THE NULL SPACE CONCEPT IN IMPLEMENTING UNITARY SYMMETRY*

J. D. Louck†

Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87544

and

L. C. Biedenharn

Department of Physics, Duke University Durham, North Carolina 27706

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ABSTRACT: The unitary groups have wide applications in physics; yet the elements (Wigner coefficients) of the unitary matrix which reduces the direct product of two unitary irreducible representations remain undetermined for the most part, despite the considerable

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importance of these coefficients and the efforts by many investigators to calculate them. The occurrence of two problems helps explain why: (a) the general principle [for $U(n)$] which distinguishes among the multiple occurrences of an irreducible representation has not been fully uncovered; (b) explicit calculations, even when (a) has been solved as is the case for $n = 3$, are exceedingly difficult. Insight into each of these problems may be obtained through the concepts of a Wigner operator and the null space possessed by it. This paper explains these concepts within the framework of $SU(2)$, and demonstrates that an $SU(2)$ Wigner operator is determined by its abstract structural form and its null space. It is suggested that the null space concept is the proper one to use in the characterization of a general canonical Wigner operator in $U(n)$.

I. INTRODUCTION

The application of symmetry techniques has played an important role in the development of a wide variety of physical theories, ranging from atomic, molecular, and nuclear spectroscopy to elementary particles. Applications are still appearing in such unsuspected areas as organic chemistry.[1] We denote these symmetry techniques generically as "unitary symmetry"; more precisely, we have in mind applications of the unitary Lie groups $SU(n)$ and their subgroups. The principal tool in such applications is the theory of tensor operators. Despite this fundamental role of the unitary groups, a fully worked out theory of tensor operators of $SU(n)\,(n > 2)$ is still not available. By "fully worked out" we mean a presentation of an explicit set of (orthonormal) Wigner coefficients such as exists for $SU(2)$. Even for $SU(3)$, where a complete theory exists[2-4], explicit calculations are apt to be difficult, and available, hence, only for selected special cases.[3-11] Part of the difficulty stems from an incomplete understanding of structural principles underlying the resolution of the multiplicity problem; part from the technical difficulty of effecting -and understanding- complicated summations.

The magnitude and scope of these problems have led to numerous possible methods for attacking them.[8, 12, 13] In our own experience, the notion of a Wigner operator[14-18] has proved to be a valuable conceptual basis for elucidating these problems; the factorization lemma[2, 16] and the pattern calculus[15] providing the principal calculational tools. A natural outgrowth of these techniques has been the development of the $U(n)$ Racah-Wigner calculus (operator algebra)[14-18] and new ways of viewing the structure of explicit matrix elements.[3,4]

The essential problem facing one in developing the canonical Racah-Wigner calculus is one of generalization: What elements of the known $U(2)$ structure are to be preserved in this generalization? What elements are to be considered as peculiar to $U(2)$ alone? Similar questions occur for the $U(3)$ case where the formal (but not completely worked out) structure is known to exist. Certain special features that in $U(3)$ imply the known canonical resolution do not exist in $U(4)$ and higher.

One aspect of a Wigner operator which has not yet been fully explored is the implications of the null space of such an operator.[14,16] In $U(3)$, for example, it has been demonstrated[4] that the null spaces corresponding to the Wigner operators belonging to the same multiplicity set are simply ordered by the inclusion property. The null space of a Wigner operator has thus begun to emerge as an important structural feature.

If indeed the null space is a significant consideration, then it would seem appropriate to re-examine the known $SU(2)$ Wigner coefficients from this viewpoint. The result, we feel, is both interesting and reassuring. In rather imprecise, but descriptive terms, the result is: *An $SU(2)$ Wigner operator is uniquely determined by its general structural form and its null space*. The purpose of this paper is to explain fully the meaning of this remark for $SU(2)$, noting that while it is quite pleasant to view $SU(2)$ from yet another perspective, it is the implications of this viewpoint for $U(n)$ that is significant.

II. $SU(2)$ WIGNER OPERATORS

In this section, we present a brief summary of the Wigner operator concept for $SU(2)$, using only the language of $SU(2)$, for the purpose of making the present viewpoint accessible to a larger audience.

Let $U \to \mathbb{O}_U$ each $U \in SU(2)$, be a representation of $SU(2)$ by unitary operators on the abstract Hilbert space possessing the orthonormal basis[19]

$$\left| \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix} \right\rangle, \quad 2j = 0, 1, 2, \ldots \quad , \tag{1}$$

where for each half-integer j the values which m can assume are $-j$, $-j+1$, $\ldots$, j. The action of $\mathbb{O}_U$ on the set of basis vectors (1) is given by

$$\mathbb{O}_U \left| \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix} \right\rangle = \sum_{m'} D^j_{m'm}(U) \left| \begin{matrix} 2j & & 0 \\ & j+m' & \end{matrix} \right\rangle \tag{2a}$$

Instead of the standard notation $D^j_{m'm}(U)$ for the elements of the irreducible unitary matrix representations of $SU(2)$, we will also use a more suggestive notation employing Gel'fand patterns:

$$D \begin{pmatrix} & j+m & \\ 2j & & 0 \\ & j+m' & \end{pmatrix} (U) = D^j_{m'm}(U) \quad . \tag{2b}$$

We next *define* each of the operators

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle , \quad \begin{matrix} \Delta = -J, -J+1, \ldots, J \\ M = -J, -J+1, \ldots, J \end{matrix} \quad , \tag{3}$$

for each $2J = 0, 1, 2, \ldots$, by giving its action on the basis (1):

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle \left| \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix} \right\rangle = C^{j\ J\ j+\Delta}_{m\ M\ m+M} \left| \begin{matrix} 2(j+\Delta) & & 0 \\ & j+\Delta+m+M & \end{matrix} \right\rangle \quad , \tag{4}$$

where the coefficient is an $SU(2)$ Wigner coefficient.[20] An immediate consequence of this definition is the property:

$$\mathbb{O}_U \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle \mathbb{O}_{U^{-1}} = \sum_{M'} D \begin{pmatrix} & J+M & \\ 2J & & 0 \\ & J+M' & \end{pmatrix} (U) \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M' & \end{matrix} \right\rangle \quad , \tag{5}$$

that is, each of the operators (3) corresponding to *particular* J *and* Δ is the component of an irreducible tensor operator of type J. A second consequence of the definition is the property:

$$T^J_M = \sum_{\Delta=-J}^{+J} \mathcal{I}^J_\Delta \left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & J+M & \end{matrix} \right\rangle \tag{6}$$

in which T^J_M denotes an arbitrary irreducible tensor operator of type J and $\mathcal{I}^J_\Delta$ denotes an invariant operator with respect to $SU(2)$. This property is just an operator expression of the Wigner-Eckart theorem. Taking matrix elements of this equation between basis states (1) of the underlying Hilbert space on which the operators (3) act, one sees that the eigenvalues of the invariant operator are, in fact, just

$$\mathcal{I}^J_\Delta(j+\Delta) = \langle j+\Delta \| T^J \| j \rangle \ , \tag{7}$$

where the right-hand side is the conventional notation for a reduced matrix element. From a vector space point of view, Eq. (6) expresses the fact that the $2J+1$ irreducible tensor operators enumerated by $\Delta = -J, -J+1, \ldots, J$ (each such operator also has $2J+1$ components, enumerated by M) are a basis for tensor operators of type J. Note, however, that the scalars in this operator basis are not complex numbers, but group invariant operators.

Using definition (4), one might now proceed[21] to reformulate the whole Racah-Wigner angular momentum calculus in terms of the Wigner operator concept, thereby obtaining new insights into the structure of the algebra. Since it is our purpose to discuss another aspect of Wigner operators -the null space properties-, we will omit this reformulation.

III. AN ILLUSTRATIVE EXAMPLE

In order to make clear the essential ideas in the null space concept, it is convenient to discuss first an elementary example which will lead us directly to the more abstract general formulation.

Let us suppose that we knew nothing of the group theoretical background of angular momentum theory, but wanted to determine explicitly the set of diagonal multipole operators: $C^{j\,J\,j}_{m\,0\,m}$. The scalar operator $J=0$ is just the unit operator. The vector operator, $J=1$, we know is proportional to m, the magnetic quantum number. How can one determine the proper normalization? The answer is to use *multiplet averaging*, a method of almost prehistoric origin in quantum mechanics. That is:

$$\sum_{\text{multiplet}} C^{jJj}_{m\,0\,m}\, C^{jJ'j}_{m\,0\,m} = \delta_{JJ'} \frac{(\text{dimension of multiplet})}{(\text{dimension of operator})} .$$

Applying this to the vector case, one finds:

$$\sum_m \left(C^{j\,1\,j}_{m\,0\,m}\right)^2 = (2j+1)/3 = N^2 \sum_m m^2 .$$

Since $\sum_m m^2 = j(j+1)(2j+1)/3$, one obtains the familiar result:

$$C^{j\,1\,j}_{m\,0\,m} = m/[j(j+1)]^{\frac{1}{2}} .$$

The complete set of diagonal multipole operators follows in this way. One recognizes that one is simply using the Schmidt orthonormalization process on the ordered set $1, m, m^2, \ldots$, and using multiplet averaging as the inner product.

To see how null space concepts can fit into this problem, let us consider the quadrupole operator. This operator has the general form:

$$C^{j\,2\,j}_{m\,0\,m} = (\text{normalization})(a_j m^2 + b_j m + c_j) ,$$

where a_j, b_j and c_j are *polynomials* in $2j+1$ of degrees 0, 1, and 2, respectively. We now introduce new information: (a) the knowledge that a quadrupole operator vanishes when connecting $j = 0$ to $j = 0$ or $j = \frac{1}{2}$ to $j = \frac{1}{2}$ – this is the null space of the diagonal quadrupole operator; and (b) the knowledge that $a_j m^2 + b_j m + c_j$ is invariant under the reflection symmetry $j \to -j-1$. (Note that this transformation leaves the dimension $|2j+1|$ invariant, as well as $j(j+1)$). This information determines the quadratic form in m up to a multiplicative constant, i.e.,

$$C^{j\,2\,j}_{m\,0\,m} = (\text{normalization})\,[3m^2 - j(j+1)] .$$

Instead of normalizing by multiplet averaging – a tedious method really–, let us apply the null space idea once again, using also the reflection symmetry:

$$\sum_{m=-j}^{j} [3m^2 - j(j+1)]^2 = \text{fifth degree polynomial in } j \text{ which vanishes for } j = 0, \tfrac{1}{2}, -1, -\tfrac{3}{2} \text{ and contains the dimension of the multiplet, } 2j+1, \text{ as a factor}$$

$$= \# \, (2j+1)(2j)(2j-1)(2j+2)(2j+3) \; ,$$

where # denotes a numerical constant. Thus, the normalization of the quadrupole operator is reduced to evaluating a numerical special case.

We conclude from this example the following fact: *Knowledge of the null space of the diagonal multipole operator,* $C^{j\,J\,j}_{m\,0\,m}$, *completely determined the explicit functional form of this operator, using only abstract properties of the* $SU(2)$ *structure.* We will show in the following sections that the more general result is also true: A Wigner operator is completely characterized by its null space properties.

IV. THE NULL SPACE CONCEPT IN THE GENERAL CASE

Let us now examine definition (4) more closely. This equation defines a Wigner operator for each Δ in the set $-J, -J+1, \ldots, J$. Recall that a Wigner coefficient vanishes unless the so-called triangle rule is satisfied by the triad $j\;J\;j+\Delta$. How then are we to interpret Eq. (4)? First let us recognize that if we wish to study the properties of a specific Wigner operator on our underlying Hilbert space, we should in our defining equation, Eq. (4), consider that J and Δ are specified (fixed), but that $2j$ in the basis vector should run over all possible integral values $0, 1, \ldots$ With this viewpoint in mind, let us now consider the origin of the triangle rule – the direct product relation

$$D^J \otimes D^j = \sum_{\Delta=-J}^{J} \oplus \, \mathcal{I}(J \otimes j; j+\Delta)\, D^{j+\Delta} \; . \tag{8}$$

This expression is just the Clebsch–Gordan series written in a form appropriate to this discussion. The intertwining number $\mathcal{I}(J \otimes j; j+\Delta)$ denotes the number of occurrences of irreducible representation (irrep) $j+\Delta$ in the direct product of irrep J by irrep j. In accordance with our viewpoint of associating J and Δ with a fixed Wigner operator, we present the intertwining number as a function of $2j$:

$$\mathfrak{I}(J \otimes j ; j + \Delta) = \begin{cases} 1 & \text{for } 2j \geqslant J - \Delta \\ 0 & \text{for } 2j < J - \Delta \end{cases} \quad . \tag{9}$$

Thus, in our language, the familiar triangle rule becomes the statement that the intertwining number is a step function which has value 0 at all points $2j$ in the set

$$\{0, 1, \ldots, J - \Delta - 1\} \tag{10}$$

and value 1 at all points $2j$ in the set

$$\{J - \Delta, J - \Delta + 1, \ldots\} \quad . \tag{11}$$

(The set (10) is the empty set for $\Delta = J$.) Returning to the study of Eq. (4), we see that the preceding observations may be stated: *The null space of the Wigner operator*

$$\left\langle \begin{matrix} & J+\Delta & \\ 2J & & 0 \\ & \bullet & \end{matrix} \right\rangle \quad \text{(lower pattern arbitrary)} \tag{12a}$$

is the set of irrep spaces $\{[2j \quad 0]\}$ *for which* $2j$ *belongs to the set*

$$\{0, 1, \ldots, J - \Delta - 1\} \quad . \tag{12b}$$

Remarks. (a) Observe that the concept of the null space of a Wigner operator is basis independent to the extent that it does not depend on how one introduces a basis into the carrier space of a given irrep j. Note that the only Wigner operator of type J which has a non-empty null space is the one having $\Delta = J$. This property may be attributed to the fact that maximal Wigner operators ($\Delta = J$) may be considered as generating the state vectors themselves from the "vacuum" ket:

$$\left|\begin{matrix}2j & & 0\\ & j+m & \end{matrix}\right\rangle = \left\langle\begin{matrix} & 2j & \\ 2j & & 0\\ & j+m & \end{matrix}\right\rangle \left|\begin{matrix}0 & & 0\\ & 0 & \end{matrix}\right\rangle \tag{13}$$

Since our principal concern is to determine the generality of the null space concept, we next examine directly the known $SU(2)$ Wigner coefficients, noting subsequently the extent to which one can turn the procedure around. The literature contains many different forms for these coefficients, but it is Racah's form[22] which is pertinent to our discussion. However, it is convenient to introduce new labels which help to distinguish between "operator labels" and "irrep space labels (variables)", and which exhibit more directly the symmetries of subsequent interest:

$$\left.\begin{matrix}\Delta_1 = J+\Delta,\ \ \Delta_2 = J-\Delta\\[2ex] \Delta'_1 = J+M,\ \ \Delta'_2 = J-M\end{matrix}\right\}\quad \text{Operator related labels} \tag{14a}$$

$$\left.\begin{matrix}z_1 = -j-1+m,\ \ z_2 = j+m\\[2ex] z = 2j+1 = z_2 - z_1\end{matrix}\right\}\quad \text{Irrep space variables}\ . \tag{14b}$$

In terms of this notation, the $SU(2)$ Wigner coefficients may be written in the following form:

$$C^{j\ \ J\ \ \ j+\Delta}_{m\ M\ \ m+M} = \left\langle\begin{matrix}2(j+\Delta) & & 0\\ & j+\Delta+m+M & \end{matrix}\right| \left\langle\begin{matrix} & J+\Delta & \\ 2J & & 0\\ & J+M & \end{matrix}\right\rangle \left|\begin{matrix}2j & & 0\\ & j+m & \end{matrix}\right\rangle$$

$$= (-1)^{\Delta'_1-\Delta_1}[(\Delta'_1)!(\Delta'_2)!/(2J)!]^{\frac{1}{2}} \times \tag{15a}$$

$$\times \left[\frac{(-z_1-1)!(z_2+\Delta'_1-\Delta_2)!}{(-z_1+\Delta_1-\Delta'_1-1)!(z_2)!}\right]^{\frac{1}{2}} \Big/ D\begin{pmatrix}\Delta_1 & \Delta_2\\ 2J & 0\end{pmatrix}(z) \times$$

$$\times \sum_{k_1+k_2=\Delta'_2} \binom{\Delta_1}{k_1}\binom{\Delta_2}{k_2}(k_1)!\binom{z_1+\Delta_2-k_2}{k_1}(k_2)!\binom{z_2}{k_2}\ ,$$

where the denominator function is given by the generalized pattern calculus rules[3]

$$D\begin{pmatrix}\Delta_1 & \Delta_2\\ 2J & 0\end{pmatrix}(z) = [(\Delta_1)!(\Delta_2)!/(2J)!]^{\frac{1}{2}}\left[\left|(\Delta_1)!\begin{pmatrix}-z+\Delta_2\\ \Delta_1\end{pmatrix}(\Delta_2)!\begin{pmatrix}z+\Delta_1\\ \Delta_2\end{pmatrix}\right|\right]^{\frac{1}{2}} \tag{15b}$$

In these results the binomial coefficients are defined for arbitrary variable γ and non-negative integer a by

$$\begin{pmatrix}\gamma\\ a\end{pmatrix} = \gamma(\gamma-1)\ldots(\gamma-a+1)/a! \quad . \tag{16}$$

Before turning to the null space aspects of Eq. (15a), we wish to analyze its structure a bit further. The motivation for doing this comes from the observation that the polynomial part (the summation over k_1 and k_2 part) is not always in *irreducible form*, i. e., unless Δ_2' is the smallest of the integers $\Delta_1, \Delta_2, \Delta_1', \Delta_2'$, the polynomial factorizes into a product of linear factors in z_1 and z_2 multiplied by a new polynomial z_1 and z_2. We wish to remove all such linear factors. The appropriate technique for carrying out this analysis is the pattern calculus.[23]

Depending on the relative magnitudes of the Δ_i and Δ_i', there are four arrow patterns which can be drawn for the arrows going between row 2 and row 1. The possible cases are illustrated below (a single arrow represents one or more arrows) along with the numerical factor which is associated with the arrow pattern by the pattern calculus rules. Our interest here is only in the factors associated with arrows going between rows, and we refer to this quantity as the *numerator pattern calculus factor* (NPCF):

Case 1 Case 2 Case 3 Case 4

In terms of the four non-negative integers defined by

$$u_i = \max(0, \Delta_i - \Delta_1'), \quad l_i = \max(0, \Delta_1' - \Delta_i), \quad i = 1, 2, \tag{17}$$

the numerator pattern calculus factor is given for each of the four cases by the expression

$$(\mathrm{NPCF})^{\frac{1}{2}} = \left[\left|\prod_{i=1}^{2} (u_i)!\,(l_i)! \begin{pmatrix} z_i \\ u_i \end{pmatrix} \begin{pmatrix} z_i + l_i \\ l_i \end{pmatrix}\right|\right]^{\frac{1}{2}}. \tag{18}$$

The point in introducing this factor is the following: We can now write

$$\left[\frac{(z_1 - 1)!\,(z_2 + \Delta_1' - \Delta_2)!}{(z_1 + \Delta_1 - \Delta_1' - 1)!\,(z_2)!}\right]^{\frac{1}{2}} = (\mathrm{NPCF})^{\frac{1}{2}} \Big/ \prod_{i=1}^{2} (u_i)! \begin{pmatrix} z_i \\ u_i \end{pmatrix}, \tag{19}$$

a result which is valid for all four cases. *It is the set of linear factors in the denominator of this expression which always factors out of the summation expression in Eq.* (15a).

Thus, we may rewrite Eq. (15a) in the following form:

$$\left\langle \begin{matrix} 2(j+\Delta) & & 0 \\ & j+\Delta+m+M & \end{matrix} \right| \left\langle \begin{matrix} & J+\Delta & \\ 2j & & 0 \\ & J+M & \end{matrix} \right\rangle \left| \begin{matrix} 2j & & 0 \\ & j+m & \end{matrix} \right\rangle$$

$$= (-1)^{l_1} \left[(\Delta_1')!\,(\Delta_2')!/(2J)!\right]^{\frac{1}{2}} \left[(\mathrm{NPCF})^{\frac{1}{2}}/D \begin{pmatrix} \Delta_1 & \Delta_2 \\ 2J & 0 \end{pmatrix} (z)\right] \times$$
$$\times\, P_k(\Delta_1 \Delta_2 \Delta_1' \Delta_2';\, z_1 z_2) \tag{20}$$

where P_k denotes a polynomial of degree $k = \min(\Delta_1, \Delta_2, \Delta_1', \Delta_2')$ in z_1 and z_2. The polynomial P_k has the following form:

$$P_k(\Delta_1 \Delta_2 \Delta_1' \Delta_2';\, z_1 z_2)$$

$$= \sum_{k_1 + k_2 = k} \begin{pmatrix} \Delta_1 \\ k_1 + u_1 \end{pmatrix} \begin{pmatrix} \Delta_2 \\ k_2 + u_2 \end{pmatrix} (k_1)! \begin{pmatrix} z_1 + \Delta_2 - u_2 - k_2 \\ k_1 \end{pmatrix} (k_2)! \begin{pmatrix} z_2 - u_2 \\ k_2 \end{pmatrix} \tag{21}$$

Further properties of these polynomials are given in Appendix A. In particular, we prove there the following symmetry property:

$$P_k(\Delta_1 \Delta_2 \Delta_1' \Delta_2' ; z_1 z_2) = P_k(\Delta_2 \Delta_1 \Delta_1' \Delta_2' ; z_2 z_1) \quad . \tag{22}$$

It is the form (20) and the symmetry (22) which is most appropriate for our discussion of the null space properties of the $SU(2)$ Wigner coefficients.

V. THE VANISHINGS OF THE POLYNOMIALS P_k

Equation (20) has a rather remarkable form. It becomes even more interesting when we recognize that the denominator

$$D \begin{pmatrix} \Delta_1 & \Delta_2 \\ 2J & 0 \end{pmatrix} (z) \tag{23}$$

is just the *normalization factor* discussed earlier in Sec. II for our motivating example. The un-normalized form of Wigner coefficient is accordingly

$$(\text{NPCF})^{\frac{1}{2}} (\text{polynomial in } z_1 \text{ and } z_2) \quad . \tag{24}$$

The form given in Eq. (24) implies that the polynomial part of this expression must vanish for each point (z_1, z_2) which belongs to the null space (12b) *and for which the numerator pattern calculus factor is nonvanishing*. Let us next give the explicit determination of these points.

Using the definitions of z_1 and z_2 given by Eq. (14b), we can enumerate the set of points $Z = \{(z_1, z_2)\}$ belonging to the null space (12b). These points are conveniently enumerated in a triangular array:

$$Z = \{(z_1, z_2)\} = \left\{ \begin{matrix} (-1,0) & & & & \\ (-1,1) & (-2,0) & & & \\ (-1,2) & (-2,1) & \ddots & & \\ \vdots & & & \ddots & \\ (-1,\Delta_2-1) & (-2,\Delta_2-2) & \cdots & \cdots & (-\Delta_2, 0) \end{matrix} \right\} \tag{25}$$

On the other hand, the subset of points of Z on which the numerator pattern calculus factor vanishes may be read off directly from Eq. (18). Denoting this set by Z', it is expressed generally by

$$Z' = \left\{ \begin{matrix} (z_1, z_2) : (z_1, z_2) \in Z, \text{ and either } z_1 \geqslant \Delta_1 - \Delta_1' \text{ or} \\ z_2 \leqslant \Delta_2 - \Delta_1' - 1 \text{ or both} \end{matrix} \right\} \tag{26}$$

For later reference, let us remark that this set of points Z' is also completely characterized by the following property: *Z' is the set of points of Z for which the final labels*

$$\begin{pmatrix} 2(j+\Delta) & & 0 \\ & j+\Delta+m+M & \end{pmatrix} \tag{27}$$

appearing in the Wigner coefficient (15a) *fail to satisfy the betweenness conditions*[19] $2(j+\Delta) \geqslant j+\Delta+m+M \geqslant 0$.

It is now apparent that the polynomial P_k must vanish on the set of points $Z'' = Z - Z'$:

$$Z'' = \{(z_1, z_2) : (z_1, z_2) \in Z, z_1 \leqslant \Delta_1 - \Delta_1' - 1, z_2 \geqslant \Delta_2 - \Delta_1'\} \ . \tag{28}$$

(In implementing the conditions appearing in the sets (26) and (27), it is useful to recall the relation $\Delta_1 + \Delta_2 = \Delta_1' + \Delta_2'$.)

Let us illustrate the preceding results for P_k by considering Case 1 where we have $u_1 = u_2 = 0$, $l_1 = \Delta_1' - \Delta_1$, $l_2 = \Delta_1' - \Delta_2$. The set Z'' becomes

$$Z'' = \{(z_1, z_2) : (z_1, z_2) \in Z, z_1 \leqslant \Delta_1 - \Delta_1' - 1\} \ , \tag{29}$$

and the polynomial

$$P_{\Delta_2'}(\Delta_1 \Delta_2 \Delta_1' \Delta_2'; z_1 z_2)$$

$$= \sum_{k_1 + k_2 = \Delta_2'} \binom{\Delta_1}{k_1} \binom{\Delta_2}{k_2} (k_1)! \binom{z_1 + \Delta_2 - k_2}{k_1} (k_2)! \binom{z_2}{k_2} \tag{30}$$

must vanish on the set of points Z''.

The remarkable property of the polynomial (30) is this:

LEMMA 1. *Up to a multiplicative factor which is independent of* z_1 *and* z_2, *the polynomial* (30) *is the unique polynomial of degree* Δ'_2 *in* z_1 *and* z_2 *which possesses the symmetry* (22) *and which vanishes on the set of points* (29).

The proof of this result is given in Appendix B. Let us illustrate the result by giving two examples:

Example 1. $\Delta'_2 = 1$. The polynomial in question has the form $Az_1 + Bz_2 + C$, and it must vanish on the point $(-\Delta_2, 0)$ from the set (29) and on the point $(0, -\Delta_1)$ because of the symmetry (22). These two conditions require that $A = \Delta_1 D$, $B = \Delta_2 D$, $C = \Delta_1\Delta_2 D$, where D is arbitrary, that is, the solution is proportional to

$$\Delta_1 z_1 + \Delta_2 z_2 + \Delta_1\Delta_2 \ . \tag{31a}$$

Example 2. $\Delta'_2 = 2$. The polynomial in question has the form $Az_1^2 + Bz_2^2 + Cz_1z_2 + Dz_1 + Ez_2 + F$. It must vanish on the six points

$$(-\Delta_2 + 1, 0) \qquad (0, -\Delta_1 + 1)$$

and

$$(-\Delta_2 + 1, 1) \quad (-\Delta_2, 0) \qquad (1, -\Delta_1 + 1) \quad (0, -\Delta_1)$$

where the first three points come from the set (30) and the second three points come from the symmetry (22). The six equations for the six unknowns $A, B, \ldots, F$ yield the information that the polynomial is proportional to

$$\Delta_1(\Delta_1 - 1)\, z_1(z_1 - 1) + \Delta_2(\Delta_2 - 1)\, z_2(z_2 - 1) + \Delta_1(\Delta_1 - 1)\, \Delta_2(\Delta_2 - 1)$$
$$+ 2\Delta_1\Delta_2 z_1 z_2 + 2\Delta_1(\Delta_1 - 1)\, \Delta_2 z_1 + 2\Delta_1\Delta_2(\Delta_2 - 1)\, z_2 \ . \tag{31b}$$

The generalization of Lemma 1 may now be stated:

LEMMA 2. *Up to a multiplicative factor which is independent of* z_1 *and* z_2, *the polynomial* $P_k(\Delta_1\Delta_2\Delta'_1\Delta'_2; z_1 z_2)$ *is the unique polynomial of degree* $k = \min(\Delta_1, \Delta_2, \Delta'_1, \Delta'_2)$ *in* z_1 *and* z_2 *which has the symmetry* (22) *and which vanishes on the set of points belonging to* Z''.

The proof may be given by a method similar to that of Appendix B, and we omit it.

It is instructive to give an example of Lemma 2 (other than for Case 1) to illustrate how one utilizes the symmetry (22). Consider Case 2 with $\Delta_2 = 1$: then $\Delta_1 \geqslant \Delta_1' \geqslant 1, \Delta_1 \geqslant \Delta_2' \geqslant 1$. The polynomial we seek is of the form

$$P_1(\Delta_1 1 \Delta_1' \Delta_2';\, z_1 z_2) = A z_1 + B z_2 + C \ , \tag{32}$$

and it must vanish on the point $(-1,0)\,[Z'' = Z = (-1,0)]$. In order to find a second point where the linear form (32) vanishes, we consider the symmetry

$$P_1(\Delta_1 1 \Delta_1' \Delta_2',\, z_1 z_2) = P_1(1 \Delta_1 \Delta_1' \Delta_2';\, z_2 z_1) \ . \tag{33}$$

The linear form on the right-hand side of this equation falls into the category under Case 3 having $\Delta_1 = 1$. Thus, we find that under the conditions of Case 3 the linear form $P_1(1\Delta_2\Delta_1'\Delta_2';\, z_1 z_2)$ must vanish at the point $(-\Delta_1', \Delta_2' - 1)$. Renaming Δ_2 to be Δ_1, z_1 to be z_2, and z_2 to be z_1, we see that $P_1(1\Delta_1\Delta_1'\Delta_2';\, z_2 z_1)$ must vanish at $(z_2, z_1) = (-\Delta_1', \Delta_2' - 1)$, i.e., at $(z_1, z_2) = (\Delta_2' - 1, -\Delta_1')$, which from Eq. (33) is a second point where the linear form (32) must vanish. The vanishing of the linear form (32) at the two points $(-1, 0)$ and $(\Delta_2' - 1, -\Delta_1')$ requires that the form be proportional to

$$\Delta_1'(z_1 + 1) + \Delta_2' z_2 \ . \tag{34}$$

VI. CONCLUSIONS

Let us now summarize and interpret the results we have obtained in Secs. IV and V. Starting with the set of Wigner coefficients and the triangle relations, we developed the equivalent notions of a Wigner operator and its associated null space. We then demonstrated that the structural form, Eq. (20), had interesting properties.

The essential remark now is that this process may be completely reversed: *the structural form given in* Eq. (20) *is abstractly deducible from general principles, including the degree and permutational symmetry of the polynomial* (permutational symmetry always refers to the symmetry under the simultaneous exchanges $\Delta_1 \leftrightarrow \Delta_2$ and $z_1 \leftrightarrow z_2$). Since the polynomial is then uniquely determined (up to an unimportant factor) by the zeroes it possesses in consequence of the null space, and since the denominator is determined by normalization, we see that an $SU(2)$ Wigner operator is uniquely determined by its abstract structure and its null space.

More particularly, let us examine each of the three parts— $(\mathrm{NPCF})^{\frac{1}{2}}$ polynomial, and denominator— of Eq. (20). We have already given a complete discussion of the polynomial part. Consider the numerator pattern calculus factor part. While the presence (and explicit form) of this factor in Eq. (20) is deducible abstractly, it is also uniquely determined by permutational symmetry and the points belonging to the null space which correspond to final labels which fail to satisfy the betweenness conditions. Finally, consider the denominator part. Its square must be of degree $\Delta_1 + \Delta_2 = 2J$ in $z = z_2 - z_1 = 2j + 1$, contain the dimension $2j+1$ as a factor, vanish on the null space points $z \in \{1, 2, \ldots, \Delta_2\}$, and be invariant under $\Delta_1 \leftrightarrow \Delta_2$, $z \leftrightarrow -z$. Hence, except for numerical factors independent of z, the form of the denominator squared is

$$\left[D \begin{pmatrix} \Delta_1 & \Delta_2 \\ 2J & 0 \end{pmatrix} (z) \right]^2 = \# \frac{z}{z + \Delta_1 - \Delta_2} \begin{pmatrix} -z + \Delta_2 \\ \Delta_2 \end{pmatrix} \begin{pmatrix} z + \Delta_1 \\ \Delta_1 \end{pmatrix} \tag{35}$$

in agreement with Eq. (15b). Thus, the numerator pattern calculus factor and the denominator as well are essentially determined by the null space. (In the final form of a Wigner coefficient, the only factor undetermined is a numerical factor which is independent of z_1 and z_2, and this factor is determined by examining a special numerical case.) While of little interest to verify all the details of the properties outlined above, it is content of our interpretation that such a procedure exists in principle.

It is our belief that the null space concept is the proper concept to abstract from the known results of $SU(2)$ and $SU(3)$, and that it is this concept which may be the proper characterization of a general canonical Wigner operator in $U(n)$.

APPENDIX A: SYMMETRY OF THE POLYNOMIALS P_k

The main purpose of this appendix is to prove the symmetry relation, Eq. (22). Consider the following expression in which the ξ_i and z_i are indeterminates:

$$\left\{\xi_1 z_1 + \xi_2 z_2 + \xi_1 \xi_2\right\}^k / k! \quad . \tag{A.1}$$

The $\{\ldots\}$ bracket around the enclosed linear form in z_1 and z_2 symbolizes the following operations: Expand the form by the usual trinomial theorem, collect together the powers of each variable, and map each power ζ^a into $a! \binom{\zeta}{a}$. Thus, the explicit definition of the expression (A.1) is[24, 25]

$$\left\{\xi_1 z_1 + \xi_2 z_2 + \xi_1 \xi_2\right\}^k / k!$$

$$= \sum_{(k)} \frac{(k_1 + k_3)!\,(k_2 + k_3)!}{(k_3)!} \binom{\xi_1}{k_1 + k_3} \binom{\xi_2}{k_2 + k_3} \binom{z_1}{k_1} \binom{z_2}{k_2} \quad , \tag{A.2}$$

where the sum is over all non-negative integers $(k) = (k_1 k_2 k_3)$ such that $k_1 + k_2 + k_3 = k$.

Observe that the symmetry relation

$$\left\{\xi_1 z_1 + \xi_2 z_2 + \xi_1 \xi_2\right\}^k = \left\{\xi_2 z_2 + \xi_1 z_1 + \xi_2 \xi_1\right\}^k \tag{A.3}$$

is an obvious property of the definition (A.2).

One of the internal summations occurring in the right-hand side of Eq. (A.2) can be carried out by using the binomial addition theorem. There are two ways of doing this, leading to the following forms:

$$\oint \xi_1 z_1 + \xi_2 z_2 + \xi_1 \xi_2 \oint^{k} / k!$$

$$= \sum_{k_1 + k_2 = k} (k_1)!\,(k_2)! \begin{pmatrix} \xi_1 \\ k_1 \end{pmatrix} \begin{pmatrix} \xi_2 \\ k_2 \end{pmatrix} \begin{pmatrix} z_1 + \xi_2 - k_2 \\ k_1 \end{pmatrix} \begin{pmatrix} z_2 \\ k_2 \end{pmatrix}$$

$$= \sum_{k_1 + k_2 = k} (k_1)!\,(k_2)! \begin{pmatrix} \xi_1 \\ k_1 \end{pmatrix} \begin{pmatrix} \xi_2 \\ k_2 \end{pmatrix} \begin{pmatrix} z_1 \\ k_1 \end{pmatrix} \begin{pmatrix} z_2 + \xi_1 - k_1 \\ k_2 \end{pmatrix} \quad . \tag{A.4}$$

Notice that while the symmetry relation (A.3) is transparent in the expression (A.2), it is lost in the individual expressions occurring in the right-hand side of Eq. (A.4), but is regained through the equality of the two summation expressions in the right-hand side of Eq. (A.4).

Making the identifications $\xi_i = \Delta_i$, we now see that the summation part of Eq. (15a) is

$$\oint \Delta_1 z_1 + \Delta_2 z_2 + \Delta_1 \Delta_2 \oint^{\Delta'_2} / (\Delta'_2)! \quad , \tag{A.5}$$

which from Eq. (A.3) is invariant under the interchanges $\Delta_1 \leftrightarrow \Delta_2$, $z_1 \leftrightarrow z_2$.

By definition, the polynomial P_k is given by

$$P_k(\Delta_1 \Delta_2 \Delta'_1 \Delta'_2;\, z_1 z_2) = \oint \Delta_1 z_1 + \Delta_2 z_2 + \Delta_1 \Delta_2 \oint^{\Delta'_2} / (\Delta'_2)! \times$$

$$\times 1 \Big/ \prod_{i=1}^{2} (u_i)! \begin{pmatrix} z_i \\ u_i \end{pmatrix} \quad . \tag{A.6}$$

The symmetry relation, Eq. (22), follows directly from this expression.

APPENDIX B: EXPLICIT CONSTRUCTION OF THE POLYNOMIALS P_k FROM THEIR ZEROES

Lemma 1 is proved in this appendix. *The method of proof is by giving the explicit construction of the polynomials from their zeroes.* We prove below that the polynomial given by Eq. (A.4) is determined up to a multiplicative factor in ξ_1 and ξ_2 by the sets of zeroes $\{(z_1, z_2)\}$ given by

$$T_\alpha = \{(k - \xi_2 - \beta, \alpha) : \beta = 1, 2, \ldots, k - \alpha\} \ , \tag{B.1}$$

where $\alpha = 0, 1, \ldots, k-1$;

$$S_\alpha = \{(\alpha, k - \xi_1 - \beta) : \beta = 1, 2, \ldots, k - \alpha\} \ , \tag{B.2}$$

where $\alpha = 0, 1, \ldots, k-1$. Under the identifications $\xi_i = \Delta_i$ and $k = \Delta'_2$, the sets $T_0, T_1, \ldots, T_{k-1}$ contain precisely the points of Z'' given by Eq. (29), and the sets $S_0, S_1, \ldots, S_{k-1}$ contain the points obtained from the T_α by permutational symmetry. Thus, the proof of Lemma 1 follows from this somewhat more general result.

Let $P_k(z_1 z_2)$ denote a polynomial of (total) degree in k in z_1 and z_2 which vanishes on the sets T_α, S_α, $\alpha = 0, 1, \ldots, k-1$. The vanishing of $P_k(z_1 z_2)$ on the set of T_0 yields the result $P_k(z_1 0) = a_0(\xi_1 \xi_2)\begin{pmatrix} z_1 + \xi_2 \\ k \end{pmatrix}$, where $a_0(\xi_1 \xi_2)$ is arbitrary. Putting $P_k(z_1 z_2) = P_k(z_1 0) + z_2 Q_{k-1}(z_1 z_2)$, where Q_{k-1} is a polynomial of degree not greater than $k-1$, we obtain $P_k(z_1 1) = Q_{k-1}(z_1 1) = 0$ on the set T_1, and therefore $Q_{k-1}(z_1 1) = a_1(\xi_1 \xi_2)\begin{pmatrix} z_1 + \xi_2 - 1 \\ k-1 \end{pmatrix}$. Putting $P_k(z_1 z_2) = P_k(z_1 0) + z_2 Q_{k-1}(z_1 1) + z_2(z_2 - 1)\, Q_{k-2}(z_1 z_2)$, where Q_{k-2} is of degree not greater than $k-2$, we obtain $P_k(z_1 2) = 2Q_{k-2}(z_1 2) = 0$ on the set T_2, that is, $2Q_{k-2}(z_1 2) = a_2(\xi_1 \xi_2)\begin{pmatrix} z_1 + \xi_2 - 2 \\ k-2 \end{pmatrix}$. Continuing this procedure in an obvious manner, we come to the conclusion that the most general polynomial of degree k which vanishes on the points in the sets $T_0, T_1, \ldots, T_{k-1}$ is

$$P_k(z_1 z_2) = \sum_{s=0}^{k} a_s(\xi_1 \xi_2) \begin{pmatrix} z_1 + \xi_2 - s \\ k - s \end{pmatrix} \begin{pmatrix} z_2 \\ s \end{pmatrix} \quad , \tag{B.3}$$

in which the $a_s(\xi_1 \xi_2)$ are arbitrary.

We now repeat the argument using the points of the sets $S_0, S_1, \ldots, S_{k-1}$ to come to the conclusion that

$$P_k(z_1 z_2) = \sum_{s=0}^{k} b_s(\xi_1 \xi_2) \begin{pmatrix} z_1 \\ s \end{pmatrix} \begin{pmatrix} z_2 + \xi_1 - s \\ k - s \end{pmatrix} \quad , \tag{B.4}$$

in which the $b_s(\xi_1 \xi_2)$ are arbitrary.

The two expressions (B.3) and (B.4) must agree identically in z_1 and z_2. Setting $z_2 = 0$ and equating the expressions gives

$$\sum_{s=0}^{k} b_s(\xi_1 \xi_2) \begin{pmatrix} \xi_1 - s \\ k - s \end{pmatrix} \begin{pmatrix} z_1 \\ s \end{pmatrix} = a_0(\xi_1 \xi_2) \begin{pmatrix} z_1 + \xi_2 \\ k \end{pmatrix} \quad . \tag{B.5}$$

We now set $z_1 = 0, 1, 2, \ldots,$ in turn, in Eq. (B.5) to obtain a triangular system of equations which uniquely yields

$$a_0(\xi_1 \xi_2) = a(\xi_1 \xi_2) \begin{pmatrix} \xi_1 \\ k \end{pmatrix} \quad , \tag{B.6}$$

$$b_s(\xi_1 \xi_2) = a(\xi_1 \xi_2)\, s!\,(k-s)! \begin{pmatrix} \xi_1 \\ s \end{pmatrix} \begin{pmatrix} \xi_2 \\ k - s \end{pmatrix} \quad , \tag{B.7}$$

in which $a(\xi_1 \xi_2)$ is arbitrary. Using this result in Eq. (B.4), we obtain the second expression in the right-hand side of Eq. (A.4), multiplied by $a(\xi_1 \xi_2)$.

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RESUMEN

Los grupos unitarios tienen amplias aplicaciones en física; aún así, los elementos (coeficientes de Wigner) de la matriz unitaria que reduce el producto directo de dos representaciones irreducibles unitarias, permanecen indeterminados en su mayoría, a pesar de su considerable importancia y de los esfuerzos de muchos investigadores por calcularlos. La existencia de dos problemas ayuda a explicar el porqué: (a) el principio general [para $U(n)$] que distingue entre las ocurrencias múltiples de una representación irreducible no se ha cubierto completamente; (b) cálculos explícitos, aún cuando (a) se ha resuelto como es el caso para $n = 3$, son excesivamente difíciles. Se puede obtener comprensión en estos dos problemas a través de los conceptos de un operador de Wigner y su espacio nulo. Este artículo explica estos conceptos dentro del marco de $SU(2)$ y demuestra que un operador de Wigner de $SU(2)$ está determinado por su forma estructural abstracta y su espacio nulo. Se sugiere que el concepto de espacio nulo es el apropiado para usarse en la caracterización de un operador de Wigner, canónico general en $U(n)$.