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DISTRIBUTIONS, FLUCTUATIONS AND SYMMETRIES IN MANY-PARTICLE SPECTROSCOPY

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ABSTRACT: The state density for a spectroscopic system of many interacting particles is determined to a large extent by the operation of a central limit theorem which fixes the averaged form, while, throughout the entire spectrum, the fluctuations about the average closely follow the laws, derived for high excitation energy, by considering random matrix ensembles. The fluctuations are small, the Dyson-Mehta description of the random ensemble spectra as "essentially crystalline" being found to apply much more generally. The same general results obtain, both for averages and fluctuations, under a restriction to states of given exact symmetry, and, for the averages, even for broken symmetries. Other properties of the system, occupancies for example, are determined similarly.

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1. INTRODUCTION

We shall consider, in a non-technical way, the very close relationships which exist between symmetries and statistical behaviour in many-particle systems. These show up particularly when, as is very often appropriate we describe the system in spectroscopic terms, the states being represented in terms of particles distributed over some finite set of single-particle states, say N in number. One knows of course that the existence of good symmetries $(J, T$ or perhaps $SU(4))$ supplies an overriding principle which may greatly simplify the problem of finding the properties of the system. But it is not so well appreciated that there is another general principle, the central limit theorem, which similarly makes for great simplicities; moreover the two simplifying principles are closely related. With the action of these principles it becomes feasible and profitable, in very strong contrast to what one usually does, to discuss in a unified way statistical behaviour (both averages and fluctuations) right down into the ground-state domain and to make use of symmetries even at high excitations and when they are badly broken. Many general questions which have hardly been explored at all arise when we do this, for example about the information content of complex spectra, and a large number of technical questions as well, for example about relationships between irreducible representations when there is a relevant chain or more general lattice of subgroups.

Let us start with an old calculation of Bethe[1] which perhaps for the first time made a connection between statistical averaging and symmetry. Assuming that residual interactions could be ignored, Bethe had calculated the nuclear level density and then asked for its decomposition according to angular momentum. He gave an answer essentially as follows; regarding the z-component of the i'th-particle angular momentum as a random variable whose distribution centers about zero with width $\sigma(1) = \frac{1}{3}\sqrt{j(j+1)}$, we have by the elementary central limit theorem that $J_z = \sum_i J_z(i)$ becomes Gaussian $(0, m\sigma)$ in the $(m \gg 1)$-particle case; then J_z^2 is $\chi^2(1)$ and $J^2 = (J_x^2 + J_y^2 + J_z^2)$ is $\chi^2(3)$, which gives for J (or better for $(J + \frac{1}{2})$) essentially the Maxwellian distribution $\sim x^2 \exp\{-x^2/2\sigma^2(m)\}$ with $\sigma^2(m) = m\sigma^2(1)$.

We immediately ask why the very powerful principle invoked here was not taken up and applied to other symmetries. But on thinking about it one sees (as Bethe himself did) many problems connected with the calculation. Statistical independence of the angular momenta for various particles, which is taken for granted in applying the CLT, is not compatible with the Pauli principle; while ignoring interactions is perhaps satisfactory for the density

itself, we cannot be confident that a strongly J-dependent interaction will not change the J-decomposition; and how really are we to know the effective number of contributing nucleons?

More significant for our present purposes is the fact that the connection with symmetry seems not to be an *essential* one in that the argument used for J^2 could have been used for the square of any one-body operator, whether or not it is connected with a relevant group. Let us turn this disappointment to our advantage and ask whether we could for example have dealt with the Hamiltonian itself to produce thereby the form of the spectrum. We might expect that ordinary H's involve not three but a much larger number, r, of degrees of freedom and, since the CLT is a very "forgiving" theorem, we might be willing to ignore cross terms and so forth. But then since $\chi^2(r) \xrightarrow{r\to\infty}$ Gaussian (once again by the CLT) we might expect the resultant distribution to be Gaussian. Then combining this with Bethe's J-result we would have that the *partial density* $\rho(E, J)$ is Gaussian in E and Maxwellian in J; we could imagine further decompositions as well.

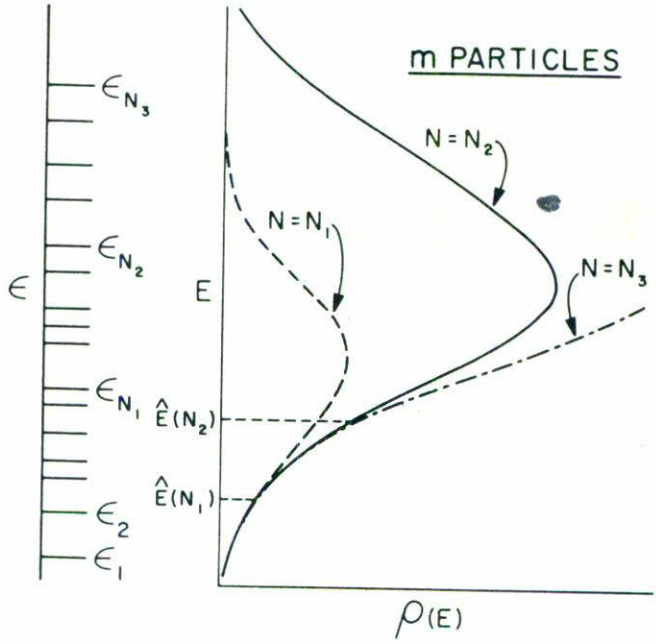

Fig. 1. Relationship between the single-particle and the m-particle spectra, shown schematically (to different energy scales). $\hat{E}(N_i)$, a function of the number, N_i, of single-particle states considered, is a limiting energy above which the spectroscopic eigenvalues and eigenfunctions do not represent those of the true H.

This result is quite wrong as it stands, for it is obvious that the level density increases indefinitely with the excitation energy; in fact Bethe's calculation gives an exp $\{a\sqrt{E}\}$ form. We can properly use the CLT only with functions which are bounded in some natural way; hence we turn to a spectroscopic description as indicated in Fig. 1, which shows the relationship between the 1-particle and m-particle spaces. To $\hat{E}$, the largest multiparticle energy of interest to us, there corresponds some minimum number $N(\hat{E})$ of single-particle states which we must take into account*; and conversely to given N there exists a maximum $\hat{E}(N)$. In this way then we have a finite spectroscopic system which is equivalent to the original many-body problem for energies $E \leqslant \hat{E}(N)$, it being agreed that appropriate methods are used to construct the corresponding Hamiltonian and other operators in this spectroscopic space. These things being understood we can reasonably speculate that the restricted density is close to Gaussian in energy as given above.

So much for the spirit of our undertaking; we are in fact considerably more ambitious than these simple considerations would suggest. Let us proceed therefore to a somewhat more formal treatment. We are dealing with an $\begin{bmatrix} N \\ m \end{bmatrix}$-dimensional m-particle vector space, m say, which has an (anti-symmetrized) direct-product structure

$$m = 1 \times 1 \times 1 \ldots\ldots \times 1$$

If H is the effective Hamiltonian which operates in this space then the eigenvalue density is defined by its centroid $\mathcal{E}$ and its higher central moments $(p \geqslant 2)$ by

$$\mu_p(m) = \int \rho(E)(E - \mathcal{E})^p dE \equiv <(H - \mathcal{E})^p>^m \qquad (1)$$

where $<0>^m$ is the expectation value of 0 averaged over any orthonormal basis of m; $\rho(E)$ is the relative density its integral being unity; $d \times \rho(E)$, where in the present case $d = \begin{bmatrix} N \\ m \end{bmatrix}$, is the true density. We shall interpret the moments as those of a smoothed distribution but, as shown in Fig. 2, we

* That $N(E)$ should increase only slowly with E is the essential requirement for the validity of the high energy shell model. Note that we do not admit continuum states though this might be feasible; we could rationalize this by thinking about the Wigner-Eisenbud states for the nuclear interior.

can recover an approximate discrete spectrum by using $F(E_j) = (j - \frac{1}{2})/d$ for $j = 1, 2, 3 \ldots$, where $F(E) = \int_{-\infty}^{E} \rho(x)\, dx$ is the distribution function.[2] We would use for $F(E)$ a low-moment approximation deriving perhaps from the moments of order $p = 1, 2$ (in which case we would assume a Gaussian ρ) but perhaps using moments to $p = 4$ (we would then use a "corrected" Gaussian[3], a Gram-Charlier or Edgeworth form); or in very large spaces we would represent ρ as a superposition of partial densities (the bases for which must in fact be defined in terms of symmetries, as we shall see).

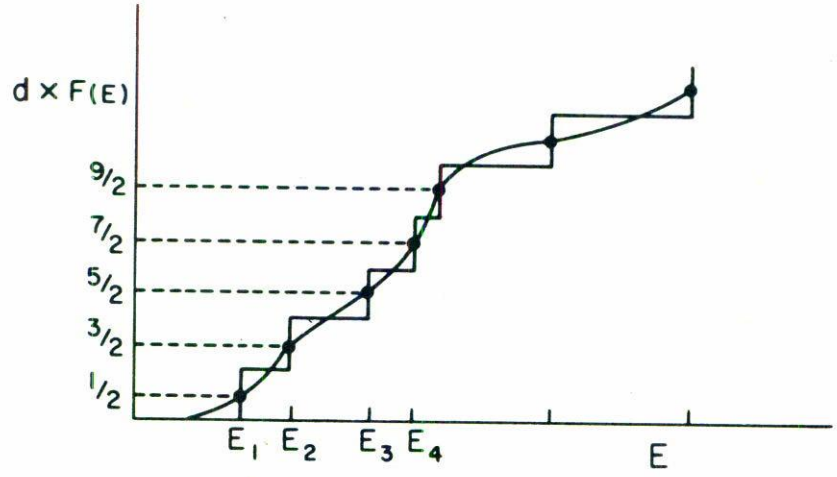

Fig. 2. The approximate spectrum as determined from a smoothed distribution (Ratcliff). The eigenvalues are chosen so that, at its discontinuities, the resultant staircase function evenly brackets the smoothed $F(E)$.

Equation (1) for the density moments is valid even when we take account of the Pauli principle and even when there are interactions, so that in a sense these difficulties with the Bethe calculations are overcome by going to moments. But we can anticipate that moment evaluation will rapidly become harder as p, the order of the moment, increases; we see then that, as a practical matter, we will have solved the problems only if the essential feature of the non-interacting-particle density is preserved, namely that there should be a characteristic form (describable now in terms of a small number of moments). This will turn out to be true (giving then validity to the low-moment approximation for $F(E)$); but beyond this a further major extension becomes feasible; if we decompose the m-particle space according to symmetry, the moments, being expressed in terms of traces, decompose similarly giving rise to a natural symmetry decomposition of the density.

We encounter now a large number of questions:

(1) Is there any mathematical basis for the assumption of Gaussian or close-to-Gaussian densities?

(2) What is the role of symmetries?

(3) The centroid energy for the spectroscopic problem will usually lie far above the energy $\hat{E}(N)$; does any essential problem arise from the fact that a description by moments "ties" everything to this energy which is outside, and often very far outside the energy range in which the true problem and the spectroscopic problem coincide; what in fact is the significance of the resulting densities? Note that some of these questions arise also in conventional shell-model calculations.

(4) Can the information which is lost when we represent the complex density in terms of a few moments be recovered in some way?

(5) How can we calculate the necessary moments?

(6) Can we deal with other quantities besides the energies?

(7) What is the nature and role of fluctuations? Do the fluctuations carry the missing information?

2. GAUSSIAN BEHAVIOUR

To see where Gaussian behaviour comes from, one should recognize at the beginning that for the "dilute" system ($m \ll N$) the Pauli principle is not liable to be important; so let us ignore it, which we can do conveniently by taking the large-N limit. For non-interacting particles we may write $H = \Sigma\, \epsilon_i n_i$, and then immediately $\rho_m(E) = \int \rho_{m-1}(E')\, \rho_1(E - E')\, dE'$ so that ρ_m, being an m-fold convolution, becomes Gaussian very rapidly as m increases. When we have interactions however the energies are no longer additive, the argument fails, and the density may or may not be Gaussian. J^2 has a non-Gaussian spectrum; a more interesting example is discussed by Quesne[4]. A sensible procedure then is to define a characteristic form of the density by considering not a single H but an ensemble of H's; we would then be inclined to regard the ensemble-averaged density as characteristic, but of course this would be proper only if we could show that all but a negligible fraction of the H's in the ensemble do give that characteristic form

Hamiltonian *matrix* ensembles were introduced by Wigner[5], the matrix elements being taken as statistically independent and similarly distributed by a Gaussian law centered about zero, these matrices forming the Gaussian

Orthogonal Ensemble (GOE). It is found in this case that the ensemble-averaged density in the large-N limit is semicircular; this is moreover the characteristic form because, when the limit is taken in an appropriate way, the probability vanishes for finding a non-circular single-matrix spectrum[6]. Since however the circular form does not reproduce itself under convolutions one can strongly suspect that, if we realize the matrices in an m-particle spectroscopic space, then the Hamiltonian must change with particle number. This would imply that H contains an m-body part; in fact, if we consider the number of independent matrix elements for m-body operators (which is $\binom{N}{m}^2$), we see that for large N this part is dominant and that the GOE then describes the situation where all the particles interact simultaneously, a case which is very rarely of interest.

Things are different if, instead of the matrix GOE, we consider the GOE ensemble of k-body operators, say with $k = 2$, acting in the m-particle space. In this case, which we shall describe as the "embedded" GOE, or EGOE for short, one finds[7] that the characteristic form goes steadily from semicircular for $m = k$ to Gaussian for large m. It is easy to see why; consider for example the fourth moment (the odd-order ones vanish on ensemble averaging):

$$\mu_4(m) = \langle H^4 \rangle^m = \langle \underline{H\,H}\;\underline{H\,H} \rangle^m + \langle \overline{H\,H\,H\,H} \rangle^m + \langle H\,\overline{H\,H\,H} \rangle^m \tag{2}$$

where the linkages indicate a correlation (the $\alpha\beta$ term with its adjoint $\beta\alpha$) between terms in the expansion of $H = \sum_{\alpha,\beta} W_{\alpha\beta}\psi_\alpha(k)\psi^+_\beta(k)$, where the ψ's are k-body state operators. An even-order correlation is necessary in order that the term should not vanish on ensemble averaging, while, in a large space, correlations of order higher than two have a weight which is negligible and goes to zero as $N \to \infty$. The first two terms of (2) are equal by cyclic invariance of traces and each gives $\{\langle H\,H \rangle^m\}^2 = \{\mu_2(m)\}^2$. The third term is quite different; for example it vanishes as $N \to \infty$ for $m = k$ because the fact that there are only k-particles available altogether demands a further correlation between the linked pairs which is not to be found in the limit. However when the number of particles is also unlimited (strictly when $N \to \infty$, $m \to \infty$, $m/N \to 0$) no further correlation is needed, the different linked pairs operate independently, and this term once again gives $\{\mu_2(m)\}^2$. Thus $\mu_4/\mu_2^2 \to 2$ for $m = k$, 3 for $m \gg k$, these defining respectively the fourth moments of a semicircle and of a Gaussian. Indeed we see immediately that, in the large-m limit defined above, $\mu_{2\nu}(m)/\mu_2^\nu(m)$ is simply the number,

$(2\nu-1)!!$, of pairwise correlations of 2ν objects, this of course being a standard way of coming upon a Gaussian distribution. For $m = k$ on the other hand we have the Wigner calculation[5,8] which gives semicircular.

Thus the ensemble-averaged density makes a semicircle → Gaussian transition as we increase the number of particles starting with $m = k$. The rate at which this transition occurs has recently been given by Mon[7] as a function of the particle rank k of the interaction. If we want the negative of the "excess" $(\mu_4/\mu_2^2 - 3)$ to be not larger than 0.3, which would correspond to a pretty good Gaussian, we need about 12 particles for $k = 2$, about 25 for $k = 3$, and about 50 for four-body interactions. By considering the ensemble variances of the moments Mon has shown moreover[7] that for each (m, k) the resulting distributions are, in the sense described above, characteristic ones.

The conclusion then is that there is a characteristic form for the m-particle density and that it approaches Gaussian when there are many particles. If however the space is very large we cannot expect that a Gaussian, even with low-moment corrections, will adequately represent the density far from the centroid, in particular near the ground state. Roughly speaking a corrected Gaussian will be adequate up to a distance 3.5σ from the centroid which will encompass the entire spectrum if the dimensionality is less than 10,000 or so.

3. DENSITIES FOR FIXED SYMMETRIES

All the m-particle states together form an irreducible representation of the group $U(N)$ of unitary transformations in the single-particle space; thus, in an almost trivial sense, we have generated above a relationship between symmetry and statistical averaging; our real interest however extends well beyond this, to symmetries defined by $U(N)$ subgroups. Many different situations arise depending on the groups involved; there are many different chains and lattices of subgroups; they may define good symmetries or broken ones; the groups may involve free parameters to be chosen in some optimal way; all states for a given symmetry may be localized or spread out in energy; all the states of a given symmetry may form together an irreducible representation of some group (as with isospin) or they may not (as with angular momentum). Moreover, in contrast to the $U(N)$ case above, there seems often no natural way to proceed to an asymptotic limit which would define characteristic forms although such forms are encountered. Very little study has been made of these things though it is clear from available calculations on the one hand, and on the other hand from general theorems which are known

about probability on algebraic structures, and in particular about the corresponding limit theorems (Grenander[6]), that the whole subject is a rich one waiting to be studied. We must content ourselves however with a few practical aspects of it.

Symmetries enter when instead of studying the density for *all* m-particle states in our "universe" defined by N we consider the distribution of the summed intensity of a subset of the states as they show up in the Hamiltonian eigenstates. There is a triple of reasons why the subset chosen should be connected with an irreducible representation of a group: only in that case can the distribution be expected to be close to a characteristic form and therefore calculable in terms of a few moments; only then also will methods emerge for calculating these moments (in terms of invariants and generators of the groups); and besides that of course the study of symmetries, which becomes feasible when the subsets are chosen in that way, is a matter of the greatest interest.

If the symmetry which labels the states is a good one we can regard the density as an intensity distribution as above, or alternatively as an eigenvalue distribution; but when the symmetry is not a good one only the first of these choices is open to us, for in that case a subspace cannot in general be spanned by a set of eigenstates. If this point is not appreciated an apparent paradox arises when the energy level fluctuations are considered; to the decomposition of the m-particle space, which we write as $m = \Sigma(m, \alpha)$, there is a corresponding decomposition of the density. If we think of the partial densities as that of eigenvalues we know that, in a region where several spectra overlap, the energy level fluctuations are drastically modified, the nearest-neighbour spacing distribution going towards Poisson instead of the very different Wigner form which displays the level repulsion. If in fact the defining symmetry is good this is precisely what is known to happen; but if it is very badly broken it has no effect on the Wigner form and we have a conflict. The resolution is that the correspondence of the partial density with eigenvalues is lost in the case of a broken symmetry. In the same way it must be understood that the density being a superposition of partial densities does not in any sense imply that one set of states has no effect on the other; when the symmetry is broken the distribution of one irrep is affected by all of the others which are connected to it by H, this showing up in the "partial widths" of the representation which go to intermediate states of different symmetries, and in higher moments as well. Thus each irrep accomodates itself to the others. We stress these matters, perhaps unnecessarily, because confusion has in the past arisen on both counts.

We can see easily how the characteristic Gaussian form can arise

when we have a specified symmetry. Going back to the beginning and accepting Bethe's argument and his results, we see that σ_J^2, the square of the spin cutoff factor, is $\frac{1}{3} J^2(E)$, this denoting an expectation value locally averaged in the neighbourhood of E; we show later how it may be calculated. If this quantity is essentially independent of E so also is the J-decomposition and then the energy variation is the same for each J. Thus in a large spectroscopic system we would, if σ_J^2 constant, encounter a Gaussian form for the distribution of states of fixed J. Moreover we would have then that $\sigma_J^2 = \frac{1}{3} < J^2 >^m = \frac{1}{3} m(N-m)(N-1)^{-1} \overline{j(j+1)}$, where we have evaluated the average by using the result that, for an operator O, whose maximum particle rank is μ, $<O>^m$ is a u'th order polynomial in m; for J^2, $u = 2$ and since $< J^2 >^0 = < J^2 >^N = 0$ the result follows. Note that the $(N-m)(N-1)^{-1}$ "blocking" or Pauli correction was missed in the Bethe calculation. If, instead of being constant, σ_J^2 varies strongly with energy then, unless the states of given J are localized in energy (which does *not* happen with J though it can for other symmetries), we must expect that the fixed-J densities will have distorted forms, for the construction of which we would need to consider moments say up to $p = 4$.

The simplest subgroup symmetry arises when we ask how the intensity of a given single-particle state is distributed throughout the many-particle spectrum. This question is obviously relevant to a large class of experiments and to questions about the validity of the shell model. An extended question which quite naturally comes up is about the intensity of a "configuration" the set of states which arise when we partition the N single-particle states into subsets $N_1, N_2 \ldots, N_l$ (which we describe as "orbits") and make a corresponding decomposition $m = \sum_r m_r$ of the particle number. A configuration then defines an irrep of a $U(N)$ subgroup $G = \sum U(N_r)$, the set of simultaneous unitary transformations separately carried out in each orbit; this symmetry is probably the most important one in many-particle systems.

The evaluation of the centroids and widths is quite straightforward[9], there entering naturally Hartree-Fock-like energies and other familiar quantities; recently[10] the third and fourth moments have been calculated by the methods of many-body perturbation theory on the one hand and of $R(2N+1)$ contractions on the other (all the m-particle states together, $m = 0, 1, \ldots, N$, form an irrep of an $R(2N+1)$ group), the evaluation being by no means trivial since we are dealing here with the third or fourth power of a complicated operator in a complicated space. Most of the applications which we mention briefly ahead have made use of these configuration distributions.

The presence in the nuclear Hamiltonian of pairing interactions in particular implies that (at least in the usual spherical-orbit representation)

the configuration symmetries will not be good. What happens then with the distributions can be pretty well answered in terms of the moments to order $p = 4$. One finds in fact that for the configurations located in the central region of the spectrum the densities are quite close to Gaussian, but if the space is large those near the extremes (the low-lying ones being the interesting ones of these) are skewed and otherwise distorted. This arises of course from excitations to high-lying orbits. For excitations from a low-lying configuration to a very high orbit one can of course use simple perturbation theory to calculate the corresponding shift in the ground-state energy. But we can also take account of the change in the width of the low-lying configuration according as we do or do not take into account these particular excitations, and thus we can make a separate calculation of the effect. Comparing the two tells us an important thing about the relationship between the statistical and perturbation methods.

As a simple case suppose that we have m particles in two orbits which are separated by energy Δ and that the interaction which connects them is a pairing interaction $H_P + H_P^+$ where H_P promotes two particles from orbit 1 to orbit 2. Then if Δ is large (compared with the width of the configurations) the ground state, α, belongs to the $(m, 0)$ configuration, and the ground-state shift is

$$\Delta E_g = (-1/2\Delta) \sum_\beta | < m-2, 2:\beta | H_P | m, 0: \alpha > |^2$$

$$= (-1/2\Delta) < m, 0: \alpha | H_P^+ H_P | m, 0: \alpha >$$

$$= (-1/2\Delta) < H_P^+ H_P >^{m,0} = (-1/2\Delta) \sigma_e^2 (m, 0) \ , \qquad (3)$$

where $\sigma_e^2 (m, 0)$ is the amount which the pairing excitations contributes to the ground-state configuration variance (σ_e is an "external partial width").

For the statistical calculation the ground-state is defined by $F(E_g) = \frac{1}{2}$, and thus when F changes because of a change in σ we have, the derivatives being taken at the ground state,

$$\delta F = 0 = \left(\delta F(E)/\delta\sigma \right) \delta\sigma + \left(\delta F(E)/\delta E \right) \Delta E_g \ . \qquad (4)$$

If however we were to assume that the form of the density is unchanged by

the $1 \rightarrow 2$ excitations we would have easily that

$$\delta F(E)/\delta\sigma = \left((\mathcal{E}-E)/\sigma\right)\rho(E) = \left((\mathcal{E}-E)/\sigma\right)\left(\delta F(E)/\delta E\right) \qquad (5)$$

$$\Delta E_g = -\,\delta\sigma\,(\mathcal{E}-E)/\sigma) = -\left(\delta\sigma^2/2\sigma^2\right)(\mathcal{E}-E_g)$$

$$= -\left((\mathcal{E}-E_g)/2\sigma^2\right)\sigma_e^2(m,0)\ , \qquad (6)$$

where $\mathcal{E}$ is the centroid of the $(m,0)$ configuration, and in common cases $(\mathcal{E}-E_g)$ is a few times the width σ.

The two results (3.6) are inconsistent, and in particular the second form for ΔE is independent of the orbit spacing Δ. The first result is of course correct when Δ is large enough, and in that case the second is in error, the assumption that the form of the density is unchanged by the inter-orbit excitations being invalid. It obviously is, for the excitations force upwards by 2Δ a small part of the $(m,0)$ intensity thus tending to generate a bimodal distribution; if there is a distribution of high orbits we have instead a long high-energy tail. As a practical matter we can keep track of what is happening in complex cases by evaluating the skewness μ_3/σ^3 which will grow rapidly when distant orbits are taken into account. But, more interesting to us at present, we see that the statistical method, which our experience tells us does have a region of validity, actually works in a domain of strong interactions with resulting "chaos", quite the opposite of the perturbation procedure.

4. SPECTRA AND SPECTRAL FLUCTUATIONS

Fig. 3 gives a coarse-grained picture of a (ds)[12] spectrum found by constructing and diagonalizing the H matrix[11] (839-dimensional) for a "realistic" (KLS) interaction.[12] The dimensionality is small enough that a Gaussian or corrected Gaussian density should be adequate over the whole spectrum span (close to 100 MeV) and we see that this is indeed so, the density being well represented by only four moments even though the number of matrix elements is $\sim$ 300,000. Observe that in this case there are exact-symmetry specifications but they give little or no modification of the density. There are many examples like this which show essentially the same result.

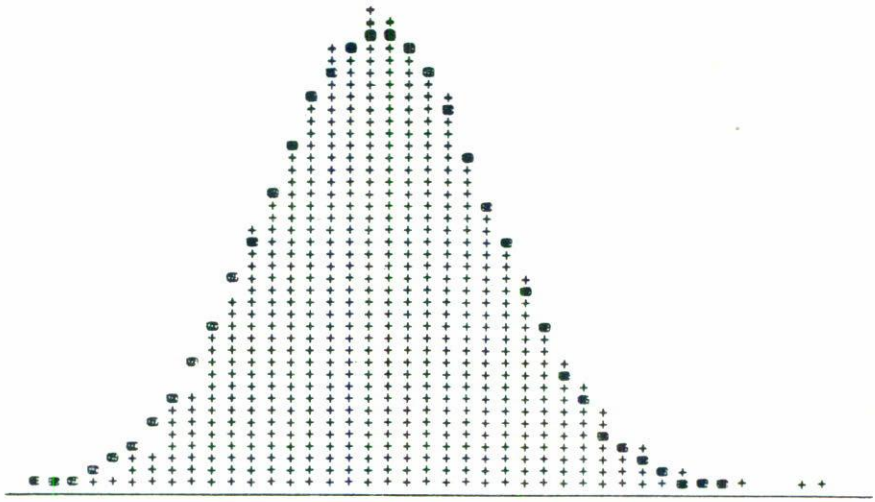

Fig. 3. A computer histogram of a $(ds)^{12}$ spectrum. The bins are 1./ MeV wide. The Gaussian values are indicated by oval "points".

Let us turn to a closer examination of the spectrum, considering particularly the "energy-level fluctuations". Recall first that in line with the general arguments of §1 indicated in Fig. 1, only the first 10 MeV of the spectrum may be regarded as representing reasonable eigenvalues for the Si^{28} nucleus. But this needn't bother us, for it will turn out that the level fluctuations at a given energy, when measured with the local level spacing as a unit, are essentially unchanged when we extend the underlying space; as a consequence it will make sense for us to consider the fluctuations throughout the entire $(ds)^{12}$ spectrum.

In Fig. 4 we take a finer view of things, comparing the ground-state segment of the spectrum, as well as a segment from the central region, with the corresponding fluctuation-free spectrum derived, as in Fig. 2, from the four-moment density. When measured in the appropriate unit the comparisons in the two segments, and indeed over the entire spectrum, are of comparable quality. The RMS deviation between the two spectra, averaged over the entire spectrum, is 1.8 spacing units,[13] and averages over restricted domains of the spectrum give essentially the same result.

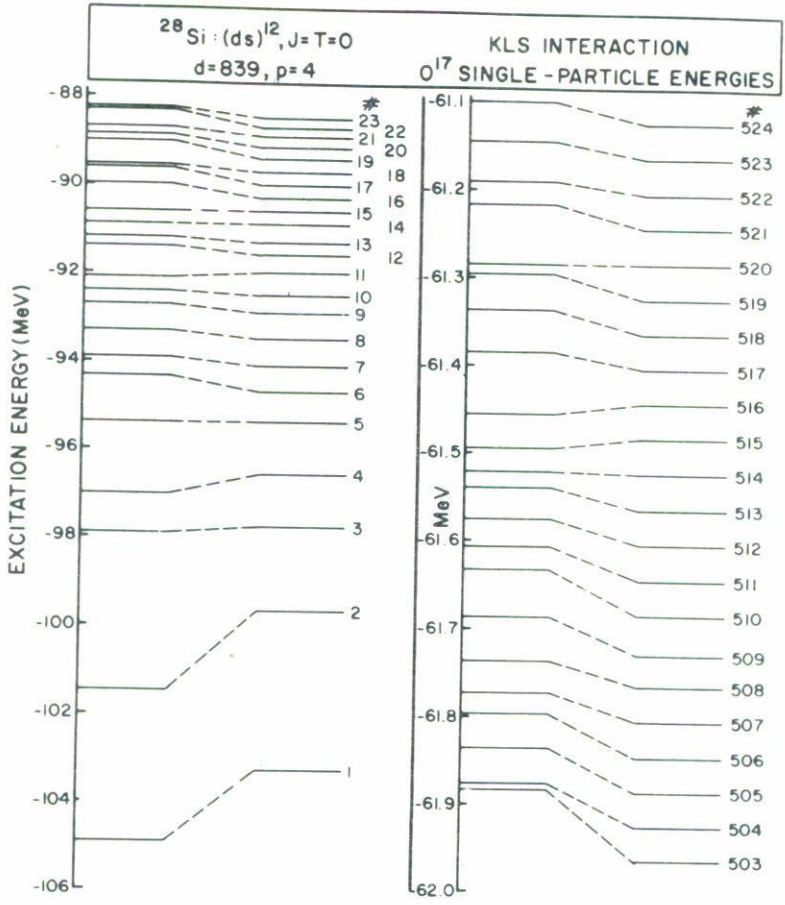

Fig. 4. A comparison between the exact shell-model $(ds)^{12}$ spectrum and the corresponding spectrum generated, as in Fig. 2, by using a four-moment distribution. Two segments of the 839-dimensional spectrum are shown, for levels 1–23 and 503–524; in each case the exact spectrum is to the left.

For a theory of these fluctuations we would naturally introduce a Hamiltonian ensemble, drawing on the standard assumption that a fluctuation measure defined by a suitable local energy average may be calculated via an ensemble average. There exists as yet no theory for (J, T)-conserving fixed-particle-rank ensembles, nor even one for the simpler EGOE, but there is a well developed theory[14] for the infinite-dimensional GOE, the most-

discussed outcome of which is the strong appearance, in the distribution of the nearest-neighbour spacings, of the well-known Wigner-von-Neumann level repulsion (which by the way is strongly moderated if there is a good hidden symmetry). By no means as well known as this striking short-range-correlation effect is the long-range correlation discovered by Dyson and Mehta[15] which led them to the conclusion that the spectra encountered in the GOE are of an "*essentially crystalline*" nature. These authors show for example how to construct the optimal linear counting statistic for the levels; they use it to show that the number of levels to be found in an energy span tD, where D is the mean spacing, can be measured with a mean square error of less than one unit (in $10^6 D$ one should find $\sim 10^6 \pm 1$ levels for example). As one would then expect, other measures of long-range order in the GOE spectra confirm the "extremely high degree of regularity" which one encounters.

The results displayed in Fig. 4 suggest immediately that the Dyson-Mehta phenomenon applies much more generally than to GOE spectra. Other shell-model calculations with reasonable interactions give the same result, which has emerged also from very extensive Monte-Carlo calculations for 2-body-interaction ensembles. Of course when one deals with finite matrices, the departures from an evenly spaced spectrum must now in part be ascribed to a slowly varying change in the density but, if we take for granted that this is well looked after by using the corrected Gaussian ($p \leqslant 4$), form, the 1.8-spacing-unit RMS error, though larger by a factor 2 than the Dyson-Mehta value, nontheless shows that the Si^{28} spectrum, for fixed (J, T) is also crystalline*.

The conclusions to be drawn from this are remarkable. The original formal results for the energy level fluctuations, derived for ensembles of quite unreasonable many-body interactions, and intended to be applied at higher excitation energies are, so far as one can see, valid for the realistic interactions encountered in the shell model; moreover they apply right down into the ground state domain**. The latter result seems particularly astonishing;

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* Values for other fluctuation measures, derived from the central region of the shell-model or two-body-ensemble spectra, are in excellent agreement with the GOE results and with experiment too; it is entirely probable that allowing more freedom in the choice of fluctuation-free density and using a more sophisticated statistical analysis[16] will reduce the 1.8 value.

** Though whether the accuracy is as high in this region is as yet unclear because of complications connected with the spectrum curvature. Besides results analogous to those of Fig. 4 there are preliminary results[17] derived for example form the GOE and TBRE ensembles of the spacings between the two lowest states, and the same procedure has been applied to the experimental results.[18]

one might rationalize the rigidity of the spectrum in the central region as being due to the "pressure" being exerted from both sides of the spectrum, but how then could one understand the rigidity at the very end-point?

The fluctuations then are small and, beyond information about symmetries, one is inclined to say that they carry little or no physical information, the physics being then carried by the density or, at the lower end, by the smoothed spectrum. Though this seems in the main to be correct, it quite probably overstates things, at least in the ground-state domain; one knows that in some simple regions (p shell, $f_{7/2}$ shell, for example) the accurate fitting of model energies to experiment is worthwhile, and besides that one sees regularities in other domains (rotational bands for example) which obtain more accurately than the fluctuation picture would permit. For the most part in these exceptional cases we are dealing with spacings between levels of different (J, T), and it might be profitable to ascribe the regularities to a general correlation between the energy level fluctuations for different symmetries. There is no theory for this kind of phenomenon nor is there any adequate microscopic theory of fluctuations for other quantities besides energies which should of course be considered also.

5. INFORMATION AND ITS PROPAGATION

We have seen in Fig. 4 a structure which displays a great microscopic complexity coupled with a macroscopic simplicity, the first being represented by the large number of matrix elements and the second by the small number of contributing moments. This kind of duality is of course common; think about the structure of a solid body! Moreover in our spectroscopic cases we know that the microscopic complexity increases with particle number*, whereas the macroscopic structure becomes simpler because of the more effective operation of the CLT. Things are as shown schematically in Fig. 5 where the numbers 67, $\sim$ 300,000 and 4 are respectively the number of *independent* (one and two-body) matrix elements, the total number of matrix elements, and the number of moments for our $(ds)^{12}$ example. We realize that the number of (independent) pieces of information is relatively small, but the information becomes enormously fragmented in the complex system and the fragmentation is worsened by the necessity of increasing N as we go up in

*Until we arrive at $m = N/2$ after which things gradually become simpler again.

excitation energy; except in a few simple cases it would become impossible to recover it from the experimental results. But beyond that, the information becomes filtered out as particle number increases, so that in a much more fundamental sense it is in fact *lost*; if the fluctuations were more characteristic of the system and the interaction then we might seek out the information there; but we know from experience (though a real theory is missing) that this is not the case. The one exception in fact is the moderation of short-range correlations when we have a hidden symmetry.

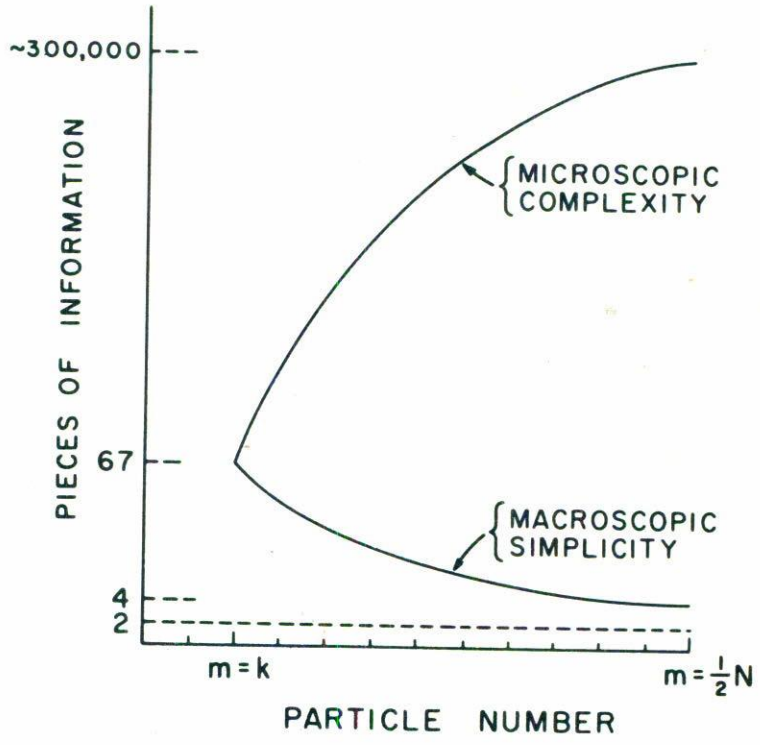

Fig. 5. A schematic representation of the effective number of pieces of information versus particle number, according as we think in microscopic or macroscopic terms. In the latter case, if N is large enough, the number is asymptotically 2 (centroid and widths of the Gaussian density), reached at $m = N/2$. The numbers 4,67, ~ 300,000 apply to the (ds)-shell example.

We consider now two questions which are suggested by Fig. 5:

(1) Is information transmitted along the macroscopic line; i.e. is the

moment information needed for m adequate for $(m+1)$, or do we encounter different pieces of information as m increases? In other words do we have a filtering action?

(2) m defines a $U(N)$ irrep so that we have propagation along a line of irreps. If we introduce $U(N)$ subgroups do we find a similar propagation throughout the lattice of subgroup irreps? And can we recover in this way our lost information?

We can expect that the answer to the first of these questions is essentially "yes", for we see easily that it is so in the $N \to \infty$ limit for non-interacting particles. In this case the CLT applies exactly, and the cumulants $K_p(m)$ (homogeneous polynomials of order p in the moments) are strictly proportional to particle number, $K_p(m) = mK_p(1)$. In fact the scale-invariant comulants, $k_p = K_p/\sigma^p$, then satisfy $k_p(m) = k_p(1)/m^{\frac{1}{2}p-1}$, so that we can see the filtering out of the information as m increases. For general k and finite N the question can be similarly answered[19] by using the fact that $< H(k)^p >^m$ is a polynomial of order pk in the variable m.

Important for both of these questions are the notions of *inclusion* and *orientation* of irreps. There is an obvious sense in which we can regard the k-particle space, k, as being contained in m $(m \geq k)$ with a weight

$$\omega(m:k) = d(k)^{-1} << \rho(k) >>^m = d(m)^{-1} d(k) < \rho(k) >^m \qquad (7)$$

where $<< >>^m$ is the m-particle trace and

$$\rho(k) = \sum_\gamma \psi_\gamma(k)\, \psi_\gamma^+(k) = \begin{bmatrix} n \\ k \end{bmatrix} \qquad (8)$$

is the operator form of the trace of the k-particle density, the γ summation being over a k-particle orthonormal basis; that $\rho(k) = \begin{bmatrix} n \\ k \end{bmatrix}$, where n is the number operator, follows from the fact that both are k-body operators with unit (diagonal) matrix elements. $\tilde{\rho}(k) = \begin{bmatrix} N-n \\ k \end{bmatrix}$, the corresponding operator for holes, is defined by interchanging ψ_γ, ψ_γ^+ in (8). We see that $\omega(m:k) = \begin{bmatrix} N-k \\ N-m \end{bmatrix}$, going properly to unity when $m = k$. Moreover the relationship between k and m is special in the sense that $\rho(k)$ is scalar in the m space. We shall say that the two spaces are *not oriented* with respect to each other.* If we have a

* The relationship is reciprocal for, if we describe things in terms of holes, k and m are interchanged; the m-hole density is then a scalar in the k hole space.

subgroup chain $U(N) \supset G_1 \supset G_2 \ldots G_l$ (or a more general subgroup lattice), and consider the irreps (m, α) of G_l, equivalent ones being separated by the intervening groups, the notions of weight and orientation have natural extensions; and similarly when we sum over all the equivalent irreps of G_l.

If now in the $U(N) \supset G_1 \ldots G_l$ case we consider a (k-body) operator $O(k, \beta)$ which is defined in an irreducible subspace (k, β) of G_l, we are led to a tentative theorem that the trace of O in other irreducible subspaces (m, α) is a multiple of that in (k, β);

$$\begin{aligned} \langle\langle O(k,\beta) \rangle\rangle^{m,\alpha} &= \langle\langle O(k,\beta) \rangle\rangle^{k,\beta} \cdot \{\text{Weight of } (k,\beta) \text{ in } (m,\alpha)\} \\ &= \langle\langle O(k,\beta) \rangle\rangle^{k,\beta} \cdot \omega(m,\alpha : k,\beta) \end{aligned} \tag{9}$$

which would give a remarkably simple and intuitively pleasing description of the propagation of information throughout the lattice. Since a general k-body operator is representable as a sum of (k, β) operators, along with operators which connect two subspaces and therefore do not contribute to traces, the extension of (9) to the more general case would then be

$$\langle\langle O(k) \rangle\rangle^{m,\alpha} = \sum_{\beta} \langle\langle O(k) \rangle\rangle^{k,\beta} \cdot \omega(m,\alpha : k,\beta) \ . \tag{10}$$

Rewriting this by means of (7), extended in the natural way, we have

$$\langle\langle O(k) \rangle\rangle^{m,\alpha} = \langle\langle \tilde{\rho}(N-m, \alpha_c)\ O(k) \rangle\rangle^{k} \tag{11}$$

where α_c denotes the m-hole subspace which is complementary to α. Eq. (11) is not quite general enough, for our interest is in H^p which, even for $p = 2$, is not readily expressible in the normal form demanded by (11); in other words we need the extension to operators which do not have a definite particle rank k but have mixed ranks up to some maximum value u ($u = 4$ for the square of the usual Hamiltonian). The extension is found to be

$$\langle\langle O \rangle\rangle^{m,\alpha} = \langle\langle \tilde{\rho}(N-m, \alpha_c) \begin{bmatrix} u-m \\ u-n \end{bmatrix}\ O \rangle\rangle^{[u]} \tag{12}$$

where $[u] = \sum_{t=0}^{u} t$, the trace then being over all states with $0, 1, 2, \ldots u$

Are equations (9–12) correct for a general lattice as we have written them? Does information in fact propagate in this remarkably simple way? A little thought will show that the basic equation (9) is valid if and only if the irreps are not oriented with respect to each other. Moreover if they are not oriented the other equations do in fact follow from (9), as well as the analog of (10) which would derive from (12). The equations are in fact valid in a number of important cases, in particular for $\Sigma U(N_i)$, the direct-sum subgroups which define partitions, for direct-product subgroups $U(N/r) \times U(r)$, and for the symplectic subgroup $Sp(N)$. In these cases the density operators are expressible in terms of Casimir invariants and simple methods are available for writing them.

In other important cases, as for example with the physically interesting $SU(3)$, the representations are oriented, the density operators are non-scalar, and eqs. (9–10) are invalid. The general forms (10, 12) are however valid for arbitrary subspaces (m, α); to use them for the more complicated subgroup structures we would need forms for the density operators which, though not expressible in terms of Casimir invariants, are functions of the generators. This problem has been dealt with for only two cases, isospin[20] and angular momentum[21], for which radically different methods have been employed, but there is a strong need for a more general understanding.

In an equation such as (10) we have a separation between the input information, represented by the k-particle trace, and the algebraic structure, represented by the weight ω. Moreover, considering moments (for which $O = H^p$) we expect that as m increases the higher-moment information is filtered away so that the resultant distribution becomes simple. The problem then, which is highly non-linear in the matrix formulation, becomes of low-order multilinearity in the interaction matrix elements in the present way of considering things. One consequence is that we can often describe the behaviour of the many-particle system in terms of quite explicit functions of the input matrix elements.

We have commented above on the difference in the domains of application of the statistical and perturbation methods. On the other hand the methods are related in that the moments are calculable by diagrammatic methods[10]; but, group-theoretical methods being available for that purpose also[10], we see the possibility of inverting the connection and using unitary-group methods in many-body perturbation theory. There are other interesting relationships among distributions, ensemble-averaged results and information theory. Ensemble averaging and spectral averaging are related by an elementary ergodic theorem[22]; if our H is a typical ensemble member then the ensemble-averaged results are relevant to the actual situation; this should

lead, on the one hand, to methods for making reasonable estimates for complicated quantities, and, on the other, to using ensemble-averaged many-body perturbation theory for studying such quantities as the effective interaction. It would be well in such applications not to carry out too broad an ensemble averaging, as one would with the GOE or EGOE, but to specify for example the strengths of various parts of the interaction, the pairing strengths (as at the end of §3) and so forth. Restrictions of this kind are accomodated very well by both the statistical and the perturbation methods, and as made clear by Balian[23], by more formal information theory as well.

6. EXPECTATION VALUES

Suppose that we add to our Hamiltonian a small multiple of an operator K, so that $H \rightarrow H(\alpha) = H + \alpha K$. If we should discover that the (smoothed) state density is unchanged, it follows by first-order perturbation theory that $K(E)$, the locally averaged expectation value of K, must vanish; if the effect is to leave the density unchanged except for a translation in energy, $K(E)$ must be a constant; if the width changes (but not the shape) that effect is representable by a scale change, implying a $K(E)$ proportional to $(E - \mathcal{E})$ where $\mathcal{E}$ is the centroid of the density ρ. But it follows from this that, if H gives an essentially Gaussian spectrum and if this feature is maintained to first order in α when $H \rightarrow H(\alpha)$ (in which case we shall say that a *strong* CLT is in operation), then $K(E)$ varies linearly with the energy.

We have for the m-particle system that

$$\partial \mathcal{E}(\alpha)/\partial\alpha = (\partial/\partial\alpha) < H + \alpha K >^m = < K >^m$$

$$[\partial\sigma^2(\alpha)/\partial\alpha]_{\alpha=0} = 2 < K(H - \mathcal{E}) >^m \qquad (13)$$

and then we find easily, in this simple but important case, that[24]

$$K(E) = < K >^m + < K(H - \mathcal{E}) >^m (E - \mathcal{E})/\sigma^2 \ . \qquad (14)$$

The averages $< >^m$ which we encounter here are parametric derivatives of moments and easily evaluated by the methods indicated above.

In dealing with the J-dependence of level densities we felt the need for the energy dependence of the (locally averaged) expectation value of J_z^2 since this would give us $\sigma_J^2(E)$. It is clear that the same need arises for operators connected with other symmetries, and for more general operators as well. In cases where a strong CLT is operative, this problem is now solved by (14), giving a considerable extension of our arguments about densities and spectra. That expectation values should be derivable from densities is not surprising; for the (smoothed) state density and partition functions are related by Laplace transformation, so that parametric derivatives on the latter function, which are standard in statistical calculations, may be replaced by derivatives on the density. We have also an analog of the result for a Gaussian stochastic process, that its Gaussian nature has a major simplifying influence on the functions which describe the system; the present requirement is strictly, not that the shape be Gaussian, but that it be stable under $H \to H(\alpha)$; but in fact that comes pretty well to the same thing. We cannot now discuss the circumstances under which a strong CLT obtains, except to say that it will not, in general, if K is of high particle rank (for then, as in §2, the approach to Gaussian is slow), and that we may reasonably expect that it will when K is an arbitrary $(0+1+2)$-body operator as long as H is a more or less realistic interaction (not a "synthetic" H with just a few degrees of freedom). As we indicate ahead we would in practical cases calculate correction terms[24] to (14) so that we would in fact know when our procedure is valid.

If we "center" the operators K, H by subtracting from them their scalar parts (scalar with respect to $U(N)$ if the averages are over all m-particle states) we are left, when (14) is valid, with a system in which only widths are of consequence. But since σ defines a proper (Euclidean) norm, our results should then be expressible in *geometrical* terms, the magnitude of K and the angle between K and H in the operator space then fixing the way in which $K(E)$ varies with $H(E) = E$. If we write $|G|^2 = \langle G^+ G \rangle^m$ for the square of the norm of any operator G, and $\vec{g}$ for the centered and renormalized G,

$$\vec{g} = (G - \langle G \rangle^m)/|G - \langle G \rangle^m| \tag{15}$$

then (14) becomes

$$k(E) = \vec{k} \cdot \vec{b}\,(E - \mathcal{E})/\sigma = \cos(\widehat{\vec{k} \cdot \vec{b}})\,(E - \mathcal{E})/\sigma \tag{16}$$

giving a very clear transcription from an operator relationship to a relative energy variation. The dependence of the expectation value on particle number is also implicit in (16); e.g. if K is a 1-body operator, as with occupancies, the reduced operator $\vec{k}$ is unitarily irreducible, the Young diagram structure being $[2,1^{N-2}]$ so that only the corresponding part of $\vec{b}$ (the renormalized single-particle part) can contribute to $\vec{k}\cdot\vec{b}$, which then has a simply specified number dependence.

In very large systems the "chaos" generated by the interaction will not span the entire spectrum and we get large departures from Gaussian; even in a small system we may be interested in the (usually small) departures. For these we need an extension of (14). Two general methods, as well as many combinations of them, are available. We may on the one hand simply continue with the argument which led to (14) but take account of shape changes as well, describing the shape by a set of translation- and scale-invariant parameters (the reduced cumulants of order $\nu \geqslant 3$ are one such set[24]). Taking account of the K-induced changes in these quantities simply adds (in the reduced cumulant case) an expression in Hermite polynomials $H_\nu\{(E-\mathcal{E})/\sigma\}$ to the $K(E)$ given by (14); in spaces of dimensionality $\leqslant 10^5 - 10^6$ only $\nu = 2$ and perhaps $\nu = 3$ contribute significantly to $K(E)$; higher terms have an effect only on the energy-level fluctuations. In huge spaces (14) becomes unsatisfactory as it stands, but then we can partition the space in the standard way, no new technical problems being encountered when we do this.

The basic equation which gives rise to the various forms for $K(E)$ is most easily derived by considering the modification which the change in H generates on the distribution function $F(E)' = \int_{-\infty}^{E} \rho(x)\,dx$ (we drop the dimensionality d so that ρ is the state density not the probability density). Suppose that E is bracketed by the eigenvalues E_t, E_{t+1}, that α is small enough that the level shifts (which are then $\alpha K(E_r)$) are small compared with the spacings $\rho^{-1}(E_r)$, and that, for simplicity of discussion, $K(E_t)$ and $K(E_{t+1})$ are positive. $F(E)$ simply counts the number of levels below E and thus, under $H \to H(\alpha)$, will change by -1 if the level shift brings the t'th eigenvalue up to or beyond the value E, and zero otherwise; i.e.

$$\delta F = -1, \qquad E_t \leqslant E \leqslant E_t + \alpha K(E_t)$$

$$= 0, \qquad E_t + \alpha K(E_t) < E < E_{t+1}$$

$$\delta F = -\alpha K(E_t)/(E_{t+1} - E_t) \to -\delta\alpha\, K(E)\,\rho(E)\ , \tag{17}$$

where, in the last equation, we have written the probable value (thereby implicitly introducing the local average) and then have proceeded to infinitesimal α. We have then, with the derivative taken at $\alpha = 0$,

$$K(E) = -\rho^{-1}(E)\,\partial F_\alpha(E)/\partial\alpha = -\rho^{-1}(E)\int_{-\infty}^{E}\left(\partial\rho_\alpha(x)/\partial\alpha\right)dx\ . \quad (18)$$

All the forms for $K(E)$ emerge from this when we express ρ in terms of the location, scale and shape parameters (or sets of such quantities if we partition the space) and carry out the indicated derivative.

The $K(E)$ equations can be used to study something about expectation value fluctuations. We can define a locally averaged variance as $K^2(E) - \{K(E)\}^2$, where $K^2(E)$ corresponds to $H \to H + \alpha K^2$; when $[K,H] = 0$ this quantity is indeed the local variance of the expectation value of K, but in the general case, when K, H do not commute, it is the sum of this quantity and the averaged single-state "quantum mechanical" variance of K; we have then a problem of separating the two effects if we want better than an upper limit to the fluctuations in the expectation value.

7. APPLICATIONS

We mention only a few of the possibilities. The most obvious application is to level densities. By using moments, and relying on the CLT, we have gotten around the limitations which are natural to the combinatorial methods used by Bethe[1], reformulated shortly after that by van Lier and Uhlenbeck,[25] and used since by just about everybody interested in the subject. In the hands of artists this theory has given a good account of observed facts,[26] but it has obvious fundamental deficiencies as a result of which it makes no contact at all with the nuclear physics of the ground-state domain, of which it should be a natural extension; moreover it is unable to deal adequately with such things as spin cut-off factors which are now beginning to be measured, parity* ratios, and the partial level densities which are needed for treating both the γ cascades and the internal cascade which leads to compound nucleus formation.

* As a symmetry, parity is interesting; it should be easily seen by considering configurations that the fixed-parity distributions, though easily calculated, are multimodal, nothing like Gaussian.

The present indications are that, by partitioning the single-particle space and using four moments to describe the low-lying configuration densities (Gaussian is adequate for the higher ones), we generate quite accurate densities which give a good treatment of the experimentally measurable quantities. It is implied by this that the densities give a satisfactory treatment of the ground-state energy, this being essential in order that we can identify the excitation energy in the calculation with that of the actual nucleus. The theory for non-interacting particles encounters no such problem (just fill up the lowest states and count the single-particle energies!), but that feature is not to its credit but instead an indication of its unreality. We have now an answer to the third question of §1; we can indeed extrapolate from the system centroid down into the domain of interest. It must be realized also that the theory of ground-state energies (nuclear binding energies) is of great interest, quite apart from the role which it plays in level densities. In such considerations the energy level fluctuations appear to be of consequence only in setting a limit to the accuracy of the calculations;[13] it is fortunate then that the fluctuations are small.

We have a family of methods for calculating the spin-cut-off factor, a quantity not accessible to the elementary theory. As a minor example we give the result derived by parametric differentiation for $A = 63$ treated as $(f_{5/2}p)^7$, for which $N = 24$, $d = 3.5 \times 10^5$. With a Brown-Kuo interaction[27] and $^{57}_{2}\mathrm{Ni}$ single particle energies we find[24], with $\sigma = 3.5$ MeV,

$$\sigma_J^2 = 10.0 - 0.073(E - \mathcal{E})/\sigma - 0.72\ \{(E - \mathcal{E})^2 - \sigma^2\}/\sigma^2 \qquad (19)$$

where the spectrum span is about 20 MeV. In this case, and in many others, J^2 is essentially orthogonal to the centered H (hence the small coefficient of the linear term) so that we would predict a good Gaussian behaviour for the states of fixed J (as discussed in §3).

Fig. 6 shows an application[28] of a spin-cut-off factor to calculation of the lowest (yrast) state for each J in the $T = 5/2$ states for the same example; the nucleus now is Cu^{63} for which shell-model calculations are available[29]. One sees a remarkable agreement between the exact results and those derived very simply by a calculation of $J_x^2(E)$; the only significant discrepancy in fact is removed by using a $J_x^4(E)$ correction to the Maxwellian distribution. Experience indicates though that this quality of agreement may be found only with odd-even nuclei.

The occupancies of single-particle states are the simplest and perhaps the most important parameters encountered in studying a many-particle system.

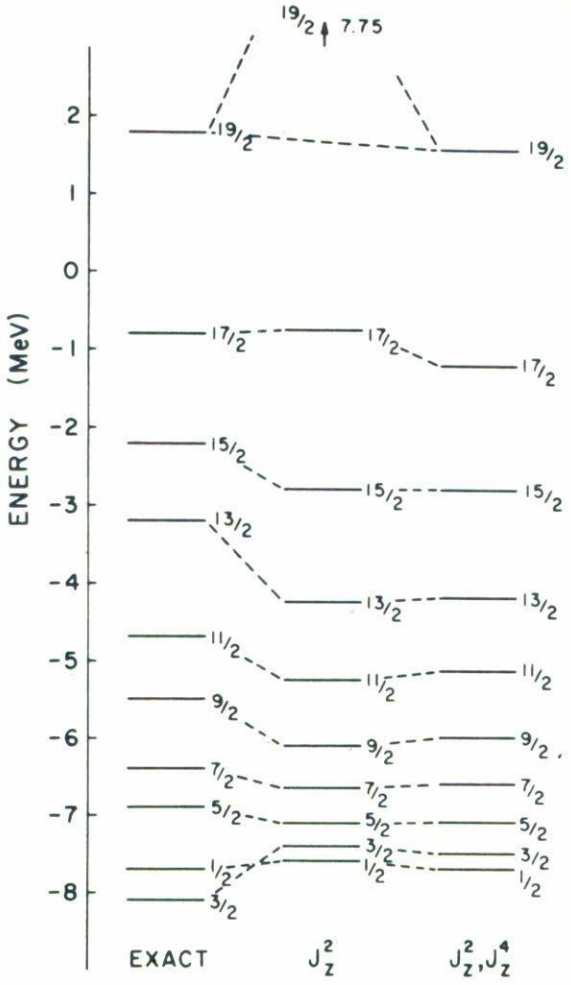

Fig. 6. The yrast spectrum for ^{63}Cu treated as $(f_{5/2}p)^7$. The exact shell-model spectrum of Wong is compared with that derived directly by Chang from the locally averaged J_z^2, with and without J_z^4 correction.

They are calculable with excellent accuracy by the method of §6. The linear equation (14) becomes[24]

$$n_s(E) = (m/n)\,\{1 + ((N-m)/(N-1))\,\zeta_s(E-\mathcal{E})\sigma^2\} \qquad (20)$$

where ζ_s is the renormalized single-particle energy; the quadratic and cubic corrections are easily calculated. A $(ds)^{12}$ example is given in Fig. 7, comparison being made with exact shell-model results. Observe that the occupancy lines are quite close to linear, the quadratic correction being small and the cubic negligible. In a very much larger space one might expect that some of the linear occupancy curves (which correspond to Gaussian densities) would give "forbidden" values (negative or larger than unity) at

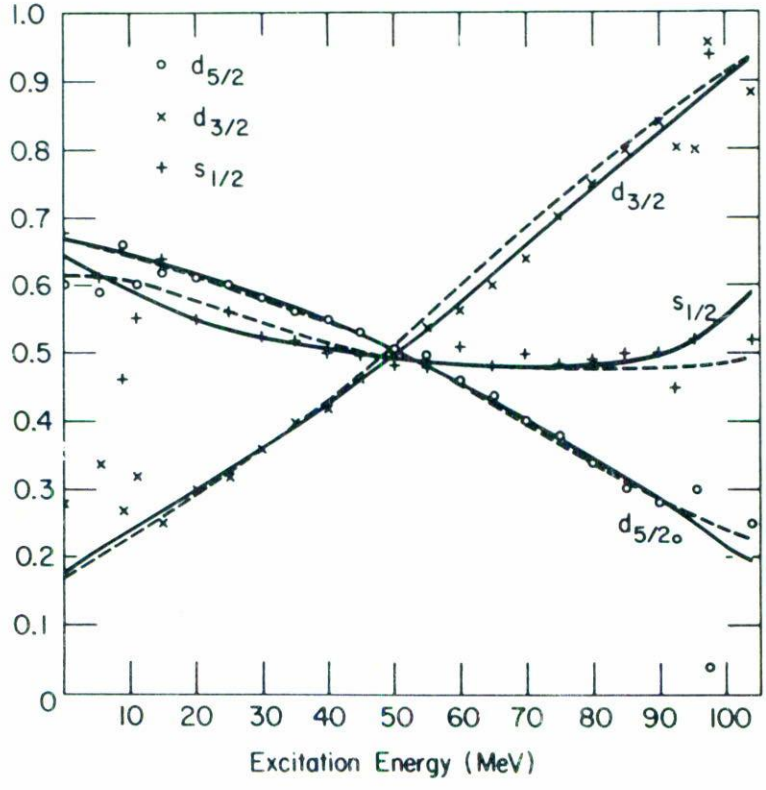

Fig. 7. Fractional occupancies (dashed line) for $(ds)^{12}$ with a *KLS* interaction, calculated via (14) with a quadratic correction; the cubic correction is negligible. The solid line shows occupancies calculated from the configuration distributions. The discrete values are for $J = T = 0$ and come by smoothing the matrix results of Soyeur and Zuker.

the spectrum extremes. This in fact does happen, another indication, that the energy range over which the density is Gaussian is limited; the occupancy curves give us a very useful way of measuring this range. The forbidden values of course disappear when we include corrections or make partitions.

There is a very wide range of other applications of the locally averaged expectation values, for example to calculating correlated occupancies, and to comparing different interactions, and different parts of the same interactions, (pairing and quadrupole-quadrupole for example), at a given excitation energy.

Considerable attention has been given to the study of symmetries by statistical methods, especially, leaving out the configuration symmetries, to spin-isospin $SU(4)$[30] and to pairing[31]. One may study a symmetry by asking first for the intensity distribution of its irreps; the two-body Casimir operator locally averaged as in §6 would tell us something about this, but in fact has not yet been really used for the purpose. Instead one has dealt separately with each irrep. The technical question then is about the calculation of the low-order moments, a problem however which has not really been faced for third and fourth-order moments, though these will in the long run be necessary; in the meantime one gets along with Gaussian densities. If, in a given domain of excitation (which in practice would be near the ground state), only one irrep has an appreciable intensity, the symmetry is necessarily good. Otherwise the symmetry may be broken, or it may be that the energy region contains states of different (good) symmetry. But in an even-even (ds)-shell nucleus, for example, one is not prepared to believe that the lowest $0^+, 2^+ \ldots$ states could have different good $SU(4)$ symmetries, so that one would ascribe the intensity admixtures, as Parikh[30] has done, to an $SU(4)$ breaking. For $SU(4)$ the spectrum of the centroids is the same for all $(0+1+2)$-body H's to with a scale factor (we ignore the possibility of a sign change); moreover, and much more generally than for $SU(4)$, the widths of the various irreps are closely equal.* One expects then that the ratio of the width to the centroid-spectrum scale factor should be the parameter which has most to do with the symmetry breaking, this then giving a simple classification of H's and for example displaying explicitly the symmetry breaking due to spin-orbit effects[30]. Nonetheless one must not really expect that a one-parameter treatment of symmetry breaking will be adequate and, for the more general treatment in terms of the densities, one would be happier to know more about the distribution shapes.

Pairing symmetries have been studied by Nissimov et al[31] in terms of BCS quasiparticles, the necessary extensions of the formalism being made to deal

* This has been used[32] for $SU(3)$ whose centroid energies are easily calculated even though the irreps are oriented. Then if we assume the widths to be the same the value follows from a calculation of the m-particle width.

with fermion non-conservation. These authors moreover have not relied on intensity considerations alone (in this case the intensities for different quasiparticle numbers) but have evaluated the partial widths (an elementary example of which we have given at the end of §3) and have used these along with the centroids in considering what truncations of the quasiparticle space are allowed, and other related matters.

Identical-particle pairing theory has been worked out[31] and applied to Ni and Sn isotopes by Hsu and, recently and in much more detail, by Quesne and Spitz, in terms of the symplectic group and $R(3)$ quasispin; there is in fact a useful combination of the two, in which propagation is carried out via the symplectic group (whose identical-particle irreps are not oriented with respect to each other), and a partical-width decomposition is made by using the $R(3)$ Racah algebra. There are two separate kinds of pairing effects (and intermediate cases as well) according as the fundamental pair (the two-particle wave function) spans all the orbits of the system or whether we consider a different pair for each orbit. In the first case the group is $Sp(N)$ and the whole structure of the theory is similar to BCS, without however the BCS degrees of freedom which permit the pair to adjust its structure to take best account of the interaction; in the $Sp(N)$ case the pair structure is fixed (to within phases) by the single-particle states considered. Little attention has yet been given to the second kind of pairing, in which the group is essentially a direct sum of symplectic groups; it is clear though that this will be very much better symmetry.

8. CONCLUSION

Much of the behaviour of spectroscopic systems is determined by the operation of symmetries and central limit theorems. There are close connections between them and with the notion of the information carried by spectroscopic spaces.

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RESUMEN

Se determina en gran medida la densidad de estados de un sistem espectroscópico de muchas partículas interactuando, por medio de la aplicación de un teorema de límite central que fija la forma promediada, mientras que, a través del espectro completo, las fluctuaciones alrededor del promedio siguen muy de cerca leyes, derivadas para alta energía, por consideración de ensambles de matrices estocásticas. Las fluctuaciones son pequeñas, la descripción de Dyson-Mehta de los espectros de ensambles estocásticos como "esencialmente cristalinos" se encuentra que es más generalmente aplicable. Los mismos resultados generales se obtienen, tanto para promedios como para fluctuaciones, al hacer una restricción a estados de una simetría exacta dada, y para los promedios, se obtienen aún para simetrías rotas. Otras propiedades del sistema, números de ocupación, por ejemplo, se determinan análogamente.