3.7).

Properties 1, 2, and 7 of §III and §IV are the only properties described that depend exclusively on the abstract group or on its algebra. The remaining properties (3–6, 8–12) enter into the discussion of the physical systems through their unitary irreducible representations.

Accordingly, to treat these remaining properties, we must contract the representations of $U(2)$ to the representations of H_4. Here we encounter a slight difficulty.[7,9] The group $U(2)$ is compact,[13] and has only finite dimensional unitary irreducible representations. The group H_4 is non-compact, and so has no faithful finite dimensional representations. Therefore, we choose a sequence of larger and larger representations of $U(2)$ as c becomes smaller and smaller. In this way ($j \uparrow \infty$ as $c \downarrow 0$) we can construct a faithful unitary irreducible representation of H_4 from the well-known[13] unitary irreducible representations of $U(2)$.

We will take this limit in a way that is transparent from a physical viewpoint, by insisting that all energies be measured from the ground state. Then

$$\mathrm{Lim}\, b_3 \left| \begin{matrix} j \\ -j \end{matrix} \right\rangle = \mathrm{Lim}\, \left(J_3 + (J_0/2c^2)\right) \left| \begin{matrix} j \\ -j \end{matrix} \right\rangle$$

$$= \mathrm{Lim}\left(-j + 1/2c^2\right)\left| \begin{matrix} j \\ -j \end{matrix} \right\rangle \qquad (5.5)$$

In order for the limit to be well defined, we demand $j \uparrow \infty$ and $c \downarrow 0$ as follows:

$$j = 1/2c^2 \quad . \qquad (5.6)$$

In this limit all the remaining properties 4.3–4.6 and 4.8–4.12 for the atomic system contract immediately to the corresponding properties 3.3–3.6 and 3.8–3.12 for the field system.

As an example of this procedure we contract the non-orthogonality relation (4.9a) to the non-orthogonality relation (3.9a):

$$\left\langle \begin{matrix} j \\ \theta'\phi' \end{matrix} \right| \left. \begin{matrix} j \\ \theta\phi \end{matrix} \right\rangle = \left\{ \cos \tfrac{1}{2}\theta' \cos \tfrac{1}{2}\theta + \left(\exp(-i\phi') \sin \tfrac{1}{2}\theta'\right)^* \left(\exp(-i\phi) \sin \tfrac{1}{2}\theta\right) \right\}^{2j}$$

$$\simeq \left\{ 1 - \tfrac{1}{2}c^2 |\alpha'|^2 - \tfrac{1}{2}c^2 |\alpha|^2 + c^2 \alpha'^* \alpha \right\}^{1/c^2} \rightarrow \exp\left(\alpha'^* \alpha - \tfrac{1}{2}(\alpha'^* \alpha' + \alpha^* \alpha)\right) \quad . \qquad (5.9)$$

The remaining contractions proceed in an analogous fashion. This contraction mechanism is summarized in Table 1.

TABLE 1

Relation between the $U(2)$ labels and the H_4 labels in the contraction of the Bloch sphere to the oscillator phase plane.

Group	Operators	Coordinates	Eigen values	Eigen states	Coherent states
$U(2)$	cJ_+, cJ_-	$(\theta/2c)\exp(-i\phi)$	$2jc^2$	$\left\| \begin{matrix} j \\ m \end{matrix} \right\rangle$	$\left\| \begin{matrix} j \\ \theta\phi \end{matrix} \right\rangle$
	$J_3 + J_0/2c^2$		$j+m$	Dicke	Bloch
H_4	$a^\dagger$ a	α	1	$\|n\rangle$	$\|\alpha\rangle$
	$a^\dagger a$		n	Fock	Glauber

VI. APPLICATIONS OF ATOMIC COHERENT STATES

The field coherent states (§III) and the atomic coherent states (§IV), which are so closely related (§V), have a number of useful properties in common. Since the usefulness of the field coherent states is firmly established[4-6], we present here some physically useful applications of the atomic coherent states.

The two applications which we discuss involve the approximate solution of the Hamiltonian (2.1), and the construction of thermodynamic partition functions for a large class of spin Hamiltonians.

Both applications depend in a crucial way on the overcompleteness of the atomic coherent states. Let G be an arbitrary operator acting within an $SU(2)$ invariant subspace of dimensionality $2j+1$. Then G can be expressed in terms of Dicke states (4.3), as

$$G = \sum_{m'}\sum_{m} \left|{j \atop m'}\right\rangle \left\langle{j \atop m'}\right| G \left|{j \atop m}\right\rangle \left\langle{j \atop m}\right| . \tag{6.1a}$$

The $(2j+1)^2$ matrix elements $\langle m'|G|m\rangle$ are in general independent.

The operator G can also be expressed in terms of the coherent states (4.5), as

$$G = \left((2j+1)/4\pi\right)^2 \int d\Omega' \int d\Omega \left|{j \atop \Omega'}\right\rangle \left\langle{j \atop \Omega'}\right| G \left|{j \atop \Omega}\right\rangle \left\langle{j \atop \Omega}\right| . \tag{6.1b}$$

It is a remarkable fact that, because of the overcompleteness of the states $\left|{j \atop \Omega}\right\rangle$ the kernel in (6.1b) can always[7] be chosen to be diagonal:

$$G = \left((2j+1)/4\pi\right) \int d\Omega \, |\Omega\rangle G(\Omega) \langle\Omega| . \tag{6.2}$$

The kernel $G(\Omega)$ is a c-number function defined on the surface of the unit sphere. As such, it can be expressed in terms of spherical harmonics.

$$G(\Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a^l_m Y^l_m(\Omega) . \tag{6.3}$$

The function $G(\Omega)$ defined by (6.2) is not unique. Any other kernel $G'(\Omega)$

$$G'(\Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} b_m^l Y_m^l(\Omega) \tag{6.3'}$$

defines the same operator G provided only that the lowest $(2j+1)^2$ coefficients appearing in (6.3) and (6.3') are equal.[7]

$$a_m^l = b_m^l \qquad -l \leqslant m \leqslant +l \qquad 0 \leqslant l \leqslant 2j \tag{6.4}$$

Two kernels $G(\Omega)$ and $G'(\Omega)$ differing only in their "Fourier coefficients" with $l > 2j$ describe exactly the same operator G. Such representations of G may be said to differ by a "gauge transformation".

The importance of the coherent state representation to physical applications lies in this: For any operator G acting within an $SU(2)$ j-invariant subspace, it is always possible to construct a diagonal representation $(G(\Omega))$ within a coherent state representation. Then all equations and manipulations involving the *operator* G become simply c-number equations and operations involving the *function* $G(\Omega)$.

In order to make contact with quantum optics experiments, it is necessary to know the implications of (2.1). This does not mean that it is necessary to solve (2.1). Rather, it is only desirable to determine the density operator $\rho(t)$ for a system governed by (2.1). The construction of the density operator from its equation of motion[16]

$$[\mathcal{H}, \rho] = i\hbar\, \partial\rho/\partial t \tag{6.5}$$

is facilitated by the coherent state representation for the field system and for the atomic system.

This comes about for the following reasons:

If the atomic system behaves classically, so that (2.1) simplifies to (3.1), then the system evolves into a coherent state (cf. Fig. 1). As a result, the density operator becomes a delta function in the field coherent state representation. If the atomic system is not "too quantum mechanical" in nature, then we should expect the field to evolve into a superposition of coherent states containing only a "small" number of coherent states.

Conversely, if the field behaves classically, so that (2.1) simplifies to (4.1), then the system evolves into a coherent state (cf. Fig. 1). As a result, the density operator becomes a delta function in the atomic coherent state representation. If the field system is not "too quantum mechanical" in nature, we should expect the atomic system to evolve into a superposition of coherent states containing only a "small" number of coherent states.

As a result of these intuitive considerations we would expect the coherent state representation to provide a useful mechanism for treating the master equation (6.5).

The density operator ρ can be described in many representations. In the diagonal representations (3.3) and (4.3) it is given by

$$\rho = \sum |n'm'\rangle\langle n'm'|\rho|nm\rangle\langle nm| \tag{6.6}$$

Since experiments are usually designed to treat only the field part of the total system, or only the atomic part of the total system, it is more useful to consider reduced density operators treating only the field subsystem or only the atomic subsystem. These reduced density operators are obtained from (6.6) by taking the trace over the uninteresting subsystem. Thus

$$\rho_A(t) = \mathrm{Tr}_F\rho(t) = \sum_{m'}\sum_{m} |m'\rangle\langle m'|\rho|m\rangle\langle m| \;,$$

where (6.7A)

$$\langle m'|\rho|m\rangle = \sum_{n} \langle nm'|\rho|nm\rangle \;.$$

$$\rho_F(t) = \mathrm{Tr}_A\rho(t) = \sum_{n'}\sum_{n} |n'\rangle\langle n'|\rho|n\rangle\langle n| \;,$$

where (6.7F)

$$\langle n'|\rho|n\rangle = \sum_{m} \langle n'm|\rho|nm\rangle \;.$$

As might be expected, the reduced density operators are not unrelated. In the superradiant regime[17] they are related through the moments of their respective shift operators according to

$$\mathrm{Tr}_F\,(a^\dagger)^r a^s \rho_F(t) = (-ig/\hbar\kappa)^{*r}(-ig/\hbar\kappa)^s\,\mathrm{Tr}_A\,J_+^r J_-^s \rho_A(t) \qquad (6.8)$$

$$t \gg \kappa^{-1} = 2l/c$$

where κ^{-1} is a photon transit time in a superradiant cavity of length l.

In many instances[18] the reduced density operator is essentially assumed to be diagonal in the Fock of Dicke representation, and then the time dependence of the diagonal elements $\rho_{F,nn}(t)$ or $\rho_{A,mm}(t)$ is determined. While this approach provides a useful first approximation, the dynamical information contained in the off-diagonal matrix elements is lost.

Under a large variety of conditions,[19] it is possible to choose the reduced density operator $\rho_F(t)$ as a diagonal matrix within the coherent state representation:

$$\rho_F(t) = \int (d^2\alpha/\pi)\,|\alpha\rangle\, R(\alpha;t)\,\langle\alpha| \quad . \qquad (6.9F)$$

We have seen above that it is always possible[7] to choose the reduced density operator $\rho_A(t)$ as a diagonal matrix within its coherent state representation:

$$\rho_A(t) = \big((2j+1)/4\pi\big)\int d\Omega\,\Big|\begin{matrix} j \\ \theta\phi \end{matrix}\Big\rangle\; P(\theta\phi;t)\;\Big\langle\begin{matrix} j \\ \theta\phi \end{matrix}\Big| \quad . \qquad (6.9A)$$

Under these diagonal ansätze, the operator equations of motion for the reduced density operator become simply ordinary c-number partial differential equations. In fact, they become Fokker-Planck equations. For the reduced density operator $\rho_F(t)$ we find

$$\partial R(\alpha,\alpha^*;t)/\partial t = \big[\big((\partial/\partial\alpha)\,\alpha + (\partial/\partial\alpha^*)\,\alpha^*\big)\{(\kappa-\gamma_l)+\gamma_{nl}\,\alpha^*\alpha\} +$$

$$+\, 4q\,(\partial/\partial\alpha^*)(\partial/\partial\alpha)\big]\,R(\alpha,\alpha^*;t) \qquad (6.10F)$$

where the parameters γ_l, γ_{nl}, and q describe the linear gain, the nonlinearity, and the fluctuations in the atomic system, respectively.[20] For the reduced density operator $\rho_A(t)$ we find[21]

$$\partial Q(\theta;t)/\partial t = [(\partial/\partial\theta)\left(j\sin\theta + \tfrac{1}{2}\sin\theta(1+\cos\theta)^{-1}\right) + $$

$$+ (\partial^2/\partial\theta^2)\,\tfrac{1}{2}(1-\cos\theta)]\;Q(\theta;t)\ , \qquad (6.10A)$$

where we have assumed an azimuthal symmetry and have set $Q(\theta;t) = \sin\theta\int_0^{2\pi} d\phi\, P(\theta\phi;t)$.

It is clear from inspection of (6.10F) and (6.10A) that the drift coefficients $2\,\mathrm{Re}\,\alpha\,\{(\kappa-\gamma_l)+\gamma_{nl}\alpha^*\alpha\}$ and $\{j\sin\theta + \tfrac{1}{2}\sin\theta(1+\cos\theta)^{-1}\}$ drive the distributions $R(\alpha;t)$ and $Q(\theta;t)$ over the surface of the oscillator phase plane and the surface of the Bloch sphere, respectively. The diffusion coefficients $4q$ and $\tfrac{1}{2}(1-\cos\theta)$ are responsible for the broadening of the respective distributions.

The coherent state representation has also been used by Lieb[22] to compute upper and lower bounds for a large class of quantum mechanical partition functions. The thermodynamic partition functions for which this technique is useful all involve the so-called spin Hamiltonians. These are Hamiltonians for N separate particles which interact only through their associated angular momentum operators J^i $(i = 1, 2, \ldots, N)$. These Hamiltonians need not be linear in the various spins, nor must they involve spins in only pairwise combinations. This class of Hamiltonians includes, as special cases, the Heisenberg model,[23] the Ising model,[24] and the Spherical model.[25]

In this particular application of the atomic coherent state representation, let G be an operator that acts within an $su(2)$ j-invariant subspace. Two kernels $g(\Omega)$ and $G(\Omega)$ are of interest

$$g(\Omega) = \langle\Omega|G|\Omega\rangle$$

$$G = \left((2j+1)/4\pi\right)\int d\Omega\,|\Omega\rangle\, G(\Omega)\,\langle\Omega| \qquad (6.11)$$

in particular, we have that

$$\mathrm{Tr}\,G = \mathrm{Tr}\left((2j+1)/4\pi\right)\int d\Omega\,|\Omega\rangle\, G(\Omega)\langle\Omega| = \left((2j+1)/4\pi\right)\int d\Omega\, G(\Omega). \qquad (6.12)$$

When G is an operator of the form $(\mathbf{v}\cdot \mathbf{J})^n$, then $g(\Omega)$ and $G(\Omega)$ are remarkably closely related. To highest order in powers of j they are given by

$$g(\Omega) = (|\mathbf{v}|)^n (j)^n + \mathbb{O}(j^{n-1}) \tag{6.13a}$$

$$G(\Omega) = (|\mathbf{v}|)^n (j+1)^n + \mathbb{O}(j^{n-1}) \quad . \tag{6.13b}$$

Thus, for example[22]

$$s_z^2(\Omega) = j^2 \cos^2\theta + \tfrac{1}{2} j\,(1-\cos^2\theta) \to j^2\cos^2\theta$$

$$S_z^2(\Omega) = (j+1)^2 \cos^2\theta - \tfrac{1}{2}(j+1)(1-\cos^2\theta) \to (j+1)^2\cos^2\theta \; . \tag{6.14}$$

The quantum partition function is

$$Z^{QM} = \mathrm{Tr}\,\exp(-\beta H) \; . \tag{6.15}$$

This can be expressed

$$\prod_{i=1}^{N} \int (d\Omega_i/4\pi) <\Omega_N | \exp(-\beta H) | \Omega_N >$$

$$= \mathrm{Tr}\,\exp(-\beta H) = \mathrm{Tr} \lim_{n\to\infty} \left(I - (1/n)\beta H\right)^n \; . \tag{6.16}$$

For the first term in this equality, we have

$$\prod_{i=1}^{N} \int (d\Omega_i/4\pi) <\Omega_N | \exp(-\beta H) | \Omega_N > \geqslant \prod_{i=1}^{N} \int (d\Omega_i/4\pi) \exp(-\beta <\Omega_N | H | \Omega_N >) \tag{6.17}$$

by the Peierls-Bogoliubov inequality $\left(<\phi|\exp(X)|\phi> \geqslant \exp(<\phi|X|\phi>)\right)$. The term in the exponential is

$$<\Omega_N | H | \Omega_N> = h(\Omega_N) \tag{6.18}$$

and under the replacement (6.13a), allowable by the assumption that H is a spin Hamiltonian, we have that the right hand side of (6.17) is simply the classical partition function

$$\prod_{i=1}^{N} \int (d\Omega_i/4\pi) \exp\left(-\beta h(\Omega_N)\right) = Z^{Cl}(j_1, j_2, \ldots, j_N) \tag{6.19}$$

to highest order in each value of j_i.

On the other hand, for the third term in (6.16), we have

$$\mathrm{Tr} \lim_{n\to\infty}\left(I-(1/n)\beta H\right)^n = \mathrm{Tr} \prod_{i=1}^{N} \int (d\Omega_i/4\pi) \lim_{n\to\infty} \{ | \Omega_N > \left(1-(1/n)\beta H(\Omega_N)\right) < \Omega_N | \}^n$$

$$\leqslant \mathrm{Tr} \prod_{i=1}^{N} \int (d\Omega_i/4\pi) \{ | \Omega_N > \lim_{n\to\infty} \left(1-(1/n)\beta H(\Omega_N)\right)^n < \Omega_N | \} \tag{6.20}$$

by the Schwartz inequality. The limit in the last term in (6.20) is trivial and leads to the classical partition function

$$\mathrm{Tr} \prod_{i=1}^{N} \int (d\Omega_i/4\pi) | \Omega_N > \exp\left(-\beta H(\Omega_N)\right) < \Omega_N | =$$

$$= \prod_{i=1}^{N} \int (d\Omega_i/4\pi) \exp\left(-\beta H(\Omega_N)\right) = Z^{Cl}(j_1+1, j_2+1, \ldots, j_N+1) \tag{6.21}$$

to the highest order in each value of j_i.

As a result of the inequalities appearing in (6.17) and (6.20), we have

$$\prod_{i=1}^{N} \int (d\Omega_i/4\pi) \exp\left(-\beta h(\Omega_N)\right) < \mathrm{Tr} \exp(-\beta H) \leqslant \prod_{i=1}^{N} \int (d\Omega_i/4\pi) \exp\left(-H(\Omega_N)\right) \tag{6.22a}$$

$$Z^{Cl}(j_1, \ldots, j_N) \leqslant Z^{QM} \leqslant Z^{Cl}(j_1+1, \ldots, j_N+1) \ . \tag{6.22b}$$

When each operator $\boldsymbol{J}^i$ in Z^{QM} is replaced by the corresponding normalized operator $\boldsymbol{J}^i/(\boldsymbol{J}^i \cdot \boldsymbol{J}^i)^{\frac{1}{2}}$ as in various Spherical model Hamiltonians, and each $j_i \to \infty$, the left and right hand side of (6.22) become equal, giving a precise value for the quantum partition function.

VII. DEFINITION AND GENERAL PROPERTIES OF COHERENT STATES

Coherent states were originally defined by Glauber[4,5] for the electromagnetic field. Glauber found three equivalent ways to define field coherent states:

M1. A coherent state is obtained by applying the unitary translation operator $U(\alpha)$ to the ground state (3.5):

$$|\alpha> = U(\alpha)|0> = \exp\left(\alpha a^\dagger - \alpha^* a\right)|0> \ .$$

M2. A coherent state is an eigenstate of the annihilation operator a (3.6):

$$a|\alpha> = \alpha|\alpha>$$

P1. A coherent state is obtained by applying a classical driving current to the ground state of the electromagnetic field.

The procedures M1 and M2 are mathematical; the procedure M1 is the mathematical representation for the physical procedure P1.

The three procedures M1, M2, and P1 are equivalent for the electromagnetic field because of the particular commutation properties of the field mode operators $a^\dagger a$, $a^\dagger$, a, I (3.2). For systems described by operators with different commutation relations, the three procedures M1, M2, and P1 are not all equivalent in general.

In order to extend the extremely useful coherent state concept to more complicated systems, it is necessary to generalize one of the two incompatible procedures M1 or M2.

It is at first sight attractive to use the concept expressed in M2 as a basis for defining coherent states for arbitrary systems. For eigenvalue equations play a prominent role in the quantum theory.[26] In this instance the eigenvalue equations are non-hermitian and the eigenvalues are complex. This approach has been adopted by Barut and Girardello[27] in discussing "new coherent states" associated with the spectrum generating algebra **su**(1, 1).

Adoption of M2 as a basis on which to generalize the coherent state concept suffers from two major drawbacks, the first mathematical, the second physical:

1. Coherent states could not be defined in Hilbert spaces of finite dimensionality. In particular, this would preclude construction of coherent states for compact Lie groups. Moreover, the states defined in this way have few useful properties, and in particular they are not computationally useful.
2. The states so defined do not correspond to physically realizable states, except under the special circumstances that the commutator of the annihilation operator a and its hermitian adjoint $a^{\dagger}$ is a multiple of the identity operator. Under these conditions we have restricted ourselves to the electromagnetic field.

In attempting to generalize the concept of coherent state, it is much more useful to use M1 as a point of departure. Then the mathematical objections raised above (#1) are automatically eliminated. In addition, since M1 is the mathematical representation for P1, the physical objections raised above (#2) are also eliminated.

We now proceed to define coherent states.

Let G be a dynamical transformation group (i.e., S-matrix) which acts by means of a unitary irreducible representation $\Gamma^{\lambda}(G)$ on a Hilbert space M^{λ}. Since $\Gamma^{\lambda}(G)$ is irreducible, M^{λ} is an invariant subspace under G. Let $|\text{ref}> \in M^{\lambda}$ be an arbitrary reference state in M^{λ}, which is normalized to unity: $<\text{ref}|\text{ref}> = 1$. Let $H \subset G$ be the stability group of $|\text{ref}>$. That is, H leaves $|\text{ref}>$ invariant up to a phase factor or modulus unity:

$$\Gamma^{\lambda}(h)|\text{ref}> = |\text{ref}> \exp\left(i\gamma(h)\right) \tag{7.1}$$

$$h \in H \subset G \ .$$

Then the action of an arbitrary group element $g \in G$ on the reference state $|\text{ref}>$ is given by

$$\Gamma^{\lambda}(g)|\text{ref}> = \Gamma^{\lambda}(ch)|\text{ref}> = \Gamma^{\lambda}(c)\,\Gamma^{\lambda}(h)|\text{ref}>$$

$$= \Gamma^{\lambda}(c)|\text{ref}> \exp\big(i\gamma(h)\big) = |c> \exp\big(i\gamma(h)\big) \tag{7.2}$$

$$g \in G \qquad c \in G/H$$

$$h \in H \qquad |c> \in M^{\lambda}$$

The states $\Gamma^{\lambda}(c)|\text{ref}> \equiv |c> \in M^{\lambda}$ exist in 1-1 correspondence with the coset representatives $c \in G/H$. The states $|c>$ are on the orbit of $|\text{ref}>$ under $\Gamma^{\lambda}(G/H)$. Moreover, since $\Gamma^{\lambda}(G)$ is unitary, the states $|c>$ are normalized to unity:

$$<c|c> = <\text{ref}|\Gamma^{\lambda}(c^{-1})\,\Gamma^{\lambda}(c)|\text{ref}> = <\text{ref}|\text{ref}> = 1\,.$$

The states $|c>$ are not orthogonal and they are overcomplete. States of the form

$$\Gamma^{\lambda}(G/H)|\text{ref}> \tag{7.3}$$

should therefore be considered as candidates for generalized coherent states. Before defining coherent states in general, we look at the spectrum of properties that the groups G, H, the representation $\Gamma^{\lambda}(G)$ and the reference state $|\text{ref}>$ may possess.

A. The group G may be an arbitrary dynamical transformation group. Or we may impose sufficient additional structure on G so as to make it a finite dimensional Lie group. We may impose further additional structure and demand that G be compact.

B. The unitary irreducible representation $\Gamma^{\lambda}(G)$ may be arbitrary. By imposing additional structure, we may demand $\Gamma^{\lambda}(G)$ be square integrable. Finally, by imposing a great deal of additional structure, we could demand that $\Gamma^{\lambda}(G)$ be finite dimensional.

C. The reference state $|\text{ref}>$ of norm unity may be an arbitrary state $|\text{arb}>$ in M^{λ}. By imposing additional constraints, we could demand that it be an eigenstate $|\text{diag}>$ of some unperturbed Hamiltonian $\mathcal{H}_0$. Finally, we can impose yet more structure and demand that it be an extremal state $|\text{ext}>$ in M^λ. An extremal state (for example, the ground state $|\text{gnd}> = |0>$) is a state annihilated by a maximal subalgebra of the algebra (not necessarily a Lie algebra) generating the dynamical transformation group G. If G is a semisimple Lie group, then

$$|\text{ext}> = |SM^b>$$

where M^b is the highest weight[28] in M^λ and $S \in W$, the Weyl group[28] of g.

D. The stability group H is completely determined by the choice of G, $\Gamma^\lambda(G)$, and $|\text{ref}>$. H is a closed subgroup of G which may be compact or non-compact when G is non-compact, but which must be compact when G is compact.

This spectrum of possibilities is summarized in Table 2.

TABLE 2

Spectrum of possibilities available for $\{G, \Gamma^\lambda, |\text{ref}>, H\}$ in constructing a useful definition of generalized coherent state. Within any given row, the amount of structure increases in going from left to right.

		Amount of Structure Assumed		
Level	Refers to	1	2	3
A	G	dynamical transformation group	Lie group	compact
B	Γ^λ	arbitrary unitary irreducible representation	square-integrable $\mathcal{L}^2$	finite-dimensional
C	$\|\text{ref}>$	arbitrary, normalized to unity $\|\text{arb}>$	eigenstate of $\mathcal{H}_0$ $\|\text{diag}>$	extremal $\|\text{ext}>$ or $\|\text{gnd}> = \|0>$

Definition: with the notation as described above, the coherent states associated with the system $\{G, \Gamma^{\lambda}, |\text{ref}\rangle, H\}$ are the states on the orbit $\Gamma^{\lambda}(G/H)|\text{ref}\rangle$ provided:

A1: G is a dynamical transformation group;

B2: Γ^{λ} is square integrable;

C3: $|\text{ref}\rangle = |\text{ext}\rangle$ is an extremal state.

With this definition for generalized coherent states, we are in a position to examine each of the properties #1-12 discussed in §III and §IV for two particular systems. Some of these properties depend only on the dynamical group G(#1, 2, 7), others depend on the representation $\Gamma^{\lambda}(G)$ and the choice of reference state. That is, some properties (#3) are valid for arbitrary Γ^{λ}, some (#10) depend on Γ^{λ} being square integrable, while yet others (#13) require Γ^{λ} to be finite dimensional. Some properties of coherent states (#9) are valid for arbitrary reference states, others (#3) require the reference state to be diagonal, while still others (#11) are valid only when the reference state is extremal. In Table 3 we summarize the properties of general coherent states, including a summary mathematical characterization for each property, as well as a statement about the amount of mathematical structure required for the property to be valid. In this Table we have included a thirteenth property suggested by the non-trivial applications of coherent states described in §VI.

Perelomov[29] has adopted a definition for coherent states similar to the one presented here. The definitions differ essentially in the amount of mathematical structure required in determining G, Γ^{λ}, and $|\text{ref}\rangle$. These differences are summarized in (7.4).

	In this work	Perelomov[29]	
G	A1	A2	
Γ^{λ}	B2	B1	(7.4)
$\|\text{ref}\rangle$	C3	C1	

These differences in detail lead to some differences between the approach presented here and that of Perelomov:

A. Perelomov's more restrictive choice for G allows always[30, 31] for the

TABLE 3

SUMMARY OF PROPERTIES OF GENERAL COHERENT STATES

Property Number	Mathematical Structure Required	Property Name	Mathematical Summary of Property
1.	A1	Model Hamiltonian	$\mathcal{H}_0 + \mathcal{H}_{pert} = iD_0(t) H_0 + i(D_+(t) E_+ + D_-(t) E_-)$
2.	A1	Commutation Relations	$H_0, E_+, E_- \quad [g,g] \not\subset g$
	A2		$\text{span } g \quad [g,g] \subset g$
3.	A1, B1, C2	Diagonal States	$\|M> \simeq (E_+ + E_-)^M \|\text{diag}>$
	A1, B1, C3		$\|M> \simeq (E_+)^M \|0>$
4.	A1, B1, C3	Ground State	$E_- \|0> = 0$
5.	A1, B1, C1	Unitary Transformation	$\|\Omega> = \exp(\Omega \cdot E) \|\text{ref}> = U(\Omega) \|\text{ref}>$
6.	A1, B1, C1	Eigenvalue Equations	$U(\Omega) \{\text{Invariant Operators}\} U^{-1}(\Omega) \|\Omega> = \text{Inv} \|\Omega>$
	A1, B1, C2		$U(\Omega) \{\text{Diagonal Operators}\} U^{-1}(\Omega) \|\Omega> = \text{Eig} \|\Omega>$
	A1, B1, C3		$U(\Omega) \{\text{Annihilation Ops.}\} U^{-1}(\Omega) \|\Omega> = 0 \|\Omega>$
7.	A2*	BCH Formulas	$\exp(\Omega \cdot E) = \exp(\Omega_+ E_+) \exp(\Omega_0 H) \exp(\Omega_- E_-)$
8.	A1, B1, C2	Eigenstate Expansion	$\|\Omega> = \sum \frac{c(M)}{M!} (E_+ + E_-)^M \|\text{diag}>$
	A2*, B1, C3		$= \exp(\Omega_+ E_+) \|\text{gnd}> N(\Omega_0)$
9.	A1, B1, C1	Non-orthogonality	$<\Omega\|\Omega'> = <\text{ref}\|U(\Omega^{-1}\Omega')\|\text{ref}>$
	A1, B1, C3		$= U_{00}(c) \exp(i\gamma(b)); \quad \Omega^{-1}\Omega' = cb$
10.	A2*, B2*, C1	Over-completeness	$\frac{\dim \Gamma^\lambda}{\text{Vol}(G/H)} \int \left\| {\lambda \atop \Omega} \right\rangle d\mu(G/H) \left\langle {\lambda \atop \Omega} \right\| = I$
11.	A2*, B1, C3	Uncertainty Relations	$\Delta(\text{Re } E'_\pm)^2 \, \Delta(\text{Im } E'_\pm)^2 = \text{minimum}$ $E'_\pm = U(\Omega) E_\pm U^{-1}(\Omega)$
12.	A2, B1, C3	Generating Functions	$f(A,\Omega) = <\Omega\| \exp A \cdot E \|\Omega>$
13.	A3*, B3*, C1	Diagonal Representation	$K = \frac{\dim \Gamma^\lambda}{\text{Vol}(G/H)} \int \|\Omega> K(\Omega) <\Omega\| d\mu(\Omega)$

*Requirements labelled with a * may possibly be relaxed somewhat.

construction of BCH formulas (#7, 10, 12). Since Perelomov does not discuss BCH formulas, his more restrictive requirement on G does not lead to any sharper results.

B. The square-integrability requirement adopted here guarantees the existence of over-completeness relations (#10). Without this requirement (B1 instead of B2), property #10 may not be valid.

C. The demand that $|\mathrm{ref}\rangle$ be an extremal state guarantees the usefulness of BCH formulas as a computational device.[30, 31] Thus, this restriction (C3 as opposed to C1) is useful in the discussion of those properties depending on the application of BCH formulas (#8, 9, 11, 12).

In practical applications, we generally adopt the more restrictive requirement A2, since we do not yet know how to construct BCH formulas simply under only the requirement A1. In practice, Perelomov adopts also the more restrictive requirements B2 and C3 in place of B1 and C1:

Present Work		Practical Applications		Perelomov
A1		A2		A2
B2	⟶	B2	⟵	B1
C3		C3		C1

In closing this section, finally, we prove the following important theorem. This theorem provides a kind of selection rule for coherent states, and is responsible for the usefulness of coherent states for describing physical processes.

Theorem: if a system is originally in a coherent state, then it will evolve into a coherent state.

Proof: Assume the original system state is $|c_1\rangle$:

$$|c_1\rangle = \Gamma^{\lambda}(c_1)|\mathrm{ref}\rangle$$

Then during a time interval $\Delta\tau = t_2 - t_1$ it will evolve under an element g of the dynamical transformation group G. The system state evolves into

$$\Gamma^{\lambda}(g)|c_1\rangle = \Gamma^{\lambda}(gc_1)|\mathrm{ref}\rangle = \Gamma^{\lambda}(c_2 h)|\mathrm{ref}\rangle = |c_2\rangle \exp\left(i\gamma(h)\right)$$

$$g \in G$$

$$h \in H$$

$$c_1 , c_2 \in G/H$$

$$|c_1\rangle , |c_2\rangle \in M^{\lambda} . \tag{7.6}$$

We point out that this proof is very general: the level of mathematical structure required for this theorem is (A1, B1, C1).

VIII. APPLICATION TO SUPERFLUID SYSTEMS

We now apply the considerations of the preceeding section to the description of a superfluid system. The Hamiltonian describing a system of N bosons interacting weakly with each other is[32]

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{pert}}$$

$$\mathcal{H}_0 = \sum_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} b^{\dagger}_{\boldsymbol{k}} b_{\boldsymbol{k}} \qquad \epsilon_{\boldsymbol{k}} = \hbar^2 k^2 / 2m$$

$$\mathcal{H}_{\text{pert}} = \tfrac{1}{2} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{p},\boldsymbol{q}} V_{\boldsymbol{k}} b^{\dagger}_{\boldsymbol{p}+\boldsymbol{k}} b^{\dagger}_{\boldsymbol{q}-\boldsymbol{k}} b_{\boldsymbol{p}} b_{\boldsymbol{q}} . \tag{8.1}$$

Here, $\mathcal{H}_0$ describes the kinetic energy of the non-interacting bosons. The term $\epsilon_{\boldsymbol{k}} = \hbar^2 k^2/2m$ is the kinetic energy of a boson with momentum $\hbar \boldsymbol{k}$ in mode $\boldsymbol{k}$. The perturbation term $\mathcal{H}_{\text{pert}}$ describes the scattering of two bosons out of the momentum states $(\boldsymbol{p}, \boldsymbol{q})$ and into the momentum states $(\boldsymbol{p}+\boldsymbol{k}, \boldsymbol{q}-\boldsymbol{k})$. The creation and annihilation operations obey the usual commutation relations:

$$[b_{\boldsymbol{p}} , b^{\dagger}_{\boldsymbol{q}}] = \delta_{\boldsymbol{p},\boldsymbol{q}} . \tag{8.2}$$

This scattering proceeds through an interaction potential $V(\boldsymbol{x})$ whose Fourier components are V_k. We will also assume $V_k = V_{-k}$.

Coherent states for this system are obtained by applying the unitary transformation operator

$$g = \tau \exp\left(-(i/\hbar) \int_{t_2}^{t_1} \mathcal{H}(t)\, dt \right) \tag{8.3}$$

to the extremal state

$$|\,\text{gnd}> = \prod_k |\, 0_k > \tag{8.4}$$

Here τ is the usual Dyson time ordering operator[33], and $|\,0_k >$ is the ground state of boson mode k under the unperturbed Hamiltonian $\mathcal{H}_0$.

The set of operators appearing in (8.1) does not close under commutation, and therefore $\mathcal{H}$ is not an element in a (finite dimensional) Lie algebra. The dynamical transformation group G is thus not a (finite dimensional) Lie group.

We therefore try to replace $\mathcal{H}$ by an approximate model Hamiltonian which is an element in some Lie algebra, and for which the associated dynamical transformations (8.3) are elements in a Lie group. Under these circumstances, the full power of the computational methods developed within the context of Lie group theory[13, 30] can be brought to bear on the simplified problem.

We make this replacement using the following observation. In a superfluid system, the $k = 0$ state is macroscopically occupied at the expense of states with $k \neq 0$. We therefore linearize the Hamiltonian $\mathcal{H}$ under the following two assumptions:[34]

1. the operators $b_0^\dagger, b_0$ can be replaced by the number $N_0^{1/2}$, where $N_0 = < b_0^\dagger b_0 >$;
2. terms higher than quadratic in operator $b_k^\dagger, b_k\ (k \neq 0)$ may be neglected.

Under these two simplifying assumptions the Hamiltonian becomes

$$\mathcal{H} = \tfrac{1}{2} V_0 N_0^2 + \Sigma' (\epsilon_k + N_0 V_k + N_0 V_0)\, b_k^\dagger b_k +$$

$$+ \tfrac{1}{2} \Sigma' N_0 V_k (b_k^\dagger b_{-k}^\dagger + b_k b_{-k}) \,. \tag{8.5}$$

In these expresions, Σ' indicates that the summation excludes the case $k = 0$. The occupation number N_0 in the ground state can be replaced by the total number N of bosons present using

$$N = N_0 + \Sigma' b_k^\dagger b_k \quad . \tag{8.6}$$

Within the terms of the approximation above the Hamiltonian can be expressed

$$\begin{aligned} \mathcal{H} &= \tfrac{1}{2} V_0 N^2 + \Sigma' (\mathcal{H}_0)_k + \Sigma' (\mathcal{H}_{\text{pert}})_k \\ (\mathcal{H}_0)_k &= (\epsilon_k + N V_k)\, b_k^\dagger b_k \\ (\mathcal{H}_{\text{pert}})_k &= \tfrac{1}{2} N V_k (b_k^\dagger b_{-k}^\dagger + b_k b_{-k}) \; . \end{aligned} \tag{8.7}$$

Aside from the constant term, the hamiltonian is the direct sum of single mode hamiltonians,

$$\mathcal{H} = \Sigma \oplus \mathcal{H}_k \tag{8.8}$$

where each single mode Hamiltonian has the structure

$$\begin{aligned} \mathcal{H}_k = (\epsilon_k + NV) &(b_{+k}^\dagger b_{+k} + b_{-k}^\dagger b_{-k}) + \\ &+ NV (b_{+k}^\dagger b_{-k}^\dagger + b_{+k} b_{-k}) \quad . \end{aligned} \tag{8.9}$$

As a result, the wave function $|\psi>$ is a direct product of single mode wave functions $|\psi_k>$:

$$|\psi> = \prod_k |\psi_k> \quad , \tag{8.10}$$

where each single mode eigenstate obeys

$$\mathcal{H}_k |\psi_k\rangle = E(k) |\psi_k\rangle \ .$$

The Hamiltonian (8.7) is now an element in a Lie algebra, for the operators appearing in (8.9) close under commutation.

$$J_{+k} = b^\dagger_{+k} b^\dagger_{-k}$$

$$J_{-k} = J^\dagger_{+k} = b_{-k} b_{+k} = b_{+k} b_{-k}$$

$$[J_{+k}, J_{-k}] = [b^\dagger_{+k} b^\dagger_{-k}, b_{+k} b_{-k}]$$

$$= -(b^\dagger_{+k} b_{+k} + b_{-k} b^\dagger_{-k}) = -2J_{3k} \tag{8.11}$$

The operators J_{3k}, $J_{\pm k}$ obey the $\mathfrak{su}(1,1)$ commutation relations.

$$[J_{3k}, J_{\pm k}] = \pm J_{\pm k}$$

$$[J_{+k}, J_{-k}] = -2J_{3k} \ . \tag{8.12}$$

Moreover, operators belonging to different modes k commute

$$[J_k, J'_{k'}] = [J, J'] \, \delta_{k,k'} \tag{8.13}$$

Since the algebraic treatment[34] of each $k\ (\neq 0)$ mode is identical, we suppress the subscript k in the algebraic computations to follow. It is useful to define the following hermitian linear combinations:

$$J_1 = \tfrac{1}{2}(J_+ + J_-) = \tfrac{1}{2}(b^\dagger_+ b^\dagger_- + b_+ b_-)$$

$$J_2 = -\tfrac{1}{2} i (J_+ - J_-) = (1/2i)(b^\dagger_+ b^\dagger_- - b_+ b_-)$$

$$J_3 = \tfrac{1}{2}(b^\dagger_+ b_+ + b^\dagger_- b_- + 1) \ . \tag{8.14}$$

These hermitian operators have the commutation relations

$$[J_3, J_1] = iJ_2$$

$$[J_3, J_2] = -iJ_1 \qquad J_k^\dagger = +J_k \ .$$

$$[J_1, J_2] = -iJ_3 \tag{8.15}$$

The difference operator Δ also has useful properties

$$\Delta = b_+^\dagger b_+ - b_-^\dagger b_- = \Delta^\dagger$$

$$[\boldsymbol{J}, \Delta] = 0 \ . \tag{8.16}$$

Since the operator Δ commutes with the J_k, it is:

1. Mathematically, an invariant which will serve to label the unitary representations of $SU(1,1)$.
2. Physically, a constant of the motion.

The single mode Hamiltonian (8.9) can be expressed as a linear superposition of the elements J_1, J_3 in the $\boldsymbol{su}(1,1)$ Lie algebra:

$$\mathcal{H}_k = 2NV\mu J_3 - \tfrac{1}{2}\mu + J_1 \qquad \mu = (\epsilon + NV)/NV) \ . \tag{8.17}$$

One of these two generators can be eliminated by applying the unitary transformation $U(\theta) = \exp(+i\theta J_2)$ to this Hamiltonian using

$$\exp(i\theta J_2)\begin{bmatrix} J_1 \\ J_3 \end{bmatrix}\exp(-i\theta J_2) = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix}\begin{bmatrix} J_1 \\ J_3 \end{bmatrix} \ . \tag{8.18}$$

The transformed Hamiltonian is

$$U(\theta)\,\mathcal{H}\,U^{-1}(\theta) = 2NV\,\{(\mu\,\cosh\theta - \sinh\theta)\,J_3 +$$

$$+(-\mu\,\sinh\theta + \cosh\theta)\,J_1\} - (\epsilon + NV)\,. \qquad (8.19)$$

By a suitable choice of θ, either J_3 or J_1 can be eliminated from the equation.
Recall that $\epsilon = \hbar^2 k^2/2m > 0$. Therefore

1. Attractive potential, $V < 0$:

$$\tanh\theta = \mu\,;\; U(\theta)\,\mathcal{H}U^{-1}(\theta) = 2NV\,\mathrm{sech}\,\theta\,J_1 - (\epsilon + NV)\;. \qquad (8.20)$$

2. Repulsive potential, $V > 0$:

$$\tanh\theta = 1/\mu\,;\; U(\theta)\,\mathcal{H}U^{-1}(\theta) = 2NV\,\mathrm{csch}\,\theta\,J_3 - (\epsilon + NV)\;. \qquad (8.21)$$

The infinitesimal generators J_1 and J_3 generate subgroups conjugate to $SO(1,1)$ and $U(1)$, respectively. Since J_1 is a non-compact generator, it has a continuous spectrum. On the other hand, J_3 generates a compact subgroup, and therefore has a discrete spectrum. As a result, there is an energy gap between the ground and first excited state in the second case ($V > 0$), which is responsible for macroscopic condensation into the ground state with concomitant superfluidity.

In the superfluid case with hamiltonian proportional to J_3, a lowest lying state must exist which obeys

$$b_+ b_- \left| \begin{matrix} ? \\ \mathrm{gnd} \end{matrix} \right\rangle = 0\;. \qquad (8.22)$$

The hamiltonian eigenstates must belong to a space which carries a semi-bounded[34, 35, 36] unitary irreducible representation of $SU(1,1)$:

$$\Big\uparrow\, j^+\;.$$

These representations are characterized by the eigenvalue $j(j+1)$ of the Casimir operator

$$\zeta_2 = J_3^2 - J_2^2 - J_1^2 \qquad [\mathcal{H}, \zeta_2] = 0 \ . \tag{8.23}$$

Since ζ_2 commutes with the Hamiltonian, it must be related to the difference operator Δ, which also commutes with $\mathcal{H}$, and j is given by

$$j = -\tfrac{1}{2}|\Delta| - \tfrac{1}{2} \ . \tag{8.24}$$

The effect of the diagonal generator J_3 and the shift operators $J_\pm$ on the basis vectors $\left|\begin{smallmatrix} j \\ n \end{smallmatrix}\right\rangle$ is given by

$$\begin{aligned} J_+\left|\begin{matrix} j \\ n \end{matrix}\right\rangle &= \left|\begin{matrix} j \\ n+1 \end{matrix}\right\rangle \sqrt{(n+1)(n-2j)} \\ J_-\left|\begin{matrix} j \\ n \end{matrix}\right\rangle &= \left|\begin{matrix} j \\ n-1 \end{matrix}\right\rangle \sqrt{n(n-2j-1)} \\ J_3\left|\begin{matrix} j \\ n \end{matrix}\right\rangle &= \left|\begin{matrix} j \\ n \end{matrix}\right\rangle (n-j) \qquad n = 0, 1, 2, \ldots \ . \end{aligned} \tag{8.25}$$

The energy eigenstates of the superfluid Hamiltonian are

$$E_n = (2n + 1 + |\Delta|)\, E - (\epsilon + NV)$$

where $E^2 = (\epsilon + NV)^2 - (NV)^2$. (8.26)

The eigenstates of the Hamiltonian (8.9) are

$$\begin{aligned} U(\theta)|\psi_n\rangle &= \left|\begin{matrix} j \\ n \end{matrix}\right\rangle \\ |\psi_n\rangle &= U^{-1}(\theta)\left|\begin{matrix} j \\ n \end{matrix}\right\rangle \ . \end{aligned} \tag{8.27}$$

In particular, the ground state of (8.21) is the coherent state, obtained by setting $n = 0$.

The ground state is constructed most easily by applying the appropriate $SU(1,1)$ BCH formula. These BCH formulas are analytic continuations of the $SU(2)$ BCH formulas[7,9,13,31] and have also been constructed explicitly.[30] Applying the appropriate BCH formula, we find for the ground state

$$\left|\begin{matrix} j \\ \text{gnd} \end{matrix}\right\rangle = U^{-1}(\theta)\left|\begin{matrix} j \\ n=0 \end{matrix}\right\rangle = \exp\left(-\tfrac{1}{2}\theta(J_+ - J_-)\right)\left|\begin{matrix} j \\ 0 \end{matrix}\right\rangle$$

$$= \exp(-\tanh\tfrac{1}{2}\theta\, J_+)\exp(-2\ln\cosh\tfrac{1}{2}\theta\, J_3)\exp(\tanh\tfrac{1}{2}\theta\, J_-)\left|\begin{matrix} j \\ 0 \end{matrix}\right\rangle \tag{8.28}$$

Since $J_-\left|\begin{matrix} j \\ 0 \end{matrix}\right\rangle = 0$,

$$\exp(\tanh\tfrac{1}{2}\theta\, J_-)\left|\begin{matrix} j \\ 0 \end{matrix}\right\rangle = 1\left|\begin{matrix} j \\ 0 \end{matrix}\right\rangle \quad . \tag{8.29}$$

Since $J_3\left|\begin{matrix} j \\ 0 \end{matrix}\right\rangle = (-j)\left|\begin{matrix} j \\ 0 \end{matrix}\right\rangle$,

$$\exp(-2\ln\cosh\tfrac{1}{2}\theta\, J_3\left|\begin{matrix} j \\ 0 \end{matrix}\right\rangle = \exp(2j\ln\cosh\tfrac{1}{2}\theta)\left|\begin{matrix} j \\ 0 \end{matrix}\right\rangle$$

$$= \left|\begin{matrix} j \\ 0 \end{matrix}\right\rangle \{\cosh\tfrac{1}{2}\theta\}^{2j} \quad . \tag{8.30}$$

The single mode ground state is therefore

$$\left|\begin{matrix} j \\ \text{gnd} \end{matrix}\right\rangle = \{\cosh\tfrac{1}{2}\theta\}^{2j}\exp(-\tanh\tfrac{1}{2}\theta\, b_+^\dagger b_-^\dagger)\left|\begin{matrix} j \\ 0 \end{matrix}\right\rangle \quad ,$$

$$\tanh\theta = NV/(\epsilon + NV) \quad . \tag{8.31}$$

The total system ground state of (8.7) is a direct product of single mode ground states (8.10), each of the form (8.31):

$$|\,\mathrm{gnd}> = \prod_{k\neq 0} \otimes \{\cosh \tfrac{1}{2}\theta(k)\}^{2j(k)} \exp(-\tanh \tfrac{1}{2}\theta(k)\, b^{\dagger}_{+k}\, b^{\dagger}_{-k}) \left|\begin{matrix} j(k) \\ 0 \end{matrix}\right\rangle \tag{8.32}$$

Once the single mode coherent states (8.31) have been obtained explicitly, it is possible to show their non-orthogonality and over-completeness explicitly. Instead of doing this directly, we first compute a useful generating function. To compute the moments of the operator J_+, it is sufficient to compute the derivatives of a simple generating function:[7, 13, 30]

$$<\theta'|(J_+)^k\,|\,\theta> = (d/d\alpha)^k <\theta'|\exp(\alpha J_+)|\,\theta>\,\Big|_{\alpha=0} \;. \tag{8.33}$$

where

$$|\,\theta> = \{\cosh \tfrac{1}{2}\theta\}^{2j} \exp(-\tanh \tfrac{1}{2}\theta J_+)\left|\begin{matrix} j \\ 0 \end{matrix}\right\rangle$$

$$<\theta'| = \{\cosh \tfrac{1}{2}\theta'\}^{2j} \left|\begin{matrix} j \\ 0 \end{matrix}\right\rangle \exp(-\tanh \tfrac{1}{2}\theta' J_-)\;. \tag{8.34}$$

Arbitrary moments (k non-integral) are computed in the usual way.

It is more convenient to determine a more general generating function than the one introduced in (8.33). This function is

$$f(\alpha,\beta,\gamma) = <\theta'|\exp(\alpha J_+ + \beta J_- + \gamma J_3)|\,\theta>\;. \tag{8.35}$$

This generating function is, moreover, easy to compute, for

$$f(\alpha,\beta,\gamma) = \{\cosh \tfrac{1}{2}\theta'\}^{2j}\{\cosh \tfrac{1}{2}\theta\}^{2j} \times$$

$$\times \left\langle\begin{matrix} j \\ 0 \end{matrix}\right| \exp(-\tanh \tfrac{1}{2}\theta' J_-) \exp(\alpha J_+ + \beta J_- + \gamma J_3) \exp(-\tanh \tfrac{1}{2}\theta J_+) \left|\begin{matrix} j \\ 0 \end{matrix}\right\rangle \tag{8.36}$$

Applying now the appropriate $SU(1,1)$ BCH relation gives

$$f(\alpha,\beta,\gamma) = \{\cosh \tfrac{1}{2}\theta' \cosh \tfrac{1}{2}\theta\}^{2j}$$

$$\left(\left\langle \begin{matrix} j \\ 0 \end{matrix} \right| \exp(x' J_+) \Big) \exp(-2\ln z J_3) \Big(\exp(x J_-) \left| \begin{matrix} j \\ 0 \end{matrix} \right\rangle \right)$$

$$= \{\cosh \tfrac{1}{2}\theta' \cosh \tfrac{1}{2}\theta\}^{2j} \left\langle \begin{matrix} j \\ 0 \end{matrix} \right| \exp(-2\ln z J_3) \left| \begin{matrix} j \\ 0 \end{matrix} \right\rangle$$

$$= \{\cosh \tfrac{1}{2}\theta' \cosh \tfrac{1}{2}\theta\}^{2j} (z)^{2j} \quad . \tag{8.37}$$

The function z is given by

$$z = \mathrm{Tr} \begin{bmatrix} -\tanh \tfrac{1}{2}\theta' \tanh \tfrac{1}{2}\theta & -\tanh \tfrac{1}{2}\theta' \\ \tanh \tfrac{1}{2}\theta & 1 \end{bmatrix} \begin{bmatrix} \cosh\omega + \tfrac{1}{2}\gamma\, \mathrm{sh}\omega/\omega & \alpha\,\mathrm{sh}\omega/\omega \\ -\beta\,\mathrm{sh}\omega/\omega & \cosh\omega - \tfrac{1}{2}\gamma\,\mathrm{sh}\omega/\omega \end{bmatrix}$$

$$\omega^2 = (\tfrac{1}{2}\gamma)^2 - \alpha\beta \quad . \tag{8.38}$$

From this generating function we easily compute

$$<\theta'|(J_-)^k \,|\, \theta> = (d/d\beta)^k f(0,\beta,0)\Big|_{\beta=0}$$

$$= \{\cosh \tfrac{1}{2}\theta' \cosh \tfrac{1}{2}\theta\}^{2j} (d/d\beta)^k \{1-(-\beta)\tanh \tfrac{1}{2}\theta - \tanh \tfrac{1}{2}\theta' \tanh \tfrac{1}{2}\theta\}^{2j}$$

$$= (\cosh \tfrac{1}{2}\theta' \sinh \tfrac{1}{2}\theta)^k \left(\Gamma(2j+1)/\Gamma(2j+1-k)\right) \{\mathrm{IN}\}^{2j-k} \tag{8.39}$$

$$\mathrm{IN} = \cosh \tfrac{1}{2}\theta' \cosh \tfrac{1}{2}\theta - \sinh \tfrac{1}{2}\theta' \sinh \tfrac{1}{2}\theta \quad . \tag{8.40}$$

In particular, the inner product $<\theta'|\theta>$ is determined by setting $k = 0$.

Now we show that the identity operator can be resolved in the coherent state representation. We construct the operator

$$\int_{\Omega \epsilon G/H} |\Omega > 1 < \Omega| \, d\mu(G/H) \, . \tag{8.41}$$

This operator commutes with the action of all elements $g' \epsilon G$:

$$\Gamma^{\lambda}(g') \int_{\Omega \epsilon G/H} |\Omega><\Omega| \, d\mu(\Omega)$$

$$= \Gamma^{\lambda}(g') \int_{\Omega} \Gamma^{\lambda}(\Omega) |0><0| \Gamma^{\lambda}(\Omega^{-1}) \, d\mu(\Omega)$$

$$= \Gamma^{\lambda}(g') \left(1/\mathrm{Vol}(H)\right) \int \Gamma^{\lambda}(\Omega h) |0><0| \Gamma^{\lambda}(h^{-1}\Omega^{-1}) \, d\mu(\Omega h)$$

$$= \left(1/\mathrm{Vol}(H)\right) \int \Gamma^{\lambda}(g'g) |0><0| \Gamma^{\lambda}(g^{-1}) \, d\mu(g)$$

$$= \left(1/\mathrm{Vol}(H)\right) \int \Gamma^{\lambda}(g'g) |0><0| \Gamma^{\lambda}(g^{-1}g'^{-1}) \, d\mu(g'g) \Gamma^{\lambda}(g')$$

$$= \{ \int_{\Omega \epsilon G/H} |\Omega><\Omega| \, d\mu(\Omega) \} \Gamma^{\lambda}(g') \, . \tag{8.42}$$

As a result, the operator (8.41) is a multiple of the identity

$$\int_{\Omega} |\Omega > 1 < \Omega| \, d\mu(\Omega) = \gamma \mathrm{Id} \tag{8.41}$$

and γ can be computed by taking the 00 matrix element

$$<0| \int_{\Omega} \Gamma^{\lambda}(\Omega) |0><0| \Gamma^{\lambda}(\Omega^{-1}) |0> d\mu(\Omega) = \gamma <0|\mathrm{Id}|0>$$

$$\int_{G/H} |\Gamma^{\lambda}_{00}(\Omega)|^2 \, d\mu(G/H) = \gamma \, . \tag{8.43}$$

This exists whenever Γ^{λ} is square integrable. When G is compact, the resolution of the identity has the form given explicitly in Table 3.

The resolution of the identity for $SU(1, 1)$ has been given explicitly by Perelomov:[29]

$$-(2j+1)\pi^{-1}\int|\Omega>1<\Omega|\,d\mu(\Omega) = \mathrm{Id} \quad . \tag{8.44}$$

The integral is over the hyperboloid $SU(1,1)/U(1)$.

The results presented above are valid for any physical system whose dynamical transformation group is $SU(1, 1)$, or a direct product $\Pi SU(1, 1)$.

IX. SUMMARY AND CONCLUSIONS

The properties of field coherent states, originally introduced as a useful system of vectors in terms of which to represent physically occurring states of the electromagnetic field, have been studied from a group theoretical point of view. We have been able to find a group theoretical interpretation for each of the properties (§III, #1-12) which make the coherent states such an attractive mathematical representation for certain physical systems.

These properties have been applied to the description of an ensemble of N identical 2-level atoms interacting with an external electromagnetic field (§IV). The treatments given in §III and §IV are extremely similar in nature. This similarity exists for 3 reasons:

1. The procedure described in §III is related to the procedure described in §IV by a group contraction process. This is shown explicitly in §V.
2. The problems described in §III and §IV are essentially duals to each other. This duality has suggested several non-trivial applications of the newer atomic coherent states. Two such applications are outlined in §VI.
3. The calculations carried out in §III and §IV are special cases of a much more general procedure for constructing coherent states. Such states are defined in §VII as states on the orbit $\Gamma^{\lambda}(G/H)|\,\mathrm{ext}>$, where $|\,\mathrm{ext}>$ is an extremal state (i. e., ground state) in a Hilbert space M^{λ} which carries a unitary irreducible representation Γ^{λ} of a dynamical transformation group G, and where H is the stability group of $|\,\mathrm{ext}>$. The properties of generalized coherent states are outlined in §VII and presented in Table 2 and Table 3.

Finally, in §VIII the coherent state concept is used to treat the Foldy model for a superfluid system in a simple and elegant way.

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RESUMEN

Se obtienen estados coherentes aplicando una transformación unitaria dinámica a un estado extremal en un subespacio invariante de un hamiltoniano mecánico cuántico. Las propiedades de los estados coherentes quedan completamente caracterizadas en términos matemáticos. Además, probamos el siguiente teorema que es muy útil. Un sistema físico inicialmente en un estado coherente, o en particular en su estado base, evolucionará a un estado coherente. Damos varios ejemplos de la utilidad de los estados coherentes.