THE HARMONIC OSCILLATOR IN A RANDOM ELECTROMAGNETIC FIELD: SCHRÖDINGER'S EQUATION AND RADIATIVE CORRECTIONS*

L. de la Peña-Auerbach and A.M. Cetto

Instituto de Física, Universidad Nacional Autónoma de México
Apartado Postal 20-364, México 20, D. F.

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ABSTRACT: We analyze the problem of a radiating harmonic oscillator in interaction with the random zero-point radiation field. We obtain a solution of the equation of motion by taking into account explicitly the fact that the particle modifies the field. This solution goes asymptotically into a stationary state, the statistical properties of which are precisely those of the ground state of a quantum-mechanical oscillator. We show furthermore, using the formalism of stochastic quantum mechanics, that the statistical behaviour of the harmonic oscillator in equilibrium with the radiation field is exactly described by the Schrödinger equation. A similar treatment using the black-body radiation field at temperatures $T>0$ yields the excited states of the system, which again coincide with the quantum-mechanical results. We conclude that the harmonic oscillator in the random

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electromagnetic field is not a classical system – as is usually assumed – but a model of the quantum-mechanical oscillator, and in fact a more detailed model than the usual one, since the approach to equilibrium is also described. From the solution of the equation of motion we obtain in a straightforward way a non-divergent expression for the Lamb shift of the harmonic oscillator; the result coincides with the usual quantum-electrodynamic result when a relativistic cut-off frequency $\omega_c = mc^2/\hbar$ is used. With the same cut-off frequency, we also obtain the mass renormalization of the electron, again without any divergences; this correction is proportional to the fine-structure constant.

I. INTRODUCTION

In the course of the last years there have appeared various attempts to reformulate quantum mechanics (QM), motivated mainly by the need for a deeper understanding of the physical principles underlying this theory. Perhaps the most widely accepted view among the slowly growing circle of dissenters from the traditional views is that the physical content of QM is a stochastic process, whose nature remains as yet unknown. Two main streams of thought share this point of view: On one hand, several authors[1,2,3] have shown that it is possible to derive the fundamental laws of QM by assigning a stochastic character to the dynamical variables of the particles. Although this description, which we call stochastic quantum mechanics (SQM), is strongly reminiscent of Brownian motion, there are obvious differences that must be taken into account. This requires in particular the introduction of specific dynamical postulates whose only justification (at this time) is pragmatic, inasmuch, as they allow us to derive Schrödinger's equation.

On the other hand, there exist several attempts to derive quantum-mechanical results by studying the motion of a harmonically bound particle with radiation reaction, which reaches a state of equilibrium through its interaction with the stochastic zero-point radiation field[4-7]. In this description, which we call stochastic electrodynamics (SED), the dynamical postulates are clear; what is not at all clear, however, is that the few results obtained up to now after considerable effort (e.g., the energy and the Lamb shift for the ground state of the harmonic oscillator) represent quantum-mechanical results, and not merely results that could be predicted for a Brownian motion-type (i.e., classical) system. (The objection that only very simple problems are tractable with this method simply stresses a technical, but not conceptual, difficulty.)

These two approaches are not mutually exclusive. On the contrary: an appropriate combination of them results in a richer theory whose basic postulates are clear and whose results are justified. In fact, SED gives us the answer to the basic question of SQM about the nature of the stochastic force responsible for the quantum-mechanical behaviour of the electron, and at the same time, SQM provides us with the necessary tools to demonstrate that the system described by SED is indeed quantum-mechanical. As a result we obtain a physically wellfounded and consistent picture of the quantum-mechanical system.

In this paper we justify this proposition using the example of the harmonic oscillator. First we solve the problem of the oscillator in its ground state using the basic postulates of SED (Section II); the treatment we use can be extended to yield the excited states as well (Section III). Once we have obtained this solution, we show that it is consistent with the description of the quantum-mechanical oscillator as provided by SQM (Section IV). In other words, we show that by starting from a Langevin-type equation for the harmonic oscillator coupled to a stochastic electromagnetic field whose spectral energy density is given by Planck's law, we may derive the corresponding Schrödinger equation as a statistical law describing the equilibrium state of the system.

The main results of our treatment are the following:

a) The harmonically bound particle of SED follows the rules of QM once the system has reached the state of equilibrium; thus we conclude that QM is an asymptotic theory[8]. Clearly, whether our solution holds also for smaller times is a question that requires further investigation.

b) The joint action of the electromagnetic forces upon the particle produces a Lamb shift, whose nonrelativistic value coincides with that predicted by nonrelativistic electrodynamics with a cut-off frequency $\omega_c = mc^2/\hbar$. [9] Moreover, a convergent result is obtained even before introducing any cut-off.

c) The treatment can be applied in particular to the free particle, to calculate the mass renormalization due to its coupling with the zero-point radiation field. Again the result obtained is convergent (though too large) even before introducing relativistic considerations

Introducing once more the relativistic cut-off frequency ω_c, we obtain the result $\delta m = \alpha m/6\pi$ ($\alpha = e^2/\hbar c$), which confirms the assumption usually made in quantum electrodynamics that the mass renormalization is of order α.

II. THE STOCHASTIC HARMONIC OSCILLATOR IN ITS GROUND STATE

1. Statement of the problem.

We set out to study the motion of a charged harmonic oscillator in one dimension, following the lines of SED[4,5,7]. According to this theory, the oscillator is acted on both by a random, stationary electromagnetic force and by the radiative reaction. In a non-relativistic treatment it suffices to write the Lorentz force simply as eE, the electric field E being a stochastic function of time only. Hence the equation of motion reads

$$\ddot{x} + \omega_0^2 x = \tau \dddot{x} + (e/m)\, E(t) \tag{1}$$

where ω_0 is the natural frequency of the oscillator and $\tau = 2e^2/3mc^3$.

For $E(t)$ to represent the zero-point radiation field, we must impose on it the following conditions: i) it is a stationary Gaussian process with zero mean; ii) its Fourier transform, defined through

$$E(t) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} f(\omega) \exp(i\omega t)\, d\omega \tag{2}$$

with

$$f(-\omega) = f^*(\omega), \tag{3}$$

has a correlation[4,5,11]

$$\langle f(\omega)\, f^*(\omega') \rangle = (2\hbar/3c^3)\, |\omega|^3\, \delta(\omega - \omega'). \tag{4}$$

Eq. (4) implies that the spectral energy density of the field is $\hbar\omega^3/2\pi^2 c^3$.

Having specified the field through its statistical properties only, we are forced to make a statistical description of the system. The ensemble averages are taken over the set of all possible $f(\omega)$.

Next we go over to the solution of Eq. (1). We impose on it more stringent requirements than earlier treatments. First of all we demand that the solution be causal, which in this instance means that x is to be computed as an integral extending from 0 to t to take into account only the retarded effects of $E(t)$. Secondly we pay attention to the effects of the radiation-reaction force. As is well known, the characteristic equation associated with Eq. (1) is usually written as an algebraic equation of third degree, one of its roots being spurious and giving rise to run-away solutions. We therefore propose a solution which a priori is related only to the physically legitimate roots of the characteristic equation, namely,

$$x = \int_0^t E_m(t') \exp\left[-\sigma(t-t')\right]\left[a \operatorname{sen} \omega_1(t-t') + b \cos \omega_1(t-t')\right] dt' \tag{5}$$

with $\sigma > 0$. The parameters a, b, σ and ω_1 are determined by introducing Eq. (5) into Eq. (1). After selecting a and b in the form

$$a = e(1+\tau\sigma)/m\omega_1 \;; \qquad b = e\tau/m \;, \tag{6}$$

we are left with two algebraic equations for σ and ω_1:

$$\omega_0^2 + (1+\tau\sigma)(\sigma^2 - \omega_1^2) - 2\tau\sigma\omega_1^2 = 0 \tag{7}$$

$$2\sigma(1+\tau\sigma) + \tau(\sigma^2 - \omega_1^2) = 0 \tag{8}$$

and a differential equation for E_m, the field as modified by its interaction with the radiating particle:

$$-\tau^2 \ddot{E}_m + \left[(1+\tau\sigma)^2 + \tau^2\omega_1^2\right] E_m = E(t). \tag{9}$$

Since $E(t)$ cannot be disconnected, we do not include the general solution of the homogeneous equation. To solve Eq. (9), we take its Fourier transform and use Eq. (2), thus obtaining:

$$E_m(t) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} \left(f(\omega)/\Delta_1\right) \exp(i\omega t)\, d\omega \tag{10}$$

where

$$\Delta_1 = (1+2\tau\sigma)^2 + \tau^2\omega^2 \;. \tag{11}$$

After introducing Eq. (10) into Eq. (5) and performing the time integration, we obtain:

$$x = (e/m\sqrt{2\pi}) \int_{-\infty}^{\infty} \left(f(\omega)/\Delta_1\Delta_2\right) \{ \exp(i\omega t)[1 + 2\tau\sigma + i\tau\omega] +$$

$$+ \exp(-\sigma t)[(\tau\omega_1 - (1 + \tau\sigma)(\sigma + i\omega)/\omega_1) \operatorname{sen} \omega_1 t - (1 + 2\tau\sigma + i\tau\omega) \cos \omega_1 t]\} d\omega$$

(12)

where

$$\Delta_2 = \omega_1^2 + (\sigma + i\omega)^2 \quad . \tag{13}$$

For times long compared with σ^{-1}, the instantaneous position of the particle is given by

$$x = (e/m\sqrt{2\pi}) \int_{-\infty}^{\infty} \left(f(\omega)/\Delta_1\Delta_2\right)(1 + 2\tau\sigma + i\tau\omega) \exp(i\omega t)\, d\omega \quad .$$

Hence we see that the oscillator gradually reaches a stationary state, in which it is in stochastic equilibrium with the vacuum field.

2. The energy of the system; the Lamb shift.

We are now in a position to calculate the average energy of the oscillator from the usual formula

$$\tfrac{1}{2} m < \dot{x}^2 + \omega_1^2 x^2 > \quad . \tag{14}$$

A word of caution is required here, however. By deriving Eq. (12) with respect to time and taking $t = 0$, we obtain:

$$\dot{x}(0) = (e\tau/m\sqrt{2\pi}) \int_{-\infty}^{\infty} \left(f(\omega)/\Delta_1\right) d\omega = (e\tau/m)\, E_m(0) \quad .$$

Hence, the initial velocity (which is a stochastic variable as well) is different from zero and contributes to the initial energy an amount

$$(e^2\tau^2/4\pi m)\left\langle\left|\int_{-\infty}^{\infty}\left(f(\omega)/\Delta_1\right)d\omega\right|^2\right\rangle . \qquad (15)$$

What we are interested in is the energy $\varepsilon(t)$ the particle has acquired since $t = 0$; this is obtained by subtracting (15) from (14). We calculate it after the transient has disappeared, i.e., for $t \gg \sigma^{-1}$. Using Eq. (4) to perform the averages, we finally get

$$\varepsilon = (\lambda e^2/2\pi m)\int_0^{\infty}\left(\omega^3/\left|\Delta_1(\omega)\Delta_2(\omega)\right|^2\right)\left[M_1 + M_2\omega^2\right]d\omega \qquad (16)$$

where

$$\lambda = 2\hbar/3c^3 \qquad (17)$$

$$M_1 = \omega_1^2(1 + 2\tau\sigma)^2 - \tau^2(\omega_1^2 + \sigma^2)^2 \qquad (18)$$

$$M_2 = (1 + 2\tau\sigma)^2 + \tau^2(3\omega_1^2 - 2\sigma^2) . \qquad (19)$$

The integral in Eq. (16) is clearly convergent. Two facts have combined to produce this convergent result: i) Eqs.(14) and (15) both contain a logarithmically divergent integral which cancels out when we take their difference to get Eq. (16); ii) E_m differs from E in that it contains the denominator $\Delta_1(\omega)$, which eliminates a quadratic divergence. Hence it is clear that the introduction of E_m is crucial, and that a perturbative approximation to it would destroy the convergence of Eq. (16).

For the calculations that follow it is convenient to introduce a set of dimensionless parameters defined by

$$q = \tau^2\omega_0^2; \quad q\mu = \tau^2\omega_1^2; \quad q\eta = 2\tau\sigma \qquad (20)$$

and a dimensionless variable defined by

$$u = \tau^2\omega^2 . \qquad (21)$$

In terms of these, Eq. (16) reads:

$$\varepsilon = (\lambda e^2/4\pi m\tau^2)\int_0^\infty (u/R\Delta_1^2)[m_1 + m_2 u]\, du \tag{22}$$

where

$$R = \tau^4|\Delta_2|^2 = q^2(\mu + \tfrac{1}{4}q\eta^2)^2 - 2q(\mu - \tfrac{1}{4}q\eta^2)u + u^2 \tag{23}$$

$$m_1 = q\mu(1+q\eta)^2 - q^2(\mu + \tfrac{1}{4}q\eta^2)^2 \tag{24}$$

$$m_2 = (1+q\eta)^2 + 3q\mu - \tfrac{1}{2}q^2\eta^2 \quad . \tag{25}$$

In terms of these new variables, Δ_1 becomes

$$\Delta_1 = (1+q\eta)^2 + u \quad . \tag{26}$$

Notice that the energy as given by Eq. (22) has a value different from zero even for $\omega_0 = 0$ (i.e., $q = 0$); this value is

$$\varepsilon_0 = (\lambda e^2/4\pi m\tau^2)\int_0^\infty \left(du/(1+u)^2\right) \tag{27}$$

and represents the energy acquired by a free particle due to its interaction with the vacuum. The net energy of the oscillator is therefore the difference between ε and ε_0, which we call $\Delta\varepsilon$:

$$\Delta\varepsilon = (\lambda e^2/4\pi m\tau^2)\int_0^\infty [((m_1 + m_2 u)/R\Delta_1^2)u - (1/(1+u)^2)] \tag{28}$$

The integration may be carried out exactly, but the algebra involved is quite complicated. We therefore prefer to introduce some approximations that simplify the expressions: we observe first that for a nonrelativistic oscillator, $q = \tau^2\omega_0^2 \ll 1$, and hence conclude from Eqs. (7) and (8) that the values of μ and η are close to unity. Performing all the calculations to first order in q (the zeroth order terms cancel out), we obtain:

$$\Delta\varepsilon = (\lambda e^2/4\pi m\tau^2)[2\pi\sqrt{q} - 2q - 3q\ln q]$$

or in terms of the physical parameters:

$$\Delta\varepsilon = \tfrac{1}{2}\hbar\omega_0 \left[1 + (\tau\omega_0/\pi)\left(3 \ln (1/\tau\omega_0) - 1\right)\right] \quad . \tag{29}$$

This result, which represents the (average) total equilibrium energy of the oscillator, may be divided into two parts: in the first place, there is the term $\frac{1}{2}\hbar\omega_0$ which coincides with the energy of the corresponding mode of the vacuum with which it is in equilibrium; the second contribution represents the Lamb shift of the oscillator due to its self-interaction via the vacuum field. The result obtained for the Lamb shift, which can be approximated as

$$\delta\varepsilon = (\alpha\hbar^2\omega_0^2/\pi mc^2) \ln (3mc^2/2\alpha\hbar\omega_0) \ , \tag{30}$$

is quite gratifying, since it is of the right order of magnitude and its computation did not involve divergent integrals. Its numerical value may still be improved by taking into account relativistic effects, such as the uncoupling of the particle from the field and its increased radiation at high frequencies. The simplest way of doing this is by cutting off the integral in Eq. (28) at a frequency ω_c of order $mc^2/\hbar$. Introducing this cut-off we obtain for the Lamb shift:

$$\delta\varepsilon = (\alpha\hbar^2\omega_0^2/\pi mc^2) \ln (mc^2/\hbar\omega_0) \ . \tag{31}$$

This result coincides exactly with the value given by nonrelativistic quantum electrodynamics for the ground state of the harmonic oscillator, when the same cut-off frequency is introduced to make the result convergent.[12]

Results similar to ours, but based on a somewhat different approach, have been obtained previously by Marshall[4] for the ground state energy and by Braffort et.al.[5] and Santos[7] for the Lamb shift of the harmonic oscillator.

3. The mass correction.

As an important by-product of our treatment of the harmonic oscillator, we obtained in the previous section an expression for the energy acquired by a free particle in the vacuum field (see Eq. 27). From this expression we may calculate the mass correction by carrying out the indicated integration; we obtain

$$\delta m = (3/8\pi\alpha) m \ . \tag{32}$$

This result, though finite, is evidently too large. To get a better estimate, we introduce the relativistic cut-off frequency ω_c, whose value we may consider to be determined by the previous calculation of the Lamb shift of the harmonic oscillator; this amounts to introducing an additional factor $(2\alpha/3)^2$ in (32):

$$\delta m = (\alpha/6\pi)\, m \quad . \tag{33}$$

Thus it is possible to obtain from elementary calculations the renormalized mass of an electron in interaction with the electromagnetic vacuum[13]; this result justifies the usual assumption of quantum electrodynamics that $\delta m \sim \alpha m$. Notice that the numerical coefficient in (33) is sensitive to the selected cut-off.

III. EXCITED STATES OF THE HARMONIC OSCILLATOR

1. Calculation of the energy.

As the temperature of the heat bath rises, the oscillators of the ensemble become excited, i.e., they go over to higher energy states. In this Section we study an ensemble of oscillators in equilibrium with the random electromagnetic field at temperatures $T > 0$, whose spectral energy density is given by Planck's law:

$$I = I_0 (1+\epsilon)/(1-\epsilon) \tag{34}$$

where $I_0 = 2\hbar|\omega|^3/3c^3$, $\epsilon = \exp(-\beta\hbar\omega)$ and $\beta^{-1} = kT$. We therefore write instead of Eq. (4):

$$\langle f(\omega) f^*(\omega')\rangle = I(\omega)\,\delta(\omega-\omega') \tag{35}$$

where $I(\omega)$ is given by Eq. (34).

Now we propose to write the Fourier component $f(\omega)$ of the electric field as a power series in ϵ:

$$f(\omega) = \sum_{k=0}^{\infty} f_k(\omega)\, \epsilon^k \quad . \tag{36}$$

It is easily seen from Eqs. (34) and (35) that the correlation of the f_k has the form

$$< f_k(\omega)\, f^*_{k'}(\omega') > = (1-\epsilon^2)\, I_0\, \delta(\omega-\omega') \tag{37}$$

Using this and Eq. (36), we may rewrite Eq. (35) as follows:

$$< f(\omega)\, f^*(\omega') > = (1-\epsilon)\, I_0\, \delta(\omega-\omega')\left[(1+\epsilon) \sum_{k,k'=0}^{\infty} \epsilon^{k+k'}\right] .$$

Since there are $n+1$ different combinations of integers k and k' such that $k+k'=n$ and n combinations for which $k+k'+1=n$, the above result can be written also in the form:

$$< f(\omega)\, f^*(\omega') > = (1-\epsilon)\, I_0\, \delta(\omega-\omega') \sum_{n=0}^{\infty} (2n+1)\, \epsilon^n . \tag{38}$$

Now we introduce this into the expressions of Section II and follow the same procedure to calculate the energy of the ensemble of excited oscillators. In performing the integrations, we observe that there is a strong resonance at $\omega \approx \omega_0$, due to the denominator Δ_2 ; hence we can make an approximate evaluation of the integrals by taking the factors $\epsilon^n = \exp(-n\beta\hbar\omega)$ out of the integrand and writing them as $\exp(-n\beta\hbar\omega_0)$ – the price we pay for this simplification is that we can no longer calculate the Lamb shift for the excited states. With this approximation, we arrive at the simple result:

$$\Delta\varepsilon = (1-\epsilon) \sum_{n=0}^{\infty} \tfrac{1}{2}\hbar\omega_0 (2n+1)\, \epsilon^n = (1-\epsilon) \sum_{n=0}^{\infty} \varepsilon_n\, \epsilon^n \tag{39}$$

where

$$\varepsilon_n = \hbar\omega_0 (n+\tfrac{1}{2}) . \tag{40}$$

$\Delta\varepsilon$ represents the average energy of the oscillators of the ensemble. Eq. (39) tells us that there are an infinite number of possible states, contributing with an energy ε_n and with statistical weight ϵ^n. The factor in front of the sum in Eq. (39) is just the inverse of the partition function Z:

$$Z = \sum_{n=0}^{\infty} \epsilon^n = 1/(1-\epsilon)$$

and therefore Eq. (39) can be rewritten in the form:

$$\Delta\varepsilon = (1/Z)\sum_n \varepsilon_n \epsilon^n \ . \tag{41}$$

Thus we have obtained two results of QM, namely: i) the energy for the n^{th} level of the harmonic oscillator, Eq. (40), and ii) the average energy for a mixed state in thermodynamic equilibrium with the black-body radiation at temperature T, Eq. (41).

2. The density of particles.

The results obtained allow us to calculate the probability density ρ as a function of x. Let us begin, for simplicity, by determining ρ_0, the density at $T=0$. To this end we calculate the mean square of $x(t)$ to first order in τ, from Eq. (12); the result is

$$\overline{x_0^2} \equiv \langle x^2 \rangle_{T=0} = (\hbar/2m\omega_0)\big(1-\exp(-\sigma t)\big) \times$$

$$\times \left[1-\exp(-\sigma t)\right| - (2\sigma/\omega_0)\exp(-\sigma t)\ \mathrm{sen}\ \omega_0 t \cos\omega_0 t\big] \ . \tag{42}$$

With the same approximation, it follows from Eqs. (7) and (8) that

$$\sigma = \tfrac{1}{2}\tau\omega_0^2 \ . \tag{43}$$

Since Eq. (1) is linear and $E(t)$ is Gaussian, it follows that $\rho(x)$ is also Gaussian; we therefore get

$$\rho_0(x,t) = (2\pi\overline{x_0^2})^{-\frac{1}{2}} \exp(-x^2/2\overline{x_0^2}) \ . \tag{44}$$

It can be easily seen from the calculations of III.1 that the density of the mixture at temperature T is Gaussian as well, its variance being

$$\overline{x^2} = \overline{x_0^2}\,(1+\epsilon)/(1-\epsilon) \quad . \tag{45}$$

Moreover, since the ensemble is in equilibrium, we have

$$\rho = Z^{-1} \sum_n \rho_n \epsilon^n \ . \tag{46}$$

Eqs. (45) and (46) together imply the formula

$$\left[\sqrt{1-\epsilon}/\sqrt{2\pi\,\overline{x_0^2}\,(1+\epsilon)}\right] \exp\left[-(x^2/2\overline{x_0^2})(1-\epsilon)/(1+\epsilon)\right] = (1-\epsilon)\sum_n \rho_n \epsilon^n$$

from which we may determine the ρ_n by iterative derivation. Marshall has shown[4] that the ρ_n obtained with this generating function coincide with the expressions of QM for all n, if $\overline{x_0^2}$ is taken as a constant parameter of value

$$\overline{x_0^2} = (\hbar/2m\omega_0) \tag{47}$$

which is precisely the value given by Eq. (42) for $\sigma t \gg 1$.

Other formulae of QM can be derived from this formulation in the asymptotic limit of large times. For instance, the mean square of $p(t) = m\dot{x}(t)$ at temperature $T = 0$ is

$$\overline{p_0^2} = \tfrac{1}{2}\hbar m\omega_0 \tag{48}$$

for $\sigma t \gg 1$; from this and Eq. (47) we obtain the Heisenberg relation

$$\overline{x_0^2}\,\overline{p_0^2} = \tfrac{1}{4}\hbar^2 \tag{49}$$

for the ground state. To derive an analogous relation for the excited states we write

$$\overline{x^2} = Z^{-1}\sum_n \overline{x_n^2}\,\epsilon^n$$

and transform this equation using Eq. (45), to get

$$\sum_n \overline{x_n^2}\, \epsilon^n = \overline{x_0^2} \sum_n (2n+1)\, \epsilon^n$$

from which

$$\overline{x_n^2} = \overline{x_0^2}\,(2n+1)\ . \tag{50}$$

In a similar way we obtain

$$\overline{p_n^2} = \overline{p_0^2}\,(2n+1)\ . \tag{51}$$

From Eq. (50) and (51) it follows that[14]

$$\overline{x_n^2}\;\overline{p_n^2} = \tfrac{1}{4}\hbar^2 (2n+1)^2\ . \tag{52}$$

Within this context, the meaning of the Heisenberg relations is precise: they relate the dispersions of a pair of dynamical variables and hence can only be interpreted in the statistical sense.

IV. THE SCHRÖDINGER EQUATION

1. Stochastic quantum mechanics.

We have shown that the solutions to the dynamical problem defined by Eqs. (1–4) or their generalization (1, 2, 34–36) coincide formally in the asymptotic limit with the corresponding results of QM. To confirm our basic proposition, namely, that what we are dealing with *is* a quantum-mechanical system, we should be able to derive the Schrödinger equation as an asymptotic equation of our theory.

A simple way to achieve this is by using the dynamical description provided by stochastic quantum mechanics. For the sake of completeness we present here a brief review of the main results of SQM[3,15].

The theory gives a statistical description of the behaviour of a stochastic particle in configuration space; hence it is valid only after local equilibrium has been attained. Two different local mean velocities are associated with the stochastic motion of the particle, namely, the forward and the backward velocities or alternatively, the systematic ($\mathbf{v}$) and the stochastic ($\mathbf{u}$) velocities. The first of these enters into the continuity equation

$$(\partial\rho/\partial t) + \nabla \cdot (\mathbf{v}\rho) = 0 \tag{53}$$

and the second one is given by

$$\mathbf{u} = D(\nabla\rho/\rho) \tag{54}$$

where D stands for the diffusion coefficient. These velocities can be obtained by applying to $\mathbf{x}$ the derivative operators

$$\mathcal{D}_c = (\partial/\partial t) + \mathbf{v}\cdot\nabla \tag{55}$$

and

$$\mathcal{D}_s = \mathbf{u}\cdot\nabla + D\nabla^2 \tag{56}$$

respectively, i.e.,

$$\mathbf{v} = \mathcal{D}_c\mathbf{x} \; ; \qquad \mathbf{u} = \mathcal{D}_s\mathbf{x} \quad . \tag{57}$$

In developing the dynamics, we refer for simplicity only to the case in which the external force is derivable from a potential. There are two dynamical equations: one of them expresses the conservation of particles and hence is equivalent to the continuity equation, (53). In fact, we may derive it by taking the gradient of Eq. (53) and rewriting the result in terms of the above definitions:

$$\mathcal{D}_c\mathbf{u} + \mathcal{D}_s\mathbf{v} = 0 \tag{58}$$

The second dynamical equation relates the accelerations to the external force $\mathbf{F}$; it can be shown that the most general linear relationship is

$$\mathcal{D}_c\mathbf{v} - \lambda\mathcal{D}_s\mathbf{u} = \mathbf{F}/m \tag{59}$$

where λ is a real parameter. The sign of λ is crucial in defining the dynamics of the system. In fact, it has been shown that Eq. (59) applies to Brownian motion when $\lambda = -1$ and to QM when $\lambda = +1$. The latter is most easily demonstrated as follows: instead of $\mathbf{v}$ and $\mathbf{u}$, take two new functions w_+ and w_- such that

$$\mathbf{v} \pm \sqrt{-\lambda}\,\mathbf{u} = \pm 2D\sqrt{-\lambda}\,\nabla w_{\pm} \tag{60}$$

and rewrite the fundamental equations (58) and (59) in terms of them. The new equations uncouple and linearize upon the further change of variable

$$\psi_{\pm} = \exp w_{\pm} \,. \tag{61}$$

In fact, we obtain after a first integration:

$$\mp 2mD\sqrt{-\lambda}\,(\partial\psi_{\pm}/\partial t) = -2mD^2\lambda\nabla^2\psi_{\pm} + V\psi_{\pm} \tag{62}$$

where $\mathbf{F} = -\nabla V$. We see that for $\lambda = 1$, Eq. (62) indeed reduces to the Schrödinger equation, if the diffusion coefficient is given the value

$$D = \hbar/2m \quad . \tag{63}$$

The probability density is given in general by

$$\rho = \psi_+\psi_- \tag{64}$$

and for $\lambda = 1$ in particular, $\psi_+ = \psi_-^*$, as follows from Eq. (62). For further details we refer the reader to the literature.[1,3,15,16,17]

In the following we apply this description to the harmonic oscillator of Sections II and III, with the aim of demonstrating that it obeys the Schrödinger equation for $t \gg \sigma^{-1}$.

2. The identification of the harmonic oscillator of SED as a quantum-mechanical system.

Let us return to the one-dimensional stochastic harmonic oscillator discussed previously. To simplify the calculations, we shall determine the velocities v and u for the ground state only. The stochastic velocity is obtained from Eqs. (44) and (54):

$$u = -(D/\overline{x_0^2})\,x \tag{65}$$

and the systematic velocity is determined by integrating the continuity equation (53):

$$v = -(1/\rho)(\partial/\partial t)\int_0^x \rho dx = -gu \tag{66}$$

where

$$g \equiv (1/2D)(d\overline{x_0^2}/dt) = (\hbar\sigma/2mD\omega_0)\big[\big((\sigma/\omega_0)\cos\omega_0 t + 2\,\mathrm{sen}\,\omega_0 t\big) -$$

$$- 2\big((\sigma/\omega_0)\cos\omega_0 t + \mathrm{sen}\,\omega_0 t\big)\exp(-\sigma t)\big]\exp(-\sigma t)\,\mathrm{sen}\,\omega_0 t\,. \tag{67}$$

We now ask ourselves if these velocities satisfy the fundamental equations of SQM. Eq. (58) is automatically satisfied for all times, since it is equivalent to the continuity equation (53). Further, we use the above results to calculate

$$\mathcal{D}_c v - \lambda\mathcal{D}_s u = -\big(D/(\overline{x_0^2})^2\big)\big[D(g^2+\lambda) - \dot{g}\,\overline{x_0^2}\big]\,x\,. \tag{68}$$

According to Eqs. (42) and (67), this reduces to

$$\mathcal{D}_c v - \lambda\mathcal{D}_s u = -\lambda(2mD\omega_0/\hbar)^2 x = \lambda(2mD/\hbar)^2(F/m) \tag{69}$$

for $t \gg \sigma^{-1}$. This expression is meaningless if we take $\lambda \neq 1$; it would describe a dynamical behaviour which is neither classical nor quantum-mechanical. On the other hand, we see that with $\lambda = 1$ Eq. (69) coincides with Eq. (59) if at the same time we take $D = \hbar/2m$. Recalling from III.1 that Eqs. (58) and (59) with $\lambda = 1$ and $D = \hbar/2m$ are equivalent to the Schrödinger equation, we arrive at the conclusion that the harmonic oscillator of SED is not classical, but quantum-mechanical. It is easy to extend the treatment to the excited states and thus show that they also satisfy the Schrödinger equation.

3. The formula for the diffusion coefficient.

The theory of SQM does not provide the means for evaluating the diffusion coefficient D and hence has admitted Eq. (63) as an empirical formula. In view of this unsatisfactory situation, several attempts have been made to estimate the value of D from general physical arguments[18, 19]. As shown above, a consistent treatment of the harmonic oscillator confirms Eq. (63), but since its value evidently does not depend on the specific problem (i.e., on the external force), we may consider this as a general result. We should like to insist here on the meaning of this result, namely, that Planck's constant enters into QM precisely through Planck's distribution law for the radiation field (with zero-point energy).

V. CONCLUSIONS

A new picture of QM has emerged from the present treatment of the harmonic oscillator, which can be essentially summarized as follows: QM is a statistical theory of particles – insofar as they obey a Langevin-type equation – that interact with the random electromagnetic field produced by the rest of the universe. The consequence of this interaction is twofold: i) the electromagnetic field becomes modified by the presence of the material system (recall that a field E_m builds up in the example of the oscillator coupled to the vacuum field E); ii) the particle acquires a stochastic motion with a statistical behaviour which reflects any regularities, such as the wave-like properties, that the field may possess. As a result of this mutual action, the structure acquired by the field as it adapts itself to the environment (including slits, borders, other particles, and so on) becomes manifest through the statistical regularities of the motion of the particle, which all along its trajectory receives via the field an integral information about the whole system. This explains the appearance of typically quantal phenomena: for instance, the electron interference pattern obtained from a double slit experiment can be thought of as a transcription of the corresponding pattern impressed upon the field by the presence of the slits. From this point of view, it is clear that interference and other wavelike phenomena in QM can be detected only by a large series of experiments, since they are predicted for the ensemble. In fact, according to our theory all the quantum-mechanical description is in terms of the ensemble and hence can only yield statistical predictions; this theory actually serves to vindicate the so-called statistical interpretation of QM[20].

As we have seen, a further consequence of the interaction between the particles and the random electromagnetic field is the appearance of additional effects such as the Lamb shift and the mass correction. The possibility of carrying out divergence-free calculations of these effects could be regarded as a motivation in itself for further research into this more general formulation of QM. Moreover, we have seen that the quantum-mechanical system admits, at least in principle, of a more complete description than the one provided by the usual theory, since: i) the present description is in terms of "hidden variables", such as the Fourier amplitudes $f(\omega)$ or the position and the velocity of the elements of the ensemble; ii) QM is obtained in the asymptotic limit, after the initial transient has died away. For the harmonically bound particle, the corresponding interval is $t \gg \sigma^{-1} = 2/\tau\omega_0^2$, from Eq. (43), or writting $\varepsilon_0 = \frac{1}{2}\hbar\omega_0 \equiv \eta mc^2$ and $\lambda_c = mc^2/\hbar$, $t \gg \lambda_c/\alpha\eta^2 c^2$. For an energetic particle this nonrelativistic calculation predicts $t \gtrsim 10^{-15} - 10^{-16}$ sec.

We would like to make a final remark about the origin of the stochastic field. In this paper we have made free use of the black-body radiation law, which was certainly the first quantum law ever established. From this point of view one could say that we are obtaining the quantization of particle systems from the quantization of the field. However, it seems more appropriate to take as the fundamental postulate the very existence of the random electromagnetic vacuum, since from this it is possible to derive both the black-body radiation law (as Boyer has done[21]) and quantum mechanics (as is done here), without explicitly introducing quantum concepts. The question of whether the zero-point radiation field itself is classical or quantal is perhaps no more than terminological; both views can be found in the literature[22].

Of course, at this primitive stage our formulation seems to pose at least as many problems as it solves; among the questions it raises, we are immediately confronted with the problem of demonstrating the general validity of the conclusions which we have drawn here based on our treatment of the harmonic oscillator. It will be also necessary to analize the solution of the general problem for short times. Another open question is the treatment of neutral particles, a problem which is being studied at present. The modification suffered by the electromagnetic field through its interaction with the particle requires further study, in particular for the problem of the mutual influence of two particles. Lastly, it should be noted that while the present attack offers certain generalisations beyond the power of the stochastic theory of quantum mechanics, this theory in several ways goes beyond the present results.

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RESUMEN

Se analiza el problema de un oscilador armónico radiante en interacción con el campo estocástico del vacío electromagnético. En la solución de la ecuación de movimiento se toma en cuenta explícitamente el hecho de que la partícula modifica el campo. La solución tiende asintóticamente a un estado estacionario, cuyas propiedades estadísticas son precisamente las del estado base del oscilador cuántico. Se demuestra además, utilizando el formalismo de la mecánica cuántica estocástica, que el comportamiento estadístico del oscilador en equilibrio con el campo de radiación, está descrito exactamente por la ecuación de Schrödinger. Un tratamiento similar usando el campo de radiación de cuerpo negro a temperaturas $T > 0$ da los estados excitados del sistema, que también coinciden con los resultados cuánticos. Se concluye que el oscilador armónico en el campo electromagnético estocástico no es un sistema clásico -como generalmente se supone- sino un modelo del oscilador cuántico, y de hecho un modelo más detallado que el usual, dado que también se describe la evolución hacia el equilibrio. De la solución de la ecuación de movimiento se obtiene en forma directa una expresión no divergente para el corrimiento Lamb del oscilador armónico; el resultado coincide con el resultado usual de la electrodinámica cuántica cuando se introduce una frecuencia de corte relativista $\omega_c = mc^2/\hbar$. Con la misma frecuencia de corte se obtiene la renormalización de la masa del electrón, también sin divergencias; esta corrección es proporcional a la constante de estructura fina.