LINEARIZED MATTER SYMMETRIES AND NON LINEAR COUNTEREXAMPLES OF EQUIVALENCE TO ISOMETRY IN GENERAL RELATIVITY

R. Berezdivin

Departamento de Física, Facultad de Ciencias
Universidad Central de Venezuela, Caracas, Venezuela

(Recibido: noviembre 15, 1974)

ABSTRACT: The extent to which various symmetries of source terms in General Relativity impose isometries is of interest. Several linearized cases of matter symmetries are shown to give rise to isometries. However several types of fluid and null electrovac symmetries in the full theory are shown not to be equivalent to isometries.

I. INTRODUCTION

It is of interest in General Relativity the extent to which various symmetries of source terms impose symmetries on the gravitational field, isometries on the space-time. The problem was introduced for a kinetic theory source term[1] and the name "matter symmetry" is here generalized for other sources and different symmetries or collineations[2]. That this would be of interest is evidenced by, for example, the degree of isotropy implied on space-time by the observed isotropy of blackbody radiation and galaxies[3]. This isotropy problem, besides being nontrivial, points to the need for a linearized treatment of, generally speaking, matter symmetries. In Sec. II we treat various different linearized matter symmetries: in Sec. A we treat the case of a kinetic theory matter symmetry which in electromagnetic theory imposes a symmetry on the electromagnetic field tensor, mainly in order to similarly treat, in Sec. B, the weak, or linearized, gravitational field. In Sec. C we see the isometry imposed by a symmetry of the stress-energy tensor, with boundary conditions, all linearized off an isometric space-time. In Sec. III se see various cases where impositions of matter, or other symmetries in the full nonlinear theory does not give rise to isometries on space-time: Sec. A uses a conformal technique to find cases where the Ricci, stress-energy, and other tensors or objects are symmetric ($\mathfrak{L}_\eta R_{\alpha\beta} = 0$, $\mathfrak{L}_\eta T_{\alpha\beta} = 0$, and others) but not so the metric (η is not Killing and there does not exist a scaling which makes it Killing). Sec. B shows that in electrovac space-time, specifically when null, an isometry and a symmetry on the electromagnetic field tensor are not equivalent so that the field is not the looked for "good" matter symmetry. The conclusion is presented in Sec. IV.

II. LINEARIZED MATTER SYMMETRIES

A. Kinetic Matter Symmetry in Electromagnetism

The distribution function f of particles with charge scaled to 1 and variable mass produces an electric current density on the space-time point x^α by, with no gravity,

$$J^\alpha = \int p^\alpha f(x^\alpha, p^\alpha)\, d^4p \quad . \tag{1}$$

The current density produces an electromagnetic field given by Maxwell's equations,

$$F^{\alpha\beta}{}_{,\beta} = -4\pi J^\alpha \tag{2a}$$

and

$$F_{\alpha\beta} = 2A_{[\alpha,\beta]} \quad . \tag{2b}$$

In the Lorentz gauge

$$A^{\alpha}{}_{,\alpha} = 0 \tag{3}$$

we get

$$\Box A^{\alpha} = -4\pi J^{\alpha} \quad , \tag{4}$$

and then, using the retarded Green's function

$$A^{\alpha}(x^{\beta}) = \int J^{\alpha}(x', t')\big|_{t_R} R^{-1} d^3x' \quad , \tag{5a}$$

where

$$R = |x - x'| \tag{5b}$$

and

$$t_R = t + R \quad , \tag{5c}$$

and where we shall write x instead of $\mathbf{x}$. The behavior of f is governed by the Liouville equation[4], for a collisionless fluid with no correlations,

$$Lf = p^{\alpha}(\partial f/\partial x^{\alpha}) + F^{\alpha}{}_{\beta}p^{\beta}(\partial f/\partial p^{\alpha}) = 0, \tag{6}$$

and a matter symmetry is given by (η^{α} is a vector field which translates to the space time point where the observed f is unchanged, and $A_{\alpha\beta}$ rotates the space time axis at that point. See Ref. 5 for a rigorous definition.)

$$Wf = \eta^{\alpha}(\partial f/\partial x^{\alpha}) + A^{\alpha}{}_{\beta}p^{\beta}(\partial f/\partial p^{\alpha}) = 0, \quad A_{(\alpha\beta)} = 0 \quad . \tag{7}$$

We will show that (7) implies, with restrictions on $A_{\alpha\beta}$ due to the fact that we work in flat space time (see later), that

$$\mathcal{L}_{\eta} A^{\alpha} = 0 \, , \tag{8}$$

and this will imply

$$\mathcal{L}_{\eta} F_{\alpha\beta} = 0 , \tag{9}$$

which is gauge invariant. From (7) one may derive[6] that

$$J^{\alpha}_{,\beta}\eta^{\beta} - A^{\alpha}{}_{\beta}J^{\beta} = 0 , \tag{10}$$

and from (6) one may get charge conservation

$$J^{\alpha}_{,\alpha} = 0 . \tag{11}$$

Now, J^{α} is a function of x^{j} and t, $J^{\alpha'}$ will be used to denote $J^{\alpha}(x^{j'}, t+R)$ which is just $J^{\alpha}(x^{j'}, t_R)$, $J^{\alpha'}_{,j'}$ will mean $\partial J^{\alpha}(x^{j'}, t_R)/\partial x^{j'}\big|_{t}$ while $J^{\alpha'}_{,j'}\big|_{t_R}$ will mean $\partial J^{\alpha}(x^{j'}, t_R)/\partial x^{j'}\big|_{t_R}$ where $\big|_{t_R}$ means "with fixed t_R"), so that remembering Eq. (5b) one will obtain

$$J^{\alpha'}_{,j'} \equiv J^{\alpha'}_{,j'}\big|_{t} = (\partial J^{\alpha'}/\partial x^{j'})\big|_{t_R} + (\partial J^{\alpha'}/\partial t_R)\, R_{,j'} \tag{12a}$$

and

$$J^{\alpha'}_{,j} \equiv J^{\alpha'}_{,j}\big|_{t} = (\partial J^{\alpha'}/\partial t_R)\, R_{,j} = -(\partial J^{\alpha'}/\partial t_R)\, R_{,j'} \quad , \tag{12b}$$

so, one has

$$(\partial J^{\alpha'}/\partial x^{j'})\big|_{t_R} = J^{\alpha'}_{,j} + J^{\alpha'}_{,j'} \quad , \tag{12c}$$

and then

$$A^{\alpha}_{,j} = \int J^{\alpha'}_{,j} R^{-1} d^3x' - \int J^{\alpha'} R_{,j} R^{-2} d^3x' = \int J^{\alpha'}_{,j} R^{-1} d^3x' +$$

$$+ \int J^{\alpha'} R_{,j'} R^{-2} d^3x' = \int J^{\alpha'}_{,j} R^{-1} d^3x' + \int J^{\alpha'}_{,j'} R^{-1} d^3x' \quad , \tag{12d}$$

so that

$$A^{\alpha}_{\ ,j} = \int (\partial J^{\alpha'}/\partial x^{j'})\big|_{t_R} R^{-1} d^3x' \ , \tag{13}$$

where we have integrated by parts and assumed J^{α} to vanish sufficiently fast at spatial infinity. Also, clearly,

$$A^{\alpha}_{\ ,0} = \int J^{\alpha'}_{\ ,0} R^{-1} d^3x' = \int J^{\alpha'}_{\ ,t_R} R^{-1} d^3x' \ . \tag{14}$$

Integrating (10) we obtain

$$\int [(\partial J^{\alpha'}/\partial x^{i'})\big|_{t_R} \eta^{i'}(t_R) + (\partial J^{\alpha'}/\partial t_R)\eta^{0'}(t_R)]\, R^{-1} d^3x' = \int A^{\alpha'}_{\ \beta'} J^{\beta'} R^{-1} d^3x' \ , \tag{15}$$

so that if $A^{\alpha'}_{\ \beta'} = A^{\alpha}_{\ \beta}$ (thus, if the $A^{\alpha}_{\ \beta}$ are constant) and if

$$\int (\eta^{j'}(t_R) - \eta^{j}))\, R^{-1} (\partial J^{\alpha'}/\partial x^{j'})\big|_{t_R} d^3x' + \int J^{\alpha'}_{\ ,t_R} R^{-1} (\eta^{0'}(t_R) - \eta^{0})\, d^3x' = 0 \tag{16}$$

we get

$$A^{\alpha}_{\ ,\beta}\eta^{\beta} - \eta^{\alpha}_{\ ,\beta} A^{\beta} = 0 \tag{17}$$

We now show that (16) does follow for the lift[5] of the inhomogeneous Lorentz group (ILG) where the $A^{\alpha}_{\ \beta}$ are constant. For the ILG,

$$\begin{aligned} &\eta^0 = 0 \ , \ \eta^j = \delta^j_i && \text{for space translations in the } i \text{ axis,} \\ &\eta^0 = 1 \ , \ \eta^j = 0 && \text{for time translations,} \\ \text{and} \quad &\eta^{\alpha} = \epsilon^{\alpha}_{\ \beta} x^{\beta} && \text{for space-time rotations,} \end{aligned} \tag{18}$$

with $\epsilon_{\alpha\beta}$ constant and antisymmetric. Thus $A^{\alpha}_{\ \beta} = \epsilon^{\alpha}_{\ \beta}$ or 0, and always

$A^{\alpha}{}_{\beta} = \eta^{\alpha}{}_{,\beta}$, and then (17) gives[7,8],

$$\mathfrak{L}_{\eta} A^{\alpha} = 0 \ .$$

To show (16) we consider three cases:

1. Translations in space-time: since η^{α} = constant, (16) holds.
2. Space rotations: $\eta^{j'} - \eta^{j} = \epsilon^{j}{}_{i}(x^{i'} - x^{i})$ and $\eta^{0} = 0$.
   Then from equations (12),

$$\int (\eta^{j'} - \eta^{j})\, R^{-1} (\partial J^{\alpha'}/\partial x^{j'})\big|_{t_R} d^3x' =$$

$$= \int \epsilon^{j}{}_{i}(x^{i'} - x^{i})\, R^{-1} (-J^{\alpha'}{}_{,t_R} R_{,j'} + J^{\alpha'}{}_{,j'})\, d^3x'$$

$$= \int \{ \epsilon_{ji}(x^{i'} - x^{i}) R^{-1} [-J^{\alpha'}{}_{,t_R}(x^{j'} - x^{j}) R^{-1}] - \epsilon^{j}_{i} [(x^{i'} - x^{i}) R^{-1}]_{,j} J^{\alpha'} \}\, d^3x'$$

$$= \int [-\epsilon^{j}{}_{i}(x^{i'} - x^{i}) R^{-1} J^{\alpha'}{}_{,t_R}(x^{j'} - x^{j}) - \epsilon_{ji}(x^{i'} - x^{i})(x^{j'} - x^{j}) R^{-3} J^{\alpha'}]\, d^3x' = 0,$$

where again we have integrated by parts and where $\epsilon_{ij} x^i x^j = 0$ since $\epsilon_{(ij)} = 0$.

3. Boosts: for simplicity we do it along the z axis. With 2, this may be rotated in any direction. Then, $\eta^{z} = x^0$, $\eta^0 = z$, $\epsilon_{01} = 1$ (with signature -2), and (16) may be written as

$$\int x^{0'} - x^{0}\big|_{x^{0'} = t_R} R^{-1} (\partial J^{\alpha'}/\partial z')\big|_{t_R} d^3x' + \int J^{\alpha'}{}_{,t_R}(z' - z) R^{-1} d^3x' =$$

$$= \int [(\partial J^{\alpha'}/\partial z')\big|_{t_R} + J^{\alpha'}{}_{,t_R} R_{,i'}]\, d^3x' = \int J^{\alpha'}{}_{,z'}\, d^3x' = 0 \ ,$$

where we have used (12a) and integrated by parts.

Finally, if $\mathfrak{L}_{\eta} A^{\alpha} = 0$, we may commute $\mathfrak{L}_{\eta}$ with covariant derivation[7] to get

$$0 = (\mathcal{L}_\eta A_\alpha)_{;\beta} - (\mathcal{L}_\eta A_\beta)_{;\alpha} = \mathcal{L}_\eta (A_{\alpha;\beta} - A_{\beta;\alpha}) = \mathcal{L}_\eta (A_{\alpha,\beta} - A_{\beta,\alpha}) = \mathcal{L}_\eta F_{\alpha\beta}$$

as announced in (19). The conclusion is that an electromagnetic matter symmetry (7) which is the lift of the ILG gives rise to an electromagnetic field symmetry (9). As before, for a Newtonian matter symmetry[1], it was necessary to impose that W be the lift of the symmetry group of space-time. This new result could, for instance, be applied to a dilute plasma where it would assert that a "symmetric" charge distribution implies the existence of the same "symmetry" in the electromagnetic field.

B. Kinetic Matter Symmetry in Weak Gravitation

With $g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon h_{\alpha\beta}$, defining $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h$ and choosing the harmonic gauge $\bar{h}^{\alpha}{}_{\beta,\alpha} = 0$, the field equations are[9],

$$\bar{h}^{\alpha\beta}{}_{,}{}^{\gamma}{}_{,\gamma} = -4\pi k \int f p^\alpha p^\beta d^4 p = -4\pi k T^{\alpha\beta} \quad , \tag{20}$$

so that

$$\bar{h}^{\alpha\beta} = k \int T^{\alpha\beta}(x', t_R) R^{-1} d^3 x' \quad . \tag{21}$$

Wf = here implies[1]

$$\mathcal{L}_\eta T_{\alpha\beta} - (A_\alpha{}^\gamma - \eta_{\alpha,}{}^\gamma)\, T_{\gamma\beta} - (A_\beta{}^\gamma - \eta_{\beta,}{}^\gamma)\, T_{\beta\gamma} = 0 \ , \tag{22}$$

which is

$$\mathcal{L}_\eta T_{\alpha\beta} = 0 \tag{23}$$

if W is the lift of the ILG. We then proceed as before, using

$$\mathcal{L}_\eta T^{\alpha\beta} = T^{\alpha\beta}{}_{,\gamma}\eta^\gamma - T^{\alpha\gamma}\eta^\beta{}_{,\gamma} - T^{\gamma\beta}\eta^\alpha{}_{,\gamma} \ , \tag{24}$$

and integrating,

$$0 = \int (\mathcal{L}_\eta T^{\alpha\beta})(x', t_R) R^{-1} d^3x' =$$

$$= \int [T^{\alpha'\beta'}{}_{,\gamma'}\eta^{\gamma'} - T^{\alpha'\gamma'}\eta^{\beta'}{}_{,\gamma'} - T^{\gamma'\beta'}\eta^{\alpha'}{}_{,\gamma'}]_{t_R} R^{-1} d^3x'$$

$$= \int R^{-1}[\partial T^{\alpha'\beta'}/\partial x^{i'}|_{t_R}\eta^{i'}(t_R) + (\partial T^{\alpha'\beta'}/\partial t_R)\eta^{0'}(t_R)]\, d^3x' +$$

$$- \eta^\beta{}_{,\gamma}\int T^{\alpha'\gamma'}(t_R) R^{-1} d^3x' - \eta^\alpha{}_{,\gamma}\int T^{\gamma'\beta'}(t_R) R^{-1} d^3x'$$

since $\eta^\alpha{}_{,\beta}$ is constant; hence we may proceed as in Sec. A to conclude

$$\mathcal{L}_\eta \overline{h}^{\alpha\beta} = 0 \ , \tag{25}$$

and since $h = h^{\alpha\beta}\eta_{\alpha\beta}$,

$$\mathcal{L}_\eta h^{\alpha\beta} = 0 \ . \tag{26}$$

This result, as in Sec. A, is not gauge invariant. With a gauge transformation $x^{\alpha'} = x^\alpha - \epsilon\sigma^\alpha$

$$h'_{\alpha\beta} = h_{\alpha\beta} + \epsilon\mathcal{L}_\sigma \eta_{\alpha\beta} \ , \tag{27}$$

and one gets[8]

$$\mathcal{L}_\eta h'_{\alpha\beta} = \epsilon\mathcal{L}_{[\eta,\sigma]}\eta_{\alpha\beta} \ . \tag{28}$$

Since f is of order ϵ, from (7) it may be seen that η^α is only defined to zero order so that one may add to it any vector field q^α,

$$\hat{\eta}^\alpha = \eta^\alpha + \epsilon q^\alpha \ , \tag{29}$$

and then, choosing q^α by

$$q^{a} = -[\eta,\sigma]^{a} \tag{30}$$

one obtains, using (29), $\mathcal{L}_{\eta}\eta_{\alpha\beta}$, and (28),

$$\mathcal{L}_{\hat{\eta}}g_{\alpha\beta} = \mathcal{L}_{\hat{\eta}}(\eta_{\alpha\beta} + \epsilon h'_{\alpha\beta}) = \mathcal{L}_{\eta}\eta_{\alpha\beta} + \epsilon[\mathcal{L}_{\eta}h'_{\alpha\beta} + \mathcal{L}_{q}\eta_{\alpha\beta}] =$$
$$= \epsilon[\mathcal{L}_{[\eta,\sigma]} + q\,\eta_{\alpha\beta}] = 0 \quad . \tag{31}$$

Thus, the first order terms of η^{α} may be used to eliminate a gauge transformation, and viceversa. We can again conclude that a matter symmetry implies an isometry if W is, to zero order, the lift of the ILG. Although it is not clear it seems to us that without the restriction to the ILG the result would not follow.

C. Linearized Ricci Collineation Off Non-flat Space-time

Using the Green's functional technique of Gilman and Sciama[10, 11] we shall see that a linearized $T^{\mu}{}_{\nu}$ symmetry will give rise, if we impose some boundary conditions, to an isometry. The Green's functional technique is particularly appropriate for the treatment of perturbations, that is, linearizations of possibly non-flat space-time, unlike A and B where the base space was Minkowsky. They have $\hat{g}^{\alpha\beta} = g^{\alpha\beta} + \delta g^{\alpha\beta}$ and a gauge such that, raising and lowering with $g_{\alpha\beta}$,

$$D^{\mu\nu}{}_{\sigma\tau}\, g^{\sigma\tau} = -kT^{\mu\nu} \quad , \tag{32}$$

$$D^{\mu\nu}{}_{\sigma\tau}\delta g^{\sigma\tau} = -k\delta K^{\mu\nu} \quad , \tag{33}$$

with

$$\delta K^{\mu\nu} = \tfrac{1}{2}[g^{\mu\lambda}(\hat{T}^{\nu}{}_{\lambda} - \tfrac{1}{2}T\delta^{\nu}_{\lambda}) + g^{\nu\lambda}(\hat{T}^{\mu}{}_{\lambda} - \tfrac{1}{2}\hat{T}\delta^{\nu}_{\lambda})] - (T^{\mu\nu} - \tfrac{1}{2}Tg^{\mu\nu}), \tag{34}$$

$$\hat{T}^{\mu}{}_{\nu} = T^{\mu}{}_{\nu} + \delta T^{\mu}{}_{\nu} \quad , \tag{35}$$

and

$$D_{\mu\nu\sigma\tau} = \tfrac{1}{4}(g_{\mu\sigma}g_{\nu\tau} + g_{\mu\tau}g_{\nu\sigma})\nabla^{\rho}\nabla_{\rho} + - \tfrac{1}{2}(R_{\mu\sigma\nu\tau} + R_{\mu\tau\nu\sigma}) \quad (36)$$

To first order the operator D has an elementary solution $E^{\alpha'\beta'}{}_{\mu\nu}(x', x)$, dependent on $g_{\alpha\beta}$ only, and integration then yields

$$g^{\alpha'\beta'} + \delta g^{\alpha'\beta'} = 2k\int_{r} E^{\alpha'\beta'}{}_{\mu\nu}(T^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}T + \delta K^{\mu\nu})\sqrt{-g}\,d^4x +$$

$$+ \int_{\partial r} g^{\rho\sigma}[E^{\alpha'\beta'}{}_{\mu\nu;\rho}(g^{\mu\nu} + \delta g^{\mu\nu}) - E^{\alpha'\beta'}{}_{\mu\nu}(\delta g^{\mu\nu})_{;\rho}]\sqrt{-g}\,dS_{\sigma} . \quad (37)$$

The equation is of course linear so that one obtains

$$\delta g^{\alpha'\beta'} = 2k\int_{r} E^{\alpha'\beta'}{}_{\mu\nu}\delta K^{\mu\nu}\sqrt{-g}\,d^4x + \int_{\partial r} g^{\rho\sigma}[E^{\alpha'\beta'}{}_{\mu\nu;\rho}\delta g^{\mu\nu} -$$

$$- E^{\alpha'\beta'}{}_{\mu\nu}(\delta g^{\mu\nu})_{;\rho}]\sqrt{-g}\,dS_{\sigma} \quad (38)$$

with

$$\delta K^{\mu\nu} = \tfrac{1}{2}[g^{\mu\lambda}(\delta T^{\nu}{}_{\lambda} - \tfrac{1}{2}\delta T^{\alpha}{}_{\alpha}\delta^{\nu}{}_{\lambda}) + g^{\nu\lambda}(\delta T^{\mu}{}_{\lambda} - \tfrac{1}{2}\delta T^{\alpha}{}_{\alpha})\delta^{\mu}{}_{\lambda}] . \quad (39)$$

It is immediately clear that if the perturbation is localized (there is some closed hypersurface ∂r where $\delta g^{\mu\nu}$ and $(\delta g^{\mu\nu})_{;\rho}$ are zero) non-matter ($\delta T^{\mu}{}_{\nu} = 0$) perturbations cannot exist. If for example we take ∂r to be a tube bounded at $t = t_0$, $t = \infty$ and at spatial infinity, and assuming that perturbations vanish at infinities fast enough, we simply have an initial value result for non-matter perturbations; that is, if we do not have the geometry perturbed at one time, and if we may assume that whatever perturbation there may be will die off fast enough at future infinity as well as spatial infinity, then there will not exist any non-matter perturbations of the space time geometry. In Sachs and Wolfe's paper[3], with $A=B=E=$ constant so as to eliminate density perturbations, if C_{μ}, $D_{\mu\nu}$ and its derivatives are zero at one time, then they will always remain zero.

Taking Lie derivatives with respect to η on equation (33), if $\mathcal{L}_{\eta}g_{\alpha\beta} = 0$, we get

$$D^{\mu\nu}{}_{\sigma\tau}(\mathcal{L}_\eta \delta g^{\sigma\tau}) = -k\,(\mathcal{L}_\eta \delta K^{\mu\nu}) \tag{40}$$

so that, since it has the same form as (33), the solution is

$$\mathcal{L}_{\eta'} \delta g^{\alpha'\beta'} = 2k \int_r E^{\alpha'\beta'}{}_{\mu\nu}(\mathcal{L}_\eta \delta K^{\mu\nu})\sqrt{-g}\,d^4x \,+$$

$$+\int_{\partial r} g^{\rho\sigma}\left[E^{\alpha'\beta'}{}_{\mu\nu;\rho}\mathcal{L}_\eta \delta g^{\mu\nu} - E^{\alpha'\beta'}{}_{\mu\nu}(\mathcal{L}_\eta \delta g^{\mu\nu})_{;\rho}\right]\sqrt{-g}\,dS_\sigma \,. \tag{41}$$

Then, with a linearized matter symmetry of the form

$$\mathcal{L}_{\eta+\delta\eta}(T^\mu{}_\nu + \delta T^\mu{}_\nu) = 0 \tag{42}$$

we must have to zero order

$$\mathcal{L}_\eta T^\mu{}_\nu = 0 \ ,$$

so that the sufficiency (but not the necessity) of η being a zero order isometry is exhibited, and to first order,

$$\mathcal{L}_\eta \delta T^\mu{}_\nu + \mathcal{L}_{\delta\eta} T^\mu{}_\nu = 0 \ . \tag{43}$$

However, $\mathcal{L}_{\delta\eta} T^\mu{}_\nu$ is a source solution to a gauge transformation[12] $\mathcal{L}_{\delta\eta} g^{\alpha\beta}$ so that if $\mathcal{L}_\eta \delta g^{\mu\nu} + \mathcal{L}_{\delta\eta} g^{\mu\nu}$ and its covariant derivatives are zero on ∂r, then they are zero on r. Hence a localized matter symmetry will, after gauge terms are transformed away, give an isometry,

$$\mathcal{L}_\eta \hat{g}^{\alpha\beta} = 0 \ . \tag{44}$$

A related problem, the initial values on $\mathcal{L}_\eta g^{\alpha\beta}$ necessary and sufficient to assure us of an isometry, for η non-lightlike, for analytic non-linear empty space-time, has been solved[13]. If some such boundary or initial conditions are not imposed in the full theory, a symmetry of $T^\mu{}_\nu$ or $T_{\alpha\beta}$ or a Ricci collineation will not generally imply an isometry.

III. NON-LINEAR EXAMPLES WHERE MATTER SYMMETRY IS NOT EQUIVALENT TO ISOMETRY

A. Fluid Examples with Conformal Techniques

We will find space-times with η not Killing and with

$$\mathfrak{L}_\eta T_{\mu\nu} = 0 \tag{45a}$$

and

$$\mathfrak{L}_\eta R_{\mu\nu} = 0 \ . \tag{45b}$$

From the Einstein equations we must then have in the space-time a conformal motion, so that with $R \equiv R(g) \equiv R^{\alpha}{}_{\alpha}(g_{\gamma\beta})$ we will have

$$\mathfrak{L}_\eta h_{\mu\nu} \equiv \mathfrak{L}_\eta (R g_{\mu\nu}) = 0 \ . \tag{46}$$

Then, from well known conformal equations[14],

$$R_{\alpha\beta}(g) = R_{\alpha\beta}(h) + 2\sigma_{\alpha\beta} + [\Delta_2\sigma + 2\Delta_1\sigma]\, h_{\alpha\beta} \ , \tag{47}$$

with

$$\sigma = -\tfrac{1}{2} \ln R, \ \ \exp 2\sigma = R^{-1} \ , \tag{48}$$

$$\Delta_1\sigma \equiv \sigma_{,\alpha}\sigma_{,\beta}h^{\alpha\beta} = \tfrac{1}{4}\, R_{,\alpha}R_{,\beta}R^{-2}h^{\alpha\beta} \ , \tag{49}$$

and

$$\Delta_2\sigma \equiv h^{\alpha\beta}\sigma_{,\alpha;\beta} = \tfrac{1}{2} h^{\alpha\beta}(R_{,\alpha}/R)_{;\beta} \ , \tag{50}$$

where metric operations are taken with respect to $h_{\alpha\beta}$, and

$$\sigma_{\alpha\beta} = \sigma_{,\alpha;\beta} - \sigma_{,\alpha}\sigma_{,\beta} \ . \tag{51}$$

Taking Lie derivatives on (47), with[2] $\mathcal{L}_\eta T'^{\alpha}_{\beta\gamma}(h) = \mathcal{L}_\eta R_{\alpha\beta}(h) = 0$,

$$\mathcal{L}_\eta R_{\alpha\beta}(g) = 2\,[(\mathcal{L}_\eta \sigma_{,\alpha})_{;\beta} - (\mathcal{L}_\eta \sigma_{,\alpha})(\mathcal{L}_\eta \sigma_{,\beta})] + [h^{\gamma\delta}(\mathcal{L}_\eta \sigma_{,\gamma})_{;\delta} +$$

$$+ 2h^{\gamma\delta}(\mathcal{L}_\eta \sigma_{,\gamma})(\mathcal{L}_\eta \sigma_{,\delta})]\, h_{\alpha\beta}\ . \tag{52}$$

We notice that a particularly simple way for $\mathcal{L}_\eta R_{\alpha\beta}(g)$ to be zero, with $\mathcal{L}_\eta \sigma \neq 0$, is for $\mathcal{L}_\eta \sigma_{,\alpha}$ to be zero. This will, by the way, make $\mathcal{L}_\eta T'^{\alpha}_{\beta\gamma}(g)$ also zero since it is homogeneous[15] in $\sigma_{,\alpha}$. In a local coordinate system adapted to η^α, $\eta^\alpha = \delta^\alpha_t$, and since $\mathcal{L}_\eta \sigma_{,\alpha} = (\mathcal{L}_\eta \sigma)_{,\alpha}$ we have

$$\dot{\sigma} = \text{constant} \tag{53}$$

so normalizing,

$$-\sigma = t + \tfrac{1}{2}\ln f(x^i)\ \ ,$$

and

$$R = \exp\,(2t)\, f(x^i) \tag{54}$$

One must only be sure that, with

$$g_{\mu\nu} = \exp\,(-2t)\, f^{-1} h_{\mu\nu}(x^i)\ \ , \tag{55}$$

$R(g)$ should equal $\exp\,(-2\sigma)$. From (47) and with[14]

$$\exp\,(-2\sigma) = R(g) = \exp\,(-2\sigma)\,[R(h) + 6(\Delta_2\sigma + \Delta_1\sigma)]\ \ , \tag{56}$$

we get[16]

$$T_{\alpha\beta}(g) = G_{\alpha\beta}(g) = T_{\alpha\beta}(h) + 2\sigma_{\alpha\beta} - h_{\alpha\beta}(2\Delta_2\sigma + \Delta_1\sigma)\ \ . \tag{57}$$

If $T_{\alpha\beta}(h)$ is due to a perfect fluid we then have

$$T_{\alpha\beta}(g) = [(\rho - 2\Delta_2\sigma - \Delta_1\sigma) + (p + 2\Delta_2\sigma + \Delta_1\sigma)] \exp(-2\sigma) \times$$

$$\times (\exp(\sigma)\mu_\alpha(h))(\exp(\sigma)\mu_\beta(h)) + [(p + 2\Delta_2\sigma + \Delta_1\sigma)\exp(-2\sigma)] g_{\alpha\beta} - 2\sigma_{\alpha\beta}(h) \tag{58}$$

with σ constrained by (56)(if it is not, (45a) will not hold but (45b) will).

That a class of solutions exists is demonstrated by taking, for $h_{\alpha\beta}$, the $k = 0$ Robertson–Walker metric. If we take $\eta^\alpha = \delta^\alpha_1 \equiv \delta^\alpha_x$ so $R(g) = \exp(2x)$, and with $\mu_\alpha = -\delta^0_\alpha$ (here sign. +2) we have

$$-\Delta_2\sigma = T^{\prime x}_{\alpha\beta} h^{\alpha\beta} = 0 \ , \tag{59}$$

and with $\sigma_{,\alpha} = -\delta^{\prime}_\alpha$ and $\Delta_1\sigma = h^{00}$ we get (with z, usually $\dot{R}(t)/R(t)$ here $\dot{P}(t)/P(t)$ to avoid confusion)

$$T_{\alpha\beta}(g) = (\tilde{\rho} + \tilde{p})\tilde{\mu}_\alpha\tilde{\mu}_\beta + \tilde{p} g_{\alpha\beta} - (4\dot{P}/P)\,\delta^0_{(\alpha}\delta^x_{\beta)} + 2\delta^x_\alpha\delta^x_\beta \quad , \tag{60}$$

with

$$\tilde{\rho} = (\rho + 1)\exp(2x)\ ,\ \tilde{p} = (p - 1)\exp(2x)\ ,\ \tilde{\mu}_\alpha = \delta^0_\alpha \exp(-x)\,.$$

To get this into a proper form[17] including recognizable heat conduction we define a tracefree stress tensor $\pi_{\alpha\beta}$ and heat conduction tensor q_α by

$$q_\alpha = (2\dot{P}/P)\,\delta^x_\alpha \exp(-x) \ , \tag{61}$$

and

$$\pi_{\alpha\beta} = q_\alpha q_\beta \tfrac{1}{2}\exp(2x)(P/\dot{P})^2 - 2/3\ g_{xx}(\tilde{\mu}_\alpha\tilde{\mu}_\beta + g_{\alpha\beta}) \ , \tag{62}$$

and redefine

$$\bar{p} = \tilde{p} - 2/3\, g_{xx} \ . \tag{63}$$

One then has trace $\pi = 0$, $\pi_{\alpha\beta}\tilde{\mu}^{\alpha} = 0$, and $q^{\alpha}\tilde{\mu}_{\alpha} = 0$. With $\exp f = P^2(t)$ we get for (56)

$$3f'' + f'^2 = R(h) = -6h^{xx} = -(6/P^2(t)) = 6 \exp(-f) \ . \tag{64}$$

One may solve (64) for P and then find ρ and p from the field equations for the Robertson-Walker metric[18]. For a proper initial choice of $P(t)$, from (63) p will be positive, and from (60) $\tilde{\rho} \gtrsim 3p$ if $\rho \gtrsim 3p$ (which can be assured by choosing $\dot{P}(t_0)$ such that $3(\dot{P}/P)^2 \gtrsim (6/P^2) - 5(\dot{P}/P)^2$, all in a neighborhood of t_0. It is not hard to check that there does not exist a scalar $r(x)$ such that $r(x)\,\eta$ is Killing so that scaling η will not help.

We have constructed a space-time where locally we have a non-perfect fluid which obeys $\mathcal{L}_{\eta} T'^{\alpha}_{\beta\gamma} = \mathcal{L}_{\eta} R_{\mu\nu} = \mathcal{L}_{\eta} T_{\mu\nu} = 0$ but which does not obey, even scaling η by a factor, $\mathcal{L}_{\eta} g_{\alpha\beta} = 0$. A similar technique has been used by Szekeres[19] to obtain what he calls "local fluids" in a Petrov type N space-time.

B. Other Examples

If in Robertson-Walker we choose $\eta^{\alpha} = \delta^{\alpha}_{\theta}$ we will have $\mathcal{L}_{\eta} g_{\alpha\beta} \neq 0$ but[18] $\mathcal{L}_{\eta} R^{\mu}{}_{\nu} = 0$ and $\mathcal{L}_{\eta} T^{\mu}{}_{\nu} = 0$. We also obtain then $\mathcal{L}_{\eta}\rho = \mathcal{L}_{\eta} p = \mathcal{L}_{\eta}\mu^{\alpha} = \mathcal{L}_{\eta}\mu_{\alpha} = 0$.

We see that, in the non-linear theory, a vector field symmetry on stress energy tensors, Ricci tensors, or macroscopic fluid properties, will not necessarily imply that the vector-field is an isometry.

C. Null Electrovac Symmetry

Recently, Woolley[20] has found that if η is Killing then $\mathcal{L}_{\eta} F_{\mu\nu}$ is zero or a duality rotation in the non-null case. Also recently[21], two different neutrino fields have been found possible in the same space-time. Not so recently Witten[22] constructed a family of null electromagnetic field tensors in the same space-time starting from a solution by Peres[23]. In his notation, $\tilde{w}^{\mu\nu} = w^{\mu\nu} \exp(i\alpha(z-t))$ describes a duality rotation, and since $\tilde{F}_{\mu\nu} = \mathrm{Re}\,(\tilde{w}_{\mu\nu})$ it will depend on $z-t$; however $g_{\mu\nu}$ does not.

A more general case is[5]

$$ds^2 = -dx^2 - dy^2 - 2du\,dv + H(x,y,u)\,du^2 \tag{65}$$

where $\eta^{\alpha} = \delta^{\alpha}_{3} \equiv \delta^{\alpha}_{v}$ is Killing, covariant constant, and null, and

$$G_{\alpha\beta} = R_{44}\delta^{4}_{\alpha}\delta^{4}_{\beta} \quad , \tag{66}$$

with

$$R_{44} = {}^{2}\nabla^{2}H \equiv (\partial^{2}H/\partial x^{2}) + (\partial^{2}H/\partial y^{2}) \ . \tag{67}$$

For a null electrovac field

$$T_{\alpha\beta} = F_{\gamma\alpha}F^{\gamma}{}_{\beta} \quad ,$$

and if $F_{\alpha\beta} = g\,[k_{\alpha}g_{\beta} - k_{\beta}g_{\alpha}]$ (which is $F_{\alpha\beta}$ up to a duality rotation[22]), with $k^{\alpha} = \eta^{\alpha}$ here and $g^{\alpha}k_{\alpha} = 0$, $g^{\alpha}g_{\alpha} = -1$, the Einstein equations reduce to

$${}^{2}\nabla^{2}H = g^{2}(x, y, u) . \tag{68}$$

The Maxwell equations reduce to solving, with $gg^{u} = h^{u}$,

$$0 = F^{\mu\nu}{}_{;\nu} = h^{\nu}{}_{;\nu}k^{\mu} - h^{\mu}{}_{;\nu}g^{\nu} = h^{\nu}{}_{,\nu}\delta^{\mu}_{3} - h^{\mu}{}_{,3} = {}^{2}\nabla \cdot \mathbf{h} \tag{69}$$

with $\mathbf{h} = (h_{1}, h_{2}) = \mathbf{h}(x, y, u)$, $h^{4} = 0$, and $g^{\alpha}g_{\alpha} = -1$ gives

$$(h^{1})^{2} + (h^{2})^{2} = g^{2} \ . \tag{70}$$

One may check that $F^{\alpha\beta}$ is independent of h^{3} and so $F^{\alpha\beta} = F^{\alpha\beta}(x, y, u)$. If we, however, perform a duality rotation through $\alpha(v)$, keeping the same metric, and changing coordinates to $v = z - t$, we see that $\alpha_{,t} - \alpha_{,z} = 0$, so that $\alpha(v)$ preserves the Maxwell equations as in Witten[22]. $F^{\mu\nu}$ would then depend on v and the metric would not. Witten's example is a special case with

$$g^{2} = 4A^{2}\cos^{2}k\,(z + t),\ h^{1} = h^{2} = \sqrt{2}\,A\cos k\sqrt{2}u \ .$$

It is perhaps noteworthy that Liouville's equation here requires the matter distribution f to be independent of v. For, with[5]

$$f = \delta(p^1)\,\delta(p^2)\,\delta(p^4)\,S(x^i, p^3),$$

$$Lf = p^\alpha \nabla_\alpha f = (\partial S/\partial v)p^3 + T'^{\alpha}_{\beta\gamma}(\partial f/\partial p^\alpha)p^\beta p^\gamma = p^3(\partial S/\partial v) +$$

$$+ T'^{3}_{33}(\partial S/\partial p^3)p^3 p^3 + p^3 p^3 T'^{i}_{33} S\,\delta'(p^i)\,\delta(p^j)\,\delta(p^k),$$

$(i,j,k) \neq 3$ and unequal. But $T'^{3}_{33} = 0$ for the metric above so that $\partial S/\partial v = 0$.

IV. CONCLUSION

Several well posed linearized matter symmetries give rise to isometries. There are, however, several types of fluid matter symmetries and null electrovac symmetries which are not equivalent to isometries. Further work is under way where several types of non-linear matter symmetries are examined.

REFERENCES

1. R. Berezdivin and R.K. Sachs, J. Math. Phys. 14 (1973) 1254.
2. G.H. Katzin, J. Levine and W.R. Davis, J. Math. Phys. 10 (1969) 617.
3. R.K. Sachs and A.M. Wolfe, Les Congres et Colloques de L'Université de Liege, Volume 41 (1967); also, J. Ehlers, P. Geren, and R.K. Sachs, J. Math. Phys. 9 (1968) 1344.
4. J. Ehlers, in *Relativitá Generale e Cosmologia*, Course 47 of the Int. School of Phys. "Enrico Fermi", ed. by R.K. Sachs, (Academic Press, New York 1971).
5. R. Berezdivin and R.K. Sachs, in *Relativity*, ed. by M. Carmeli, S.I. Fickler, and L. Witten, (Plenum Press, New York 1970).
6. R. Berezdivin, Ph.D. Thesis, Univ. of Calif., Berkeley (1972).
7. K. Yano, *The Theory of Lie Derivatives and Its Applications*, (North-Holland, Amsterdam 1955).
8. R.L. Bishop and S.I. Goldberg, *Tensor Analysis on Manifolds*, (MacMillan, New York 1968).

9. C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation*, preliminary version, U. of Maryland, College Park (1970).
10. D.W. Sciama, P.C. Waylen, and R.C. Gilman, Phys. Rev. 187 (1969) 1762.
11. R.C. Gilman, Phys. Rev. D2 (1970) 1400; D.W. Sciama, ICTP Preprint IC/73/94, Miramare-Trieste (1973). See also references in 10.
12. S. Weinberg, *Gravitation and Cosmology*, (Wiley, New York 1972).
13. R. Berezdivin, preprint Univ. Central de Venezuela, Depto. de Física (1973), to be submitted for publication.
14. A.Z. Petrov, *Einstein Spaces*, (Pergamon Press, New York 1969).
15. J.L. Synge, *Relativity: The General Theory*, (North-Holland, Amsterdam 1960).
16. Here with signature + 2 as in 15.
17. J.M. Stewart, *Non-Equilibrium Relativistic Kinetic Theory*, (Springer-Verlag, New York 1971).
18. R. Adler, M. Bazin, and M. Schiffer, *Intro. to General Relativity*, (McGraw-Hill, New York 1965).
19. P. Szekeres, J. Math. Phys. 7 (1966) 751.
20. M.L. Woolley, Commun. Math. Phys. 31 (1973) 75.
21. C.D. Collinson and P.B. Morris, J. of Phys. A6 (1973) 915.
22. L. Witten, *Gravitation: An Introduction to Current Research*, ed. by L. Witten, (Wiley, New York 1962).
23. A. Peres, Phys. Rev. 118 (1960) 1105.

RESUMEN

Es interesante el grado al que varias simetrías de términos fuente en Relatividad General, imponen isometrías. Se muestra que varios casos linearizados de simetrías de la materia dan lugar a isometrías. Sin embargo, se muestra que varios tipos de simetrías de fluido y de electrovac nulo en la teoría completa, no son equivalentes a isometrías.