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VARIATIONS OF MAGNETIC MIRRORS DETERMINED BY A SURFACE OF SECTION

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ABSTRACT: In many astrophysical and geophysical problems, the invariance of magnetic mirrors is correct only to first order. Their slow variations are important for maintaining radiation belts. They could be determined by a method inspired by the classical surface of section. The conditions for the existence of a double surface of section are generally fulfilled in the motion of charged particles in an axial magnetic field. It is then only necessary to determine the trajectories for the finite interval of time between two successive crossings of the surface of section and to infer the law of transformation of the surface of section. The slow variation of adiabatic invariants as well as the characteristics of the general motion can then be deduced. The dipole case and the method for determining the secular variations of the magnetic mirrors are studied especially.

ACKNOWLEDGEMENT

Some 35 years ago, Professor Vallarta accepted me as a research fellow in his department at MIT. Under his dynamic leadership, I studied the motion of cosmic-radiation particles in the vicinity of the earth, I was lucky also to discuss the problem of the motion of charged particles in the field of magnetic dipole with Professor G. Birkhoff. From both these inspirations came the idea that the magnetic equator of the earth was a surface of section. The war broke out and this work was never published. Afterwards engaged in other activities, I was unable to continue the study of the properties of the surface of section and in particular, the determination of the singular periodic orbits. Happily, I communicated my results to Professor R. de Vogelaere, he took over[1], refined my proofs and showed how useful this method could be in the Störmer problem. I thought it suitable on this solemn occasion to show my gratitude to Professor Vallarta by sending a paper concerning the same subject for a more general type of dynamical problem.

1. MAGNETIC FIELD WITH AXIAL SYMMETRY

For studying the dynamics of charged particles in an axially symmetric field, we write the Hamiltonian of the motion:

$$2H = p_R^2 + p_z^2 + \left[\frac{p_\phi}{R} - A_\phi\right]^2 = 1 \quad . \tag{1}$$

The mass of the charged particle has been put equal to 1. R, z, ϕ are the usual cylindrical coordinates and p_R, p_z, p_ϕ their conjugate momenta. A_ϕ, the only component of the vector potential different from zero, into which the electric charge of the particle is absorbed, is a function of R and z but not ϕ.

The Hamiltonian does not depend on time; we choose its constant value to be ½. With this choice of units, time intervals are numerically equal to arc lengths along the trajectories.

When the motion in the meridian plane R, z is known, the ignorable variable ϕ can be determined by the quadrature of the equation

$$R^2 \frac{d\phi}{ds} = p_\phi - RA_\phi = U(R, z) \quad . \tag{2}$$

Each line of force will lie in a meridian plane and be defined by U = constant and ϕ = constant. The square of the particle's speed in the meridian plane will obey the relation:

$$\left[\frac{dR}{ds}\right]^2 + \left[\frac{dz}{ds}\right]^2 = 1 - \frac{U^2}{R^2} \quad . \tag{3}$$

The second member of (3), never negative, limits the motions to an allowed region of the R, z plane defined by

$$|U| < R \quad . \tag{4}$$

Let us suppose that in that domain, except possibly at some points on the axis $R = 0$, U is twice continuously differentiable.

The motion in the meridian plane will be determined by the equations:

$$\frac{d^2R}{ds^2} = -\frac{U}{R}\frac{\partial}{\partial R}\left[\frac{U}{R}\right] \qquad \frac{d^2z}{ds^2} = -\frac{U}{R}\frac{\partial}{\partial z}\left[\frac{U}{R}\right] \quad . \tag{5}$$

Motions of charged particles in a magnetic field can be decomposed into:

1) a gyration about a guiding centre

2) a slow drift of this centre.

For this purpose, we must determine the curvature centre of the trajectory. By defining cartesian coordinates, $x = R\cos\phi$, $y = R\sin\phi$, it is possible to evaluate their second derivatives with respect to arc length as functions of x, y, z from the equations (2) and (5). From the Frenet relations, the value of the curvature is:

$$\frac{1}{\rho} = \sqrt{\left[\frac{d^2x}{ds^2}\right]^2 + \left[\frac{d^2y}{ds^2}\right]^2 + \left[\frac{d^2z}{ds^2}\right]^2} \quad ; \tag{6}$$

the coordinates of the centre of curvature are:

$$X = x + \rho^2 \frac{d^2x}{ds^2}, \quad Y = y + \rho^2 \frac{d^2y}{ds^2}, \quad Z = z + \rho^2 \frac{d^2z}{ds^2} \quad .$$

The centre of curvature at a point will not necessarily lie in the meridian plane of this point. Introducing instead of X, Y, Z, the cylindrical coordinates R_0, z_0, ϕ_0 one gets from (5) and (6)

$$\frac{1}{\rho^2} = \frac{1}{R^2} \left\{ \left[\frac{\partial U}{\partial R}\right]^2 + \left[\frac{\partial U}{\partial z}\right]^2 \right\}, \quad R_0 \sin(\phi - \phi_0) = -\frac{1}{R}\frac{dU}{dS} \quad , \quad (7)$$

$$z_0 = z - U\frac{\partial U}{\partial z} \Big/ \left\{ \left[\frac{\partial U}{\partial R}\right]^2 + \left[\frac{\partial U}{\partial z}\right]^2 \right\}, \quad R_0 = R - U\frac{\partial U}{\partial R} \Big/ \left\{ \left[\frac{\partial U}{\partial R}\right]^2 + \left[\frac{\partial U}{\partial z}\right]^2 \right\}. \quad (8)$$

In place of R, z it is sometimes convenient to use as coordinates lines $V =$ constant. Then by definition:

$$\left[\frac{\partial R}{\partial U}\right]_V \left[\frac{\partial R}{\partial V}\right]_U + \left[\frac{\partial z}{\partial U}\right]_V \left[\frac{\partial z}{\partial V}\right]_U = 0 \quad . \quad (9)$$

Consider now the differential relations

$$dR = \left[\frac{\partial R}{\partial V}\right]_U dV + \left[\frac{\partial R}{\partial U}\right]_V dU$$

$$dz = \left[\frac{\partial z}{\partial V}\right]_U dV + \left[\frac{\partial z}{\partial U}\right]_V dU \quad .$$

Multiplying the first one by $\left[\frac{\partial R}{\partial U}\right]_V$, the second one by $\left[\frac{\partial z}{\partial U}\right]_V$ one obtains by addition:

$$\left[\frac{\partial R}{\partial U}\right]_V dR + \left[\frac{\partial z}{\partial U}\right]_V dz = \left\{ \left[\frac{\partial R}{\partial U}\right]_V^2 + \left[\frac{\partial z}{\partial U}\right]_V^2 \right\} dU \quad .$$

Here, the orthogonality relation (9) has been taken into account. From this equation, we get:

$$\left[\frac{\partial U}{\partial R}\right]_z = \left[\frac{\partial R}{\partial U}\right]_V \Big/ \left\{\left[\frac{\partial R}{\partial U}\right]_V^2 + \left[\frac{\partial z}{\partial U}\right]_V^2\right\}$$

and

$$\left[\frac{\partial U}{\partial z}\right]_R = \left[\frac{\partial z}{\partial U}\right]_V \Big/ \left\{\left[\frac{\partial R}{\partial U}\right]_V^2 + \left[\frac{\partial z}{\partial U}\right]_V^2\right\}$$

and finally $R_0 = R - U \left[\frac{\partial R}{\partial U}\right]_V , \; z_0 = z - U \left[\frac{\partial z}{\partial U}\right]_V$. (10)

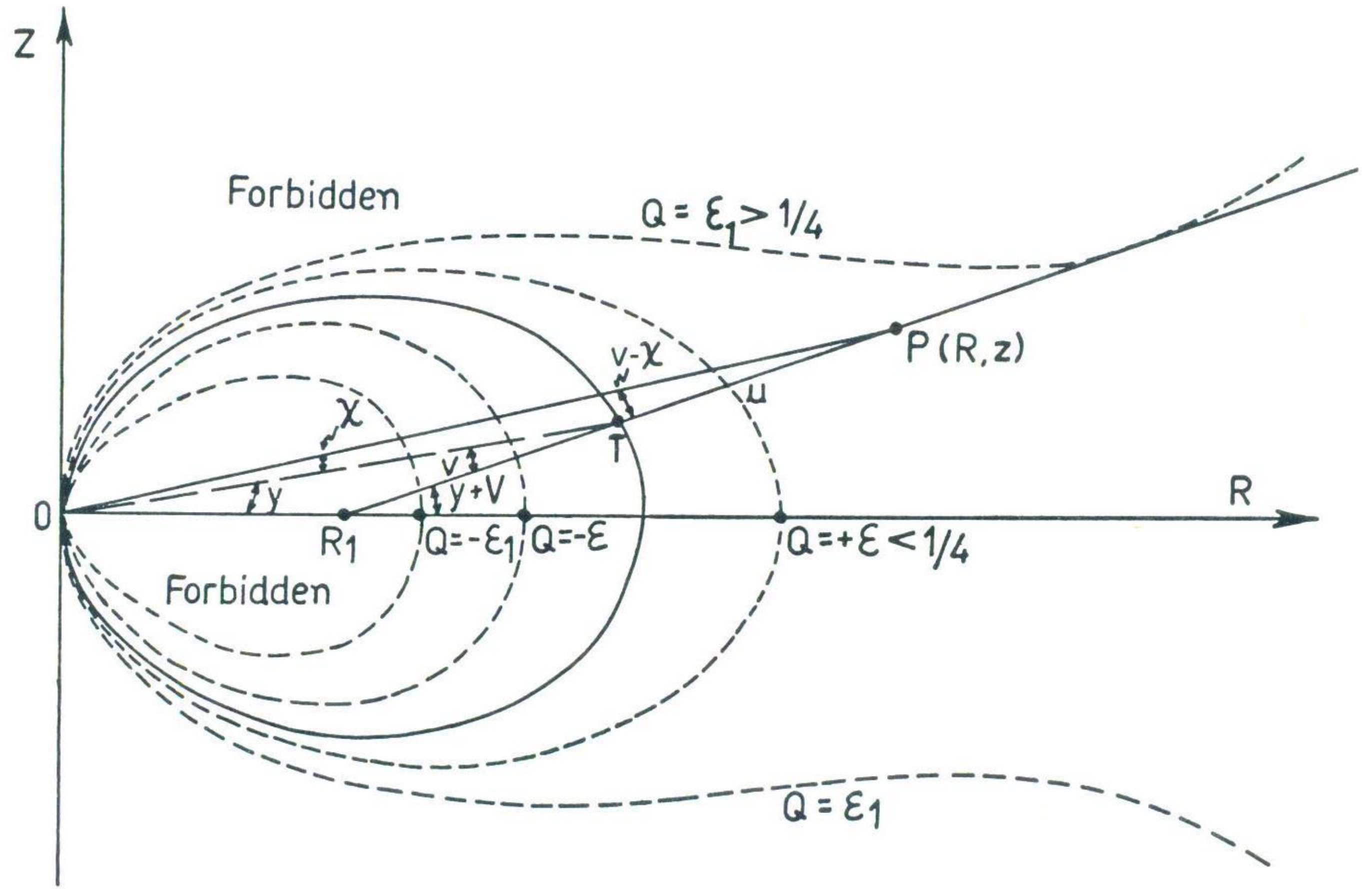
If we expand $R(U, V)$ and $z(U, V)$ in power series of U and write $R(0)$, $z(0)$ for the coordinates of the lines $U = 0$, we get, up to second order in U:

$$R_0 = R(0) - \frac{U^2}{2}\left[\frac{\partial^2 R}{\partial U^2}\right]_{V,0} + \ldots, \quad z_0 = z(0) - \frac{U^2}{2}\left[\frac{\partial^2 z}{\partial U^2}\right]_{V,0} + \ldots \tag{11}$$

Following Störmer, we call this line[2] the bottom of the valley or "thalweg". Along the thalweg there is also a line of force, particles and meridian projections of their centres of curvature coincide. For gyrations for which U remains small (U^2 negligible) the thalweg can be considered as the guiding centre.

To avoid the consideration, as centre of gyration, of a point outside the meridian plane of the particle, it is advantageous to abandon the strict magnetodynamical point of view. We shall take maximum advantage of the rigorous integral of angular momentum. Instead of considering the circulatory motion of Larmor, let us envisage an oscillatory motion in the meridian plane following the particle and, as a guiding centre, a mean of the meridian projections of the true guiding centre, that is the points of coordinates R_0, z_0 which generally differ little from $R(0)$, $z(0)$. Then let us take the thalweg ($U = 0$) as a line of reference for the positions of the particles. Moreover, let us restrict our choice of A_ϕ so that along the axis of symmetry, one can choose an origin such that $z = 0$ is also a plane of symmetry: $U(+z) = U(-z)$. We can then define an even periodic function $g(y)$ by means of

$$R(0) = e^{g(y)} \cos y, \quad z(0) = e^{g(y)} \sin y \tag{12}$$



ig. 1. Limits, in the R,z plane of the region of allowed motions for two cases $\epsilon < \frac{1}{4}$ and $\epsilon_1 > \frac{1}{4}$ of the dipole problem. The point P could have been chosen anywhere in the allowed area.

where y is the latitude of the point $(R(0), z(0))$ on the thalweg.

From any point $P(R, z)$ (fig. 1) in the allowed region of motion, let us draw the normal to the section of the thalweg lying in the same hemisphere, supposed to be unique. The straight line joining the crossing of the thalweg T to the origin will form an angle v with the normal and angle y with the equatorial axis.

Let u be the distance TP; since the angle of TP with the R axis is $(v + y)$, one can express the coordinates R, z:

$$R = e^{g(y)} \cos y + u \cos (y + v), \quad z = e^{g(y)} \sin y + u \sin (y + v) . \tag{13}$$

From the orthogonality of PT with the thalweg, one gets $d(e^{g(y)} \cos y) \cos (y + v) + d(e^{g(y)} \sin y) \sin (y + v) = 0$, giving a definition of v in analytic form:

$$\tan v = - \frac{dg}{dy} \quad ; \tag{14}$$

v is then an odd function of y such that $v = 0$ for $y = 0$. Eliminating u and v from (13) and (14), we get

$$(R \sin y - z \cos y) = \frac{dg}{dy} \left[(R \cos y + z \sin y) - e^{g(y)} \right] . \tag{15}$$

When the thalweg is defined ($g(y)$ given) for values of R and z, y can be determined from (15) and u from (13).

For evaluating the Jacobian of the transformation let us notice that

$$\left[\frac{\partial R}{\partial y}\right]_u = e^{g(y)} \left[\cos y \frac{dg}{dy} - \sin y\right] - u \sin (y + v) \left[1 + \frac{dv}{dy}\right] .$$

The relation (14) could be used to eliminate dg/dy

$$\left[\frac{\partial R}{\partial y}\right]_u = - \sin (y + v) \left\{ \frac{e^g}{\cos v} + u \left[1 + \frac{dv}{dy}\right] \right\} .$$

Performing the other partial derivation and noticing that

$$\frac{dv}{dy} = - \frac{d^2 g}{dy^2} \cos^2 v$$

we get finally

$$J = \left[\frac{\partial R}{\partial u}\right]_y \left[\frac{\partial z}{\partial y}\right]_u - \left[\frac{\partial R}{\partial y}\right]_u \left[\frac{\partial z}{\partial u}\right]_y = \frac{e^g}{\cos v} + u \left[1 - \frac{d^2 g}{dy^2} \cos^2 v\right]. \quad (16)$$

The singular points where $J = 0$ will have to be specially discussed.

In the following developments, it is sometimes interesting to change the scales of length and time in order to decrease the number of physical parameters.

For example, in the case where A_ϕ is proportional to an homogeneous function of R and z of order $-n$,

$$A_\phi = C F_{-n}(R, z) \quad (17)$$

which is the case of the dipole field ($n = 2$), let us change the scale of length through $R = R' p_\phi \epsilon$, $z = z' p_\phi \epsilon$, and the scale of time through $ds = dt\, p_\phi \epsilon^2$ where the constant angular momentum p_ϕ and the new constant ϵ are related to the physical constant C by:

$$C = p_\phi^n \epsilon^{n-1} . \quad (18)$$

The Hamiltonian (1) can then take the form

$$2H' = p_{R'}^2 + p_{z'}^2 + Q^2(R', z') = \epsilon^2 . \quad (19)$$

In these cases, the thalweg defined by $Q \equiv 1/R' - F_{-n}(R', z') = 0$ will be the same for any constant of motion appearing explicitly only in the new constant energy ϵ^2. When possible, we shall consider the Hamiltonian (19), dropping, further on, the primes.

Changing the variables R and z into u and y with the canonical transformation $p_R\, dR + p_z\, dz = p_u\, du + p_y\, dy$

and using
$$Jp_R = p_u \frac{\partial z}{\partial y} - p_y \frac{\partial z}{\partial u}$$

$$Jp_z = p_y \frac{\partial R}{\partial u} - p_u \frac{\partial R}{\partial y}$$

we get the new Hamiltonian

$$2H = p_u^2 + p_y^2/J^2 + Q^2 = \epsilon^2 \quad . \tag{20}$$

Q will then become a function of u and an even periodic function of y. A consequence of the parity of Q is the existence of a solution in the equatorial plane $z = 0$ or its equivalent $y = 0$. In fact $d^n z/dt^n$ [computed by succesive differentiations of the equations of motion deduced from (19)] can be expressed as polynomials of dz/dt such that the independent term is an odd function of z. They will be all zero when

$$z = dz/dt = 0 \quad . \tag{21}$$

We get the equatorial orbit by integrating

$$\left[\frac{dR}{dt}\right]^2 = \epsilon^2 - Q^2 \qquad (R, z = 0) \quad . \tag{22}$$

However, the thalweg is not in general a solution of the dynamical problem (20). In fact,

$$\frac{dp_u}{dt} = \frac{d^2u}{dt^2} = -Q\frac{\partial Q}{\partial u} + \frac{p_y^2}{J^3}\frac{\partial J}{\partial u} \quad . \tag{23}$$

If the first term is nil on the thalweg, to annul the second for $u = du/dt = 0$ we must have $\partial J/\partial u = 1 + dv/dy = 0$ because, in that case, we get from the Hamiltonian $p_y^2 = \epsilon^2 J^2 \neq 0$. This will happen only in the very special case where $g(y) = -\log(\cos y/R_0)$ which defines the straight line $R = R_0$ as thalweg.

2. THE APPROXIMATE PROBLEM AND ITS SURFACE OF SECTION

Let us consider the dynamical system defined by (19) where $Q(R,z)$ is any function twice differentiable of R and z even in z, such that the transformation (13) is permissible, expressing the dynamical system in the form (20). Recalling that the allowed region for the movement will be defined by $-\epsilon < Q < +\epsilon$, the extent of the "valley" around the thalweg will increase with ϵ. For the case of high angular moment, where ϵ is very small, the limiting solutions are points of equilibrium on the thalweg, and p_u, p_y and R are small quantities of the order ϵ. As Q is nil on the thalweg, its power expansion is

$$Q = u\omega(y) + \theta(u^2). \tag{24}$$

Then u will be also of the order of ϵ and

$$\left[\frac{\partial Q}{\partial u}\right]_y = \omega(y) + O(\epsilon) \quad , \qquad \left[\frac{\partial Q}{\partial y}\right]_u = u\frac{d\omega}{dy} + O(\epsilon) \ .$$

From the relations

$$\begin{aligned} \frac{\partial Q}{\partial R} &= \frac{\partial Q}{\partial u} \cos(v+y) - J \sin(v+y)\frac{\partial Q}{\partial y} \\ \frac{\partial Q}{\partial z} &= \frac{\partial Q}{\partial u} \sin(v+y) + J \cos(v+y)\frac{\partial Q}{\partial y} \end{aligned} \tag{25}$$

and noting that $\partial Q/\partial y = 0$ on the thalweg, we get

$$\omega(y) = \sqrt{\left[\frac{\partial Q}{\partial R}\right]_z^2 + \left[\frac{\partial Q}{\partial z}\right]_0^2}$$

computed at $Q = u = 0$.

In the case of axially symmetric problems, Q will possess a term in R^{-1}, and $\omega(y)$ tends to infinity for y tending towards $\frac{1}{2}\pi$. As $dy/dt = p_y/J^2 = 0 + O(\epsilon)$, equations (23) become $d^2u/dt^2 = -u\omega^2(y) + O(\epsilon^2)$ and we obtain an harmonic oscillation of frequency $\omega(y)$. This suggests a transformation:

$$u = [2p_1/\omega(y)]^{\frac{1}{2}} \sin q_1 \quad ; \quad p_u = [2p_1\omega(y)]^{\frac{1}{2}} \cos q_1 \tag{26}$$

where $q_1 = \omega(y)t + O(\epsilon)$ and $p_1 = \text{constant} + O(\epsilon^2)$. Now y appears in this transformation; to make it canonical, it is convenient to change y and p_y by introducing an odd function of y

$$s = \int_0^y e^{g(y)} \sqrt{1 + \left[\frac{dg}{dy}\right]^2}\, \omega(y)\, dy \ . \tag{27}$$

The conjugate canonical momentum is then

$$p_s = \left[p_y e^{-g(y)} \cos v - p_1 \sin 2q \frac{d\omega}{ds}\right] \omega^{-1} \ .$$

We get for the transformed Hamiltonian

$$2H \equiv 2p_1\omega(s) + p_s^2\omega^2(s) + O(\epsilon^3) = \epsilon^2 \ . \tag{28}$$

To an approximation of ϵ^3, p is a constant and the motion in s, or its equivalent y, can be determined by the elimination of p_s and dt from the Hamiltonian and the equation of motion in s:

$$\frac{ds}{dt} = p_s\omega^2 , \quad \frac{ds}{dq} = \sqrt{\epsilon^2 - 2p_1\omega(s)} \ . \tag{29}$$

If we call ω_0 the minimum of $\omega(y)$, there exist possible values of the square of the gyration amplitude such that $0 \leqslant 2p_1 < \epsilon^2/\omega_0$. To avoid complicated structures, we shall suppose that there is only one such minimum; it must then be necessarily the equator, $y = 0$, as $\omega(y)$ is an even function of y reaching infinity for $y = \pm \frac{1}{2}\pi$. We are then able to find a "mirror-latitude" y_m solution of the equation $\epsilon^2 = p_1\omega(y)$.

The charged particles will oscillate between latitudes y_m and $-y_m$ following the differential equation

$$\left[1 - \frac{\omega(s)}{\omega(s_m)}\right]^{-\frac{1}{2}} \epsilon^{-1} ds = dq_1 \quad . \tag{30}$$

Integrating in s from 0 to s_m we get $2/\pi$ of the number $k(y_m)$ gyrations during a complete oscillation in latitude. This number depends on y_m and also on ϵ, and it becomes infinite when ϵ tends to 0.

Except for the equatorial orbit, all trajectories will cross the equator between two consecutive mirror points and will do it indefinitely.

From each regular point of the surface in three dimensions defined by $2H = \epsilon^2$ (eq. (20)) there begins one and only one orbit. Choosing the three-dimensional space u/ϵ, p_u/ϵ, y in the approximate problem, orbits with a given mirror latitude y_m will cover finite two-dimensional surfaces with the three planes of coordinates as planes of symmetry.

The sections with y = constant, existing only for $-y_m < y < y_m$, are ellipses of axis

$$\left[\frac{\omega(y)}{\omega(y_m)}\right]^{\frac{1}{2}} \quad \text{and} \quad \left(\omega(y)\,\omega(y_m)\right)^{-\frac{1}{2}} \quad .$$

The area of these sections remains constant: $\pi/\omega(y_m)$. As $y\,d\omega/dy < 0$ they will fit inside one another, the surfaces having higher mirror latitudes being inside of those having smaller ones. The equatorial-orbit ($y_m = 0$) collapsed limiting surface surrounds all the others (fig. 2). The limiting surface when y_m tends to $\pi/2$ is a line: the y axis where $p_1 = 0$. Thus the infinite number of trajectories on this surface differing from one another by their phase q will collapse to a line of non-gyration: the thalweg solution of the dynamical problem to this approximation $Q(\epsilon^3)$ only. It might be useful here to recall what is meant by a surface of section.[3] If a dynamical problem is represented by n independent autonomous equations of the first order, its solution can be represented by the steady motion of an n-dimensional fluid, of which the moving point has the dependent variables as coordinates. Now, suppose that a closed $(n-1)$-dimensional analytic surface S can be constructed in this manifold of states of motion in such a way that, within any sufficiently large interval of time, every stream cuts S at least once and always in the same sense. Then S is called a "surface of section".

In this approximation to the dynamical problem, the equatorial plane is a surface of section. It cuts all the surfaces with a given mirror latitude y_m

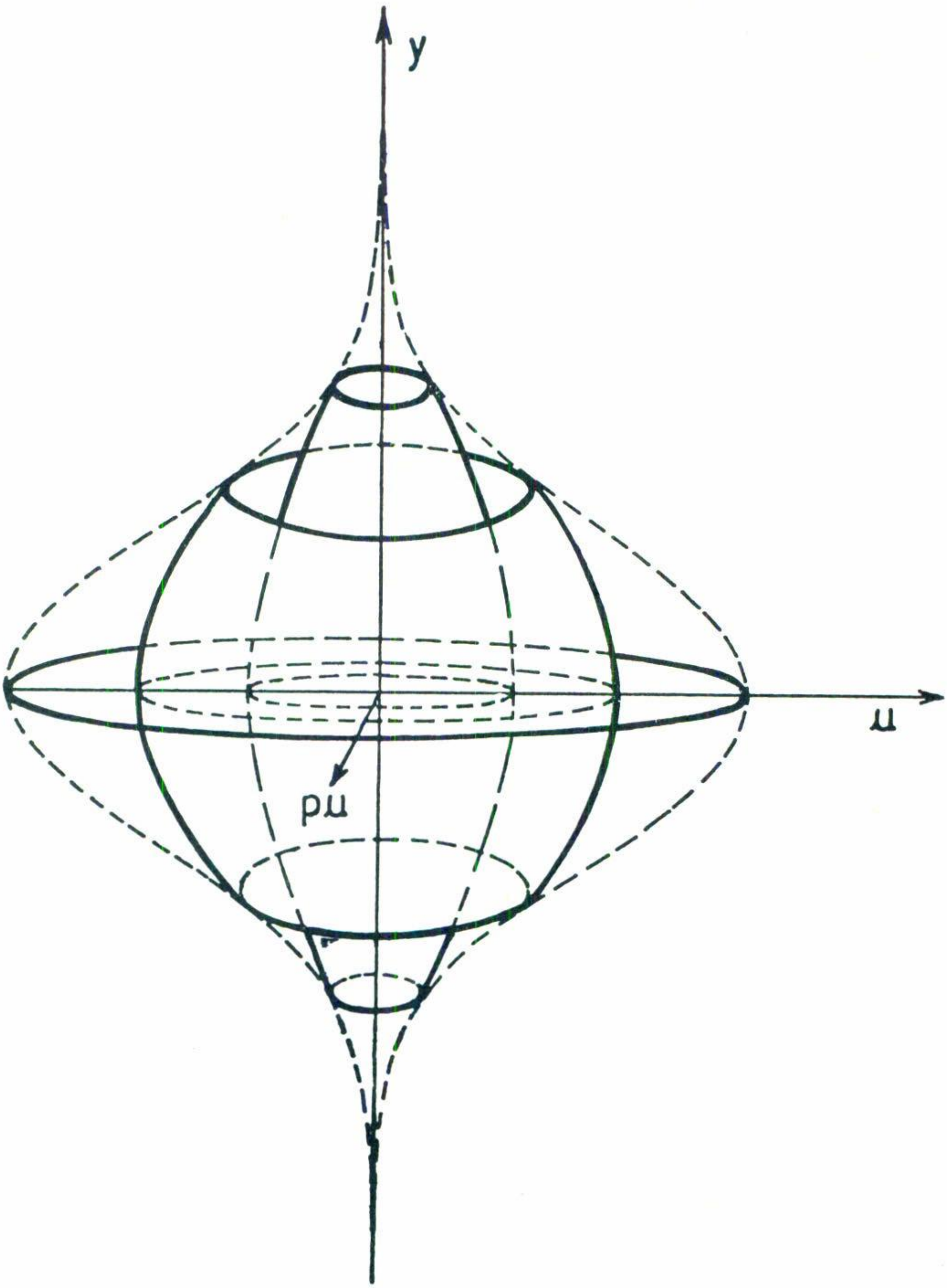


Fig. 2. Perspective of the mirror surfaces for $y_m = 0°$, $30°$ and $45°$. The envelope of those surfaces has been drawn in broken lines.

along what we shall call mirror curves. The equatorial orbit encloses all of them, limiting the allowed region; as the mirror latitude increases continuously from 0 to $\pi/2$, the corresponding mirror curve will shrink continuously up to the central limit point for $y_m = \pi/2$, filling in that way the whole surface of section. To a point of those curves which do not cross each other, could be assigned a unique value of a parameter, such as a phase of gyration, in such a way as to define some kind of polar coordinate system. However the canonical variables u, p_u have the advantage of giving a mapping of the surface of section[3] which preserves the area for the solution of the dynamical system. Their expressions of the n^{th} crossing of the equator by a trajectory (q_m, y_m) are given by :

$$\frac{u_n}{\epsilon} = U_n = \big(f(y_m)/f(0)\big)^{-\frac{1}{2}} \sin\left[q_m + (n - \tfrac{1}{2})\, k(y_m)\, \pi\right]$$

$$\frac{p_{un}}{\epsilon} = P_n = f(0) f^{-\frac{1}{2}}(y_m) \cos\left[q_m + (n - \tfrac{1}{2})\, k(y_m)\, \pi\right] \quad . \tag{31}$$

If $k(y_m)$ is not a rational number, the phase angles will be all distinct when n increases indefinitely. But in the other case, the number of distinct phases will be finite and the trajectory periodic. For each of the corresponding values of the mirror latitude y_m we shall get, in the approximate problem, an infinite number of ordinary periodic orbits covering the corresponding mirror surface in space y, u, p_u. On the surface of section, when y_m varies from 0 to $\pi/2$, the dense set of mirror curves will contain an infinite denumerable subset of continuous curves formed of periodic points only.

To fix the choice of the parameter to be used as coordinate, let us consider the segment of an orbit in the R, z plane between two successive crossings of the equator, lying thus completely in one hemisphere. Because it could be followed in both senses, it will define two trajectories with initial points in the equatorial plane U_A , P_A and U_B , P_B . In the three-dimensional space u, y_m , p_u that segment with its two boundary points A, B will be called a semi-orbit. It will have a mirror latitude y_m . But if one trajectory, let us say the one starting from A, has a mirror phase q_m, the other, starting from B, will have a mirroring phase $\pi - q_m$. The operation of passing from one trajectory of the semi-orbit to the other will be called the mapping M. In particular, we have $M(A) = B$, $M(B) = A$. The operator M is symmetrical: $M^{2n} = 1$. It produces simply a change of q_m to $\pi - q_m$ in the equations (31) for $n = 0$. The elimination of q_m between these equations gives the law of mapping:

$$U_B = U_A \cos \pi k(y_m) + \frac{P_A}{\omega(0)} \sin \pi k(y_m)$$

$$P_B = P_A \cos \pi k(y_m) - U_A\, \omega(0) \sin \pi k(y_m) \; . \tag{32}$$

On each of the mirror curves there will be two invariant points for M:

$$U'_B = U'_A,\; P'_B = P'_A$$

such that

$$\frac{U'_A}{P'_A} = \frac{1}{\omega(0)} \tan \tfrac{1}{2}\pi\, k(y_m) \; .$$

These points will correspond to the mirroring phases $q_m = \pi/2$ and $3\pi/2$, where both speeds $du/dt = p_u$ and dy/dt are nil; they will then be on the lines $Q = \pm \epsilon$, the boundary of the allowed region, and will appear in the R, z or u, y plane as self-reversing orbits.

As $k(y_m)$ varies from one mirror to another, these invariant points will be aligned on some kind of twisted radii of the surface of section, dividing it in two parts. As the mapping is continuous, the representative points with intermediate phases will move from one segment to the other. The prolongation into the other hemisphere of the trajectory starting at A will be on another semi-orbit of that hemisphere, with a starting point at the same location, but with an opposite direction from B:

$$U_{B'} = U_B \; ; \; P_{B'} = -P_B \quad .$$

B′ will play the same role as A for this new semi-orbit, which will have another boundary point A′. Because of the symmetry of the problem, the mapping operation will be the same $M(B') = A'$. The equator is then a surface of section in the sense of Poincaré–Birkhoff[3] and A′ is the canonical mapping of A. In fact, the surface of section is double because the trajectories cross the equatorial plane in both senses. To pass from one surface of section to the other, it is sufficient to apply the reflection operator R, which effects a change of the sign of the coordinate P. Because of symmetry, the two will coincide and could be considered as one surface of section always crossed in the same sense, and the canonical mapping will consist of two identical transformations T of the surface of section onto itself, each one decomposed into two operations:

$$T(\mathrm{A}) = RM(\mathrm{A}) = \mathrm{B}'.$$

The mirror curves y_m = const. are invariant under the operator M. In the approximate problem the mirror curves are symmetric with respect to the axis U, and so they will also be invariant under R and thus under T, which will maintain invariant the mirror latitude.

Along the elliptical mirror curve, the angular parametric coordinate of the boundary point will increase by the constant angle $\pi k(y_m)$.

3. THE GENERAL PROBLEM

Coming back to the general dynamical problem where ϵ is not negligibly small, a condition sufficient for the maxima in y to lie in the upper hemisphere and the minima in the lower, is that

$$y \frac{\partial Q^2}{\partial \mu} > 0 \tag{33}$$

for the allowed region outside the thalweg. It is quite easy to see that for $dy/dt = 0$, $d^2y/dt^2 = -\tfrac{1}{2}\partial Q^2/\partial y$. If the condition (33) is fulfilled, y and its second derivative will have opposite sign, and the equator will still be crossed by all the orbits; the condition (34) includes the relation (21) which is the condition for the existence of a surface of section for sufficiently small ϵ.

For the extrema on the thalweg, $Q = 0$, $d^2y/dt^2 = 0$.

The calculation of higher derivatives of y, taking into account the value of $du/dt = p_u = \epsilon$, and $d^2u/dt^2 = 0$, gives:

$$\frac{d^3y}{dt^3} = -\frac{\partial}{\partial u}\left[Q\,\frac{\partial Q}{\partial y}\right]\frac{\epsilon}{J^2} = 0 \tag{34}$$

$$\frac{d^4 y}{dt^4} = -\frac{\partial^2}{\partial u^2}\left[Q\,\frac{\partial Q}{\partial y}\right]\frac{\epsilon^2}{J^2} = -\frac{\epsilon^2}{J^2}\,\omega(y)\,\frac{d\omega}{dy}\,. \tag{35}$$

From the condition (21), d^4y/dt^4 also will have a sign opposite to y, and the extrema on the thalweg will not be an exception. Two situations may arise: the allowed region of motion (u, y) may be finite or infinite; the equatorial section is filled by the boundary points of the semi-orbits in the first case but not when the allowed area is infinite. In fact, the trajectories coming from infinity, after reaching their first extremum in y, will cross the equator at a point that is not the boundary of a semi-orbit. We shall not deal with this case in the present paper.

As in the approximate problem, the boundary points of the semi-orbits having the same mirror latitude will define the mirror curves which, in the same way, fit on each other. However, their limiting point for $y_m = \pi/2$, the crossing of the equator by the singular orbit, now distinct from the thalweg (see (23)), will not necessarily be at the origin. This implies that the mirror curves are not necessarily symmetric with respect to the axis U. As a consequence, although the mapping M of A will still change the phase q_m to $-q_m$, the reflection R will not necessarily carry the boundary point on to the same mirror curve y_m (see fig. 3), and the mirror latitude will generally change from one crossing to the other. Moreover, the variant mirror surfaces, in the space u, p_u, y, will no longer exist. Even so, certain trajectories will have a limited number n of distinct mirror points; they are those for which $T^n(\mathrm{A}) = \mathrm{A}$ and $T^m(\mathrm{A}) \neq \mathrm{A}$ for $m < n$. These points will be called periodic points of class n. Their determination is important for following the general dispersion of the mirror latitudes.

In the application of the last geometrical theorem of Poincaré[4] near the periodic mirror curves of the approximate problems where k is a rational number, there will still exist periodic singular points. A theorem of Kolmogorov[5] also proves the analytic continuation of almost all the quasi-periodic solutions of the approximate problem to the general problem. As the ratio of the frequencies of the quasi-periodicities changes continuously, isolated cases of commensurability will give rise to periodic motion.

In the general problem, the self-reversing semi-orbits ($q_m = \pi/2, 3\pi/2$) continue to exist: their boundary points still form an invariant curve for the mapping M, which we shall call $\mathfrak{M}_1$. Applying the transformation T to this curve n times, we obtain a resulting curve which we call $\mathfrak{M}_{2n}$. It is easy to see that it is formed of points invariant under the transformation RT^{2n+1}.

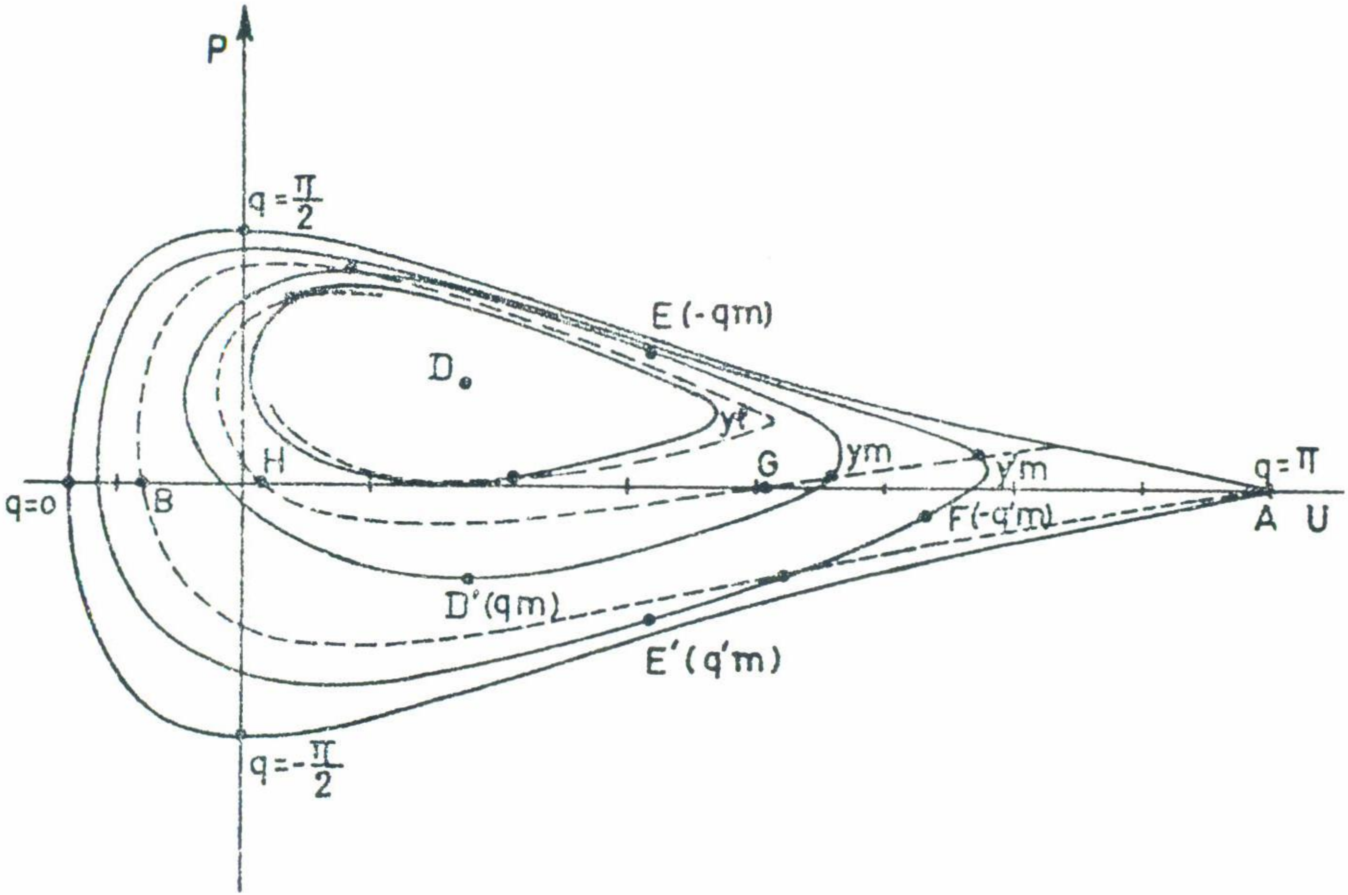


Fig. 3. Surface of section for $\epsilon = \frac{1}{4}$; the M_1 curves spiralling around D have been drawn up to the mirror curve y_e. The points A, B, G, H are periodic points of class 1. The point D is the dipole point transformed in the surface of section to E and F.

Let us notice that the axis $P = 0$ is invariant under the operation R. We shall call it $\mathfrak{M}_0$. If we apply T to those points n times, we get a curve $\mathfrak{M}_{2n}$ which is invariant under the transformation RT^{2n}.

Moreover, let us notice that all the periodic orbits symmetric with respect to the equation will have at least one point on $\mathfrak{M}_0$; we shall discover all the different symmetric periodic orbits of class r by considering the crossings of $\mathfrak{M}_r$ with $\mathfrak{M}_0$ with the condition that $\mathfrak{M}_k (k < r)$ does not include the same point. T_0 determines $\mathfrak{M}_{2n}$, and $\mathfrak{M}_{2n+1}$, we have to calculate a double infinity of trajectories up to $n+1$ crossings of the equator. This is usually done by numerical integration, in which the precision decreases with increasing n.

To give an illustration of the surface of section, it is useful to proceed graphically (see fig. 3). The point D is the singular point ($y_m = \pi/2$). When y_m decreases, the mirror curves for $y > y_l$ will not cut the axis $P = 0$ and will have their next mirror latitude less than y_l (the singular orbit y_m with a phase q_m); the mapping $M(D^r)$ gives $E(-q_m)$ on the same mirror curve. Applying R, we see that the next mirror will be at a latitude y_m, and so on.

The difficulty of representing parametrically the surface of section in the general case is due to the fact that the orbit starting from the singular point $y_m = \pi/2$ is no longer the thalweg but a complicated curve that, in general, will only be determinable by numerical integration. It seems better for its representation to go back to polar coordinates in the original R, z plane by putting $R = r \cos \lambda$, $z = r \sin \lambda$. The Hamiltonian will take the form

$$2H = p_r^2 + \frac{p_\lambda^2}{r^2} + Q^2(r, \lambda) = \epsilon^2 . \qquad (36)$$

One could deduce from (20) the following relations:

$$\begin{aligned} p_r &= p_u \cos[\lambda - (y + v)] - \frac{p_y}{J} \sin[(y + v) - \lambda] \\ \frac{p_\lambda}{r} &= \frac{p_y}{J} \cos[\lambda - (y + v)] + p_u \sin[(y + v) - \lambda] \quad . \end{aligned} \qquad (37)$$

If we consider a section λ of the manifold of motions, we could represent it on a sphere of radius ϵ:

$$\frac{p_\lambda}{r} = \epsilon \sin\theta \;;$$

$$p_r = \epsilon \cos\theta \cos\phi \;; \tag{38}$$

$$Q = \epsilon \cos\theta \sin\phi \quad ,$$

where θ and ϕ are the spherical coordinates. There will be a one-to-one relation between r, p_r, p_λ and $\theta\,(-\pi/2 \leqslant \theta \leqslant \pi/2)$ and $\phi\,(0 \leqslant \phi \leqslant 2\pi)$. If the representative point of the trajectory coming from the singular point has for coordinates β and χ, let us make a double rotation of coordinates in such a way that this point is on the axis of symmetry of a new system of spherical coordinates p, q.

$$p_r = \epsilon \cos p \cos q \cos\chi - \epsilon \cos p \sin q \sin\chi \cos\beta + \epsilon \sin p \sin\beta \sin\chi$$

$$Q = \epsilon \cos p \cos q \sin\chi + \epsilon \cos p \sin q \cos\chi \cos\beta - \epsilon \sin p \cos\chi \sin\beta$$

$$\frac{p_\lambda}{r} = \epsilon \cos p \sin q \sin\beta + \epsilon \sin p \cos\beta \quad . \tag{39}$$

For the different latitudes λ, such a transformation could be envisaged and the limiting line will be defined by $p = \pi/2$:

$$p_r = 2 \sin\chi(\lambda) \sin\beta(\lambda) \;;\; Q = -\epsilon \cos\chi(\lambda) \sin\beta(\lambda) \;;$$

$$\frac{p_\lambda}{r} = \epsilon \cos\beta(\lambda) \;. \tag{40}$$

On the surface of section, $\lambda = y = v = z = 0$ and $p_u = p_r$; $p_\lambda / r = p_y / J$; $R = e^{g(0)} + u$ and $Q(R, 0)$ could be solved for μ. The surface will then be represented by the two parameters p and q ; q is a phase angle approximately equal to q_1 and p = constant is nearly a mirror curve. Instead of considering families of trajectories reflecting at a constant y, let us consider those reflecting at a constant λ and define the phase at the mirror by putting $Q = \epsilon \sin q_{1m}$, $p_u = \epsilon \cos q_{1m}$ where the q_{1m} are chosen to allow easy harmonic analysis; for example, taking values of $q_{1m} = \pi k/n$ where n and k are integers, the first fixed, the other taking all values from 0 to $2n-1$.

It is then possible to obtain r from the expression for $Q(r,\lambda)$ and y is determined by noting that (15) can be written in the form

$$\sin[\lambda - (v+y)] = \frac{e^{g(y)}}{r} \quad .$$

Finally, using (37) and (39) we can determine p_m and q_m.

It will then be sufficient to calculate how p and q vary with λ to obtain the two crossing points of the semi-orbit envisaged. This can be done by numerical integration of the differential equations of the first order of p and q obtained by taking the time variations of (39) and using the Hamiltonian equation of motion to eliminate dp_r/dt and dp_λ/dt, and equations (40) themselves to eliminate p_r and p_λ. We shall then get the curves of crossing points of the semi-orbits reflecting at the same latitude λ, different from y except on the thalweg.

4. THE DIPOLE CASE

To apply the theory of the previous sections to the magnetic dipole case, a suitable choice of coordinates gives[6]

$$Q = \frac{1}{R} - \frac{1}{(R^2 + z^2)^{3/2}} \tag{41}$$

with

$$\epsilon = \frac{eM}{cp_\phi^2} \tag{42}$$

(e charge of the particle, M magnetic moment). Equations (12) and (14) become*

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*The same angle y has been used by Lemaître and Bossy[7] for the study of orbits in the valley. They also used the variable u with a meaning different from the one of this paper.

$e^{g(y)} = \cos^2 y$, $\tan v = 2 \tan y$. The expression for the Jacobian (17) is

$$J = \frac{\cos^2 y}{\cos v} + 3u \left[\frac{1 + \sin^2 y}{1 + 3 \sin^2 y} \right] . \tag{43}$$

To have one-to-one correspondance between R, z and u, y as defined by (13), taking into account that $\sin (y + v) = 3 \sin y \cos v$, we must have

$$\frac{z}{\sin y} = \cos^2 y + 3u \cos v > 0 \quad .$$

To guarantee that all the points on the equator can be represented by only one value, the intersection with the equator of the normal to the thalweg must be in the forbidden zone $|Q| > \epsilon$. As it is given by $R_1 = \frac{2}{3} \cos y$, from the expression for Q it can be found that the maximum value of ϵ is $\frac{3}{4}$. This is large enough to cover most of the cases of physical interest. As for this limit on u, the Jacobian *a fortiori* will not be nil except at the singular point $y = \pi/2$, and the transformation (13) can be applied.

From (21), we get $\omega(y) = (\cos^5 y \cos v)^{-1}$, satisfying the condition for the existence of the surface of section in the approximate problem.

To verify the general condition $\partial Q^2/\partial \mu^2 < 0$, let us compute it in the form

$$\frac{\partial Q}{\partial y} \times \frac{1}{Q \sin y} > 0 \quad .$$

Now

$$\frac{\partial Q}{\partial y} = \frac{\partial Q}{\partial R} \frac{\partial R}{\partial y} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial y} = J \left[\frac{\partial Q}{\partial z} \cos (v + y) - \frac{\partial Q}{\partial R} \sin (v + y) \right] .$$

Noting that the relation (15) can be written

$$\frac{z \cos (v + y)}{\sin y} = \cos v \, (3R - 2 \cos y)$$

and performing the partial derivatives, one gets

$$\frac{1}{\sin y}\frac{\partial Q}{\partial y} = 3J\cos v\left[\frac{1}{R^2} + \frac{1}{(R^2+z^2)^{3/2}} - \frac{2R\cos y}{(R^2+z^2)^{5/2}}\right] =$$

$$= 3J\cos v\left\{\frac{Q}{R} + \frac{2}{(R^2+z^2)^2}\left[(R^2+z^2)^{1/2} - \frac{R\cos y}{(R^2+z^2)^{1/2}}\right]\right\} .$$

In fig. 1, lines have been drawn for two typical constant values of Q, with the dipole problem as an example. It is quite clear from the definitions that u/Q is always positive. Instead of u, a parametric angle X can be used, being defined as the difference of the position angle of the point P and its guiding centre R. From the properties of the triangles OTP, one deduces:

$$u = \frac{\cos^2 y \sin X}{\sin(v - X)}(R^2+z^2)^{1/2} = \frac{\cos^2 y \sin v}{\sin(v-X)}$$

$$R = (R^2+z^2)^{1/2}\cos(y+X) \quad . \qquad (46)$$

Then

$$\frac{(R^2+z^2)^{1/2}}{\cos y} - \frac{R}{(R^2+z^2)^{1/2}} = \frac{\sin X\cos(y+X-v)}{\sin(v-X)}$$

and finally

$$\frac{1}{\sin y}\frac{\partial Q}{\partial y}\frac{1}{Q} = 3J\cos v\left\{\frac{1}{R} + \left[\frac{u}{Q}\right]\frac{2\cos(y+X-v)}{(R^2+z^2)^2\cos y}\right\} . \qquad (47)$$

Since the latitude of the particle $|y+X| \leq \pi/2$ and has the same sign as the parameter v and $|v| < \pi/2$, we have that the angle $|y+X-v| < \pi/2$ and the second member of (47) is always positive. The equator separates maxima and minima of the y latitude of the guiding center along all the orbits. A real surface of section must however have a finite extent in the allowed region of motion, and this will happen for $\epsilon < 1/4$. This is the geophysical case for the particles of the Van Allen Belt. In the problem of cosmic radiation $1/4 < \epsilon < 3/4$; the equator will still have most of the characteristics of a surface of section. This was studied by the author in an unpublished paper. Part of this work, improved and extended by de Vogelaere, has been published[2]. The theorems of that paper could easily be extended to prove the conditions of continuity tacitly assumed in the present more general case. Although introduced in another way, the same curves $\mathfrak{M}_n$ of the last paragraph were studied in great detail, and the most interesting periodic orbits were located. The study of the geophysical case and in particular of the variations of the mirror of latitude is in progress. The approximate problem has been solved[8].

It was found that the ratio of the number of gyration of latitude oscillations can be expressed as

$$k(y_m) = \sum_n \frac{k_n \cos 2n\, y_m}{\epsilon \cos^4 y_m}$$

where y_m is the thalweg mirror latitude and k_n the coefficient of a Fourier series given in Table I.

Table I

n	0	1	2	3	4	5
k	0.802 639	−0.312 062	−0.017 952	−0.000 970	−0.000 154	−0.000 013

The structure of the surface of sections can be determined when the canonical coordinates are expressed for a given ϵ as a function of the parameters y_m and q_m. This function being periodic in both parameters, it is sufficient to calculate a discrete number of semi-orbits for values of y_m, q_m chosen in such a way that one can perform an harmonic analysis of the results. The main difficulties met with in the numerical integration of the semi-orbit are due to its strongly oscillating character.

A modification of the classical one-to-one step method has been introduced to correct for the quasi-periodicity in $\omega(y)$ of the solution of the primitive equations

$$\frac{d^2R}{dt^2} = -Q\frac{\partial Q}{\partial R}, \quad \frac{d^2z}{dt^2} = -Q\frac{\partial Q}{\partial z} \quad . \tag{48}$$

But when y is near $\pi/2$, this method requires a prohibitive number of steps.

However, it ought to be noticed that in this very narrow valley of the allowed region in R, z the approximate solution is a very good approximation because the neglected terms are of the order of $\epsilon \cos^3 y$. In that solution, the center of gyration is also the trajectory starting from the singular point $y = \pi/2$ which is not any more the thalweg but rather the limit of the locally non-gyrating trajectories when the point of non-gyration y_0 tends to $\pi/2$. An asymptotic expansion of this solution has been calculated by defining, from the coordinates R_a and z_a of this solution, the related parameters

$$C = \frac{R^2}{R^2 + z^2} \quad , \quad x = \frac{C}{4-3C} \quad , \quad \alpha = \frac{\epsilon^2 C^4}{4(4-3C)} \quad . \tag{49}$$

We can then obtain for r an expansion as a power series in α with polynomials in x as coefficients:

$$r = (R^2 + z^2)^{1/2} = C\left[1 + 2 \sum \alpha^m R_{3m-2}(x)\right] , \tag{50}$$

where $3m - 2$ is the degree of the polynomial. In Table II are listed the integer coefficients of these polynomials as well as those of the first derivatives expressed in the form:

$$\frac{d}{dC}(r - C) = 2\sum \alpha^m P_{3m-1}(x) = P \quad . \tag{51}$$

These series diverge near the equator but are certainly valid down to $45°$; here $\alpha = \epsilon^2/160$, $x = 1/5$. From there, the gyrations being less rapid, it is possible to proceed with numerical integrations down to the equator.

The two angular functions $\beta(\lambda)$ and $\chi(\lambda)$ can be evaluated from the relations

$$r\frac{dr}{d\lambda} = \frac{dr}{rdC}\, 2 \sin\lambda \cos\lambda = \frac{4 \tan\lambda\, P}{1 + R} = \sin\lambda \tan\beta \tag{52}$$

$$Q = \frac{1}{r\sqrt{C}}\left(1 - \frac{C}{r}\right) = \frac{2R}{C^{3/2}(1 + 2R)} = \epsilon \cos\chi \sin\beta \quad . \tag{53}$$

TABLE II

a) Polynomials R_n

Order of polynomials / Power of x	1	2	3	4	5
0	3	-225	81270	-59687685	72810408330
1	3	-522	281610	-275252580	418713709050
2		-324	333774	-483969195	976837762260
3		702	-505638	458013204	-420278955156
4		945	-1933470	3745676682	-8924656265226
5			-1330722	5652939564	-21316241372130
6			1475658	-1933429338	-6280145858520
7			1818126	-13767970788	61785741068856
8				-9127789317	103025486714022
9				7604933832	-8432824263930
10				8438353605	-162263802551868
11					-100791968025060
12					71261633813274
13					71434447876386

b) Polynomials P_n

Order of polynomials / Power of x	1	2	3	4	5
0	15	-2025	1056510	-1014690645	1529018574930
1	27	-6570	4673970	-5670798660	10303857724050
2	18	-8262	8385930	-13324203405	30004115294880
3		4536	-3083598	448818570	-10426392501350
4		22815	-41970474	88277487606	-233203101554394
5		17010	-64555866	214260910776	-795187994836482
6			-3899826	108160493454	-809051179343940
7			76205286	-388434179052	1522755936596808
8			54543780	-682537768929	5212025793185454
9				-130872135780	3765009253928958
10				524427966783	-5384356498192968
11				354410851410	-10527214091635980
12					-2486380549364838
13					6063114552274098
14					3857460185324844

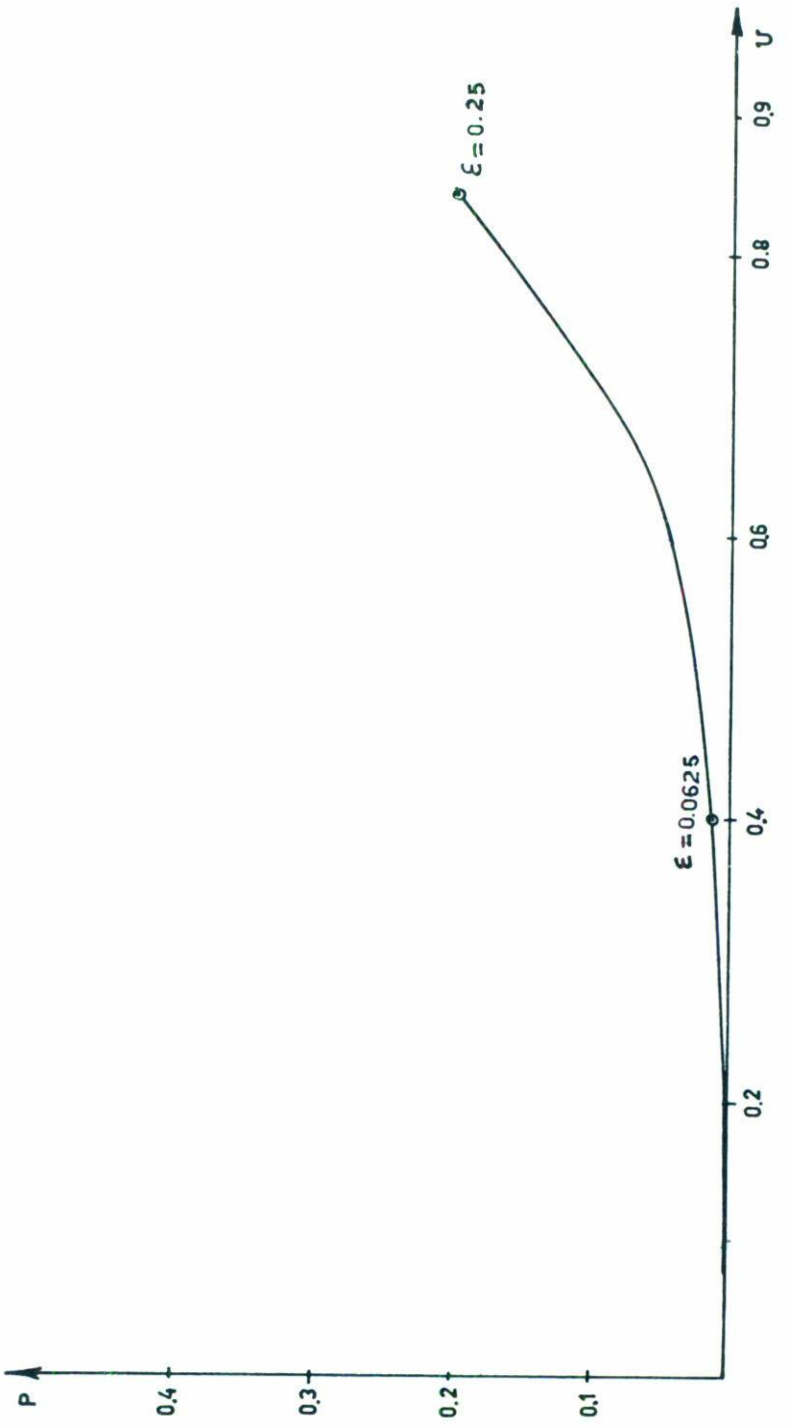

Fig. 4. Crossings of the singular orbit with the equator center of the surface of section.

As P and R are of the order of ϵ^2, χ and β are small quantities of the order of ϵ, showing that $q \approx q_1$, and

$$\cos p \approx \left(\omega(y_m)/\omega(y)\right)^{1/2}.$$

In fig. 4 we show the crossings of the singular orbit with the equator center of the surface of section.

We are proceeding at present to compute the families of orbits which parametrize the surface of section for $\epsilon = 1/8$. The results will be published elsewhere.

CONCLUSION

As we have seen, a great number of axi-symmetric dynamical problems possess a surface of section. The knowledge of the transformation of the surface of section in itself enables one to determine complete trajectories by passing from one semi-orbit to another. Usually it will be sufficient to compute a limited number of semi-orbits properly chosen to describe the complete problem.

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RESUMEN

En muchos problemas astrofísicos y geofísicos, la invariación de los espejos magnéticos es correcta sólo a primer orden. Sus variaciones lentas son importantes para mantener los anillos de radiación. Se pudieron determinar por un método inspirado en la superficie de sección clásica. En general, las condiciones para la existencia de una doble superficie de sección se cumplen para el movimiento de partículas cargadas en un campo magnético axil. Entonces sólo se necesita determinar las trayectorias para el intervalo finito de tiempo entre dos cortes sucesivos con la superficie de sección e inferir la ley de transformación de la superficie de sección. Se pueden entonces deducir la variación lenta de los invariantes adiabáticos, así como las características del movimiento en general.

Se estudian de modo especial el caso del dipolo y el método para determinar las variaciones seculares de los espejos magnéticos.