Revista Mexicana de Física 25 (1976) 31-45

FINITE $SL(2,R)$ REPRESENTATION MATRICES OF THE D_k^+ SERIES FOR ALL SUBGROUP REDUCTIONS

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(Recibido: septiembre 9, 1975)

ABSTRACT: Using canonical transform techniques, we find the unitary irreducible representation matrix elements, or integral kernels, for finite $SL(2,R)$ transformations when the generator of the diagonal subgroup is any of the one-parameter inequivalent subgroups $SO(2)$, $SO(1,1)$ or $E(2)$. The method reduces the problem to the solution of a single integral and the results are given in terms of hypergeometric functions. The mixed basis representation matrices are also given.

I. INTRODUCTION

The most direct approach in finding the representation matrix elements of finite transformations

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,R), \quad ad-bc=1,$$

is as follows. First one chooses a Hilbert space $\mathcal{H}$ of functions f of x in some space X where the action of $SL(2,R)$ is well defined and onto

$$f(x) \xrightarrow{g} [\mathfrak{I}(g)f](x) . \tag{1.1}$$

Then one builds a complete orthonormal basis $\{\psi_\lambda^\alpha\}$, $(\lambda \in \Lambda \subset \boldsymbol{R}$, the real field) for $\mathcal{H}$, generalized eigenfunctions of an element J_α of the Lie algebra $sl(2,R)$ of $SL(2,R)$, whose spectrum is Λ and which belongs to a given irreducible representation D_k of $sl(2,R)$.[1] Finally, one computes

$${}^\alpha D^k_{\lambda'\lambda}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \left(\psi^\alpha_{\lambda'}, \mathfrak{I}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\psi^\alpha_\lambda\right) = \int_X d\mu(x)\, \psi^\alpha_{\lambda'}(x)^* \left[\mathfrak{I}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\psi^\alpha_\lambda\right](x), \tag{1.2}$$

where $d\mu(x)$ is the appropiate measure over X.

In reviewing the literature we see that a number of papers[1] have implemented this method for the principal[2,3] and discrete[2,4] series. A somewhat different approach calls for finding the ${}^\alpha D^k_{\lambda'\lambda}$ functions (1.2) for one-parameter subgroups as solutions of differential equations with boundary conditions imposed by $\mathfrak{I}(1) = 1$ for the group identity.[3,4] All these results are well known for the case when the diagonal element of the algebra is the generator of the $SO(2)$ subgroup of $SL(2,R)$. When the operator chosen diagonal is the generator of a non-compact subgroup, the results are not so easy to obtain[4,5] due mainly to difficulties in evaluating (1.2). The simplest supporting space X for the principal series is the circle and the Hilbert space is $\mathcal{L}^2(-\pi, \pi)$. For the "discrete" series, however, most of the literature concerns itself with the unit disk,[1,6,7] the integration (1.2) being performed over a two-dimensional manifold. The latter is more difficult when the diagonal operator is non-compact.

It has come to our attention that the techniques of canonical transforms of References 6 and 8 considerably simplify the evaluation of (1.2) since one realizes the action (1.1) of $SL(2,R)$ as unitary mappings of $\mathcal{L}^2(\boldsymbol{R}^+)$ through a non-local (or integral transform) action,[6–9] as

$$\left[\mathfrak{I}\begin{pmatrix} a & b \\ c & d \end{pmatrix}f\right](r) = \int_0^\infty dr' \,[\exp(-i\pi b)]\, b^{-1} (rr')^{\frac{1}{2}} \times$$
$$\times \exp\left[(i/2b)(ar'^2 + dr^2)\right] J_{2k-1}(rr'/b)\, f(r'). \tag{1.3a}$$

When $b \to 0$, we see from the asymptotic properties of the Bessel function that (1.3a) becomes the geometric action.

$$[\mathfrak{I}\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} f](r) = |a|^{-\frac{1}{2}} \exp[(ic/2a)r^2] f(|a|^{-1} r). \tag{1.3b}$$

This can also be seen from the direct exponentiation of the operators. It should be mentioned that for $2k$ non-integer, (1.3) yields a ray representantion of $SL(2,R)$ which can be extended to a true representation of the universal covering group $\overline{SL(2,R)}$ as will be given below. This corresponds to the exponentiation of the second-order differential operator realization of the $sl(2,R)$ algebra given by

$$J_1 = (1/4)[-(d^2/dr^2) + (\mu/r^2) - r^2], \tag{1.4a}$$

$$J_2 = -(i/4)[r(d/dr) + (d/dr)r], \tag{1.4b}$$

$$J_3 = (1/4)[-(d^2/dr^2) + (\mu/r^2) + r^2], \tag{1.4c}$$

which belong to the "discrete" series D_k^+ of representations of $sl(2,R)$, where

$$\mu = (2k-1)^2 - (1/4) \tag{1.4d}$$

and the value of the Casimir invariant $J_3^2 - J_1^2 - J_2^2$ is $k(1-k)$.

The association of one-parameter subgroups generated by (1.4a-c) and group elements $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

$$\exp(i\alpha J_1): \begin{pmatrix} \cosh\frac{1}{2}\alpha & -\sinh\frac{1}{2}\alpha \\ -\sinh\frac{1}{2}\alpha & \cosh\frac{1}{2}\alpha \end{pmatrix}, \quad \exp(i\beta J_2): \begin{pmatrix} \exp(-\frac{1}{2}\beta) & 0 \\ 0 & \exp\frac{1}{2}\beta \end{pmatrix}, \tag{1.5a,b}$$

$$\exp(i\gamma J_3): \begin{pmatrix} \cos\frac{1}{2}\gamma & -\sin\frac{1}{2}\gamma \\ \sin\frac{1}{2}\gamma & \cos\frac{1}{2}\gamma \end{pmatrix}, \tag{1.5c}$$

$$\exp(ib[J_1 + J_3]): \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}, \quad \exp(ic[J_3 - J_1]): \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}. \tag{1.5d,e}$$

The representation (1.3) can be extended to $\overline{SL(2,R)}$ through the application of $\exp(-2\pi i J_3) = \exp(-2\pi i k)$, extending thus (1.5c) to the full γ-line.

The integral kernel in (1.3) is thus a ${}^{\delta}D$ function of the kind (1.2) in the generalized basis $\psi_{r'}^{\delta}(r) = \delta(r - r')$, $r, r' \in R^+$, which is complete and orthonormal for $\mathfrak{L}^2(R^+)$. Our task is to find the corresponding expressions in terms of eigenbases of other operators of the algebra (1.4). Of particular interest are the eigenbases of $H^b = 2J_3$ (quantum harmonic oscillator plus centrifugal potential) given in Section II which have been obtained in a different way by Moshinsky and Quesne.[9] The eigenbases of $H^r \equiv 2J_1$(repulsive oscillator plus centrifugal potential) are given in Section III. We devote proportionately more space to them since these expressions are, to our knowledge, new. For $H^f \equiv J_1 + J_3$ (pure centrifugal potential or, when $\mu = 0$, the Schrödinger free particle) we develop Section IV. Mixed-basis elements are given in Section V. We concentrate on the space $\mathcal{L}^2(\boldsymbol{R}^+)$ rather than $\mathcal{L}^2(\boldsymbol{R})$ since the $r > 0$ and $r < 0$ subspaces are left invariant under the group generated by (1.4). The matching of the two will be made in Section VI where we also examine the two representations $D^+_{k_1}$ and $D^+_{k_2}$ to which (1.4) belong, namely, for

$$k_1 = \tfrac{1}{2}(1 + [\mu + \tfrac{1}{4}]^{\frac{1}{2}}), \quad k_2 = \tfrac{1}{2}(1 - [\mu + \tfrac{1}{4}]^{\frac{1}{2}}). \tag{1.6}$$

We would like to stress that the novelty of our method hinges on the fact that the action (1.3a) of the group can be given entirely in terms of geometric transforms (1.3b) and the appropiate one-parameter subgroup (1.5) generated by the chosen diagonal operator.[10] This means that the integration (1.3a) can be entirely circumvented (for eigenfunctions of H^b, H^r and H^f) and that the evaluation of (1.2) reduces to a single integral which can be found in tables such as Reference 11, and the result expressed in terms of a hypergeometric function.

II. THE HARMONIC OSCILLATOR BASIS

The eigenfunctions of

$$H^b \equiv 2J_3 = \tfrac{1}{2}[-(d^2/dr^2) + (\mu/r^2) + r^2] \tag{2.1}$$

for $\mu > -\frac{1}{4}$ are well known to be

$$\psi^b_n(r) = [(2n!)/\Gamma(n+2k)]^{\frac{1}{2}} r^{2k-\frac{1}{2}} \exp(-\tfrac{1}{2}r^2) L_n^{(2k-1)}(r^2), \tag{2.2a}$$

with eigenvalues

$$\lambda = 2(n+k), \; n = 0,1,2,\ldots \tag{2.2b}$$

and $k > (1/2)$ is related to μ through (1.4d)-(1.6). Now, we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{a} & 0 \\ \bar{c} & \bar{a}^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.3a}$$

and find, solving for $\bar{a}$, $\bar{c}$ and θ,

$$\tan\theta = -b/a,\ \bar{a} = (a^2+b^2)^{\frac{1}{2}} > 0,\ \bar{c} = (ac+bd)(a^2+b^2)^{-\frac{1}{2}}. \tag{2.3b}$$

The decomposition (2.3) can always be made since it is just the Iwasawa decomposition into the compact ($\exp[2i\theta J_3]$) and solvable (geometric) subgroups. The former only multiplies (2.2a) by a factor $\exp(i\lambda\theta)$ while the latter is given by (1.3b). Hence, we can write

$$\begin{aligned}[\mathfrak{I}\begin{pmatrix} a & b \\ c & d\end{pmatrix} \psi_n^b](r) &= [\exp(i\lambda\theta)]\, \bar{a}^{-\frac{1}{2}} \exp[(i\bar{c}/2\bar{a})r^2]\, \psi_n^b(\bar{a}^{-1}r) \\ &= [(2n!)/\Gamma(n+2k)]^{\frac{1}{2}} \exp[2i(n+k)\arg(a-ib)] \times \\ &\times (a^2+b^2)^{-k}\, r^{2k-\frac{1}{2}} \exp[(-r^2/2)(d-ic)/(a+ib)]\, L_n^{2k-1}(r^2/[a^2+b^2]).\end{aligned} \tag{2.4}$$

Taking the scalar product of (2.2a) with (2.4) gives rise to a single integral (Ref. 11, Eq. 7.414.4) and we obtain

$$\begin{aligned}{}^bD^k_{n'n}\begin{pmatrix} a & b \\ c & d\end{pmatrix} &\equiv (\psi^b_{n'}, \mathfrak{I}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\psi^b_n) = 2^{2k}\,\Gamma(n'+n+2k)\,[n'!\,n!\,\Gamma(n'+2k)\Gamma(n+2k)]^{\frac{1}{2}} \times \\ &\times ([d-a]-i[b+c])^{n'}([a-d]-i[b+c])^{n}([a+d]+i[b-c])^{-n'-n-2k} \times \\ &\times {}_2F_1(-n',-n;-n'-n-2k+1;\,[a^2+b^2+c^2+d^2+1]/[a^2+b^2+c^2+d^2-1]).\end{aligned} \tag{2.5}$$

The hypergeometric function is a polynominal of degree $\min(n',n)$. We can check that ${}^bD^k_{n'n}(1) = \delta_{n'n}$, as it should, and that unitarity is mainfest, i.e.,

$${}^bD^k_{n'n}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = [{}^bD^k_{n'n}\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}]^*. \tag{2.6}$$

(Compare with the result of Moshinsky and Quesne.[9])

The ${}^bD$ functions (2.5) take a somewhat simpler form if we redefine the matrix elements as

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^*\end{pmatrix} = g_B\begin{pmatrix} a & b \\ c & d\end{pmatrix} g_B^{-1},\quad g_B \equiv (1/\sqrt{2})\begin{pmatrix} 1 & -i \\ -i & 1\end{pmatrix}. \tag{2.7}$$

We can then rewrite (2.5), using λ as in (2.2b), as

$$
{}^{h}D^{k}_{n'n}\begin{pmatrix} a & b \\ c & d \end{pmatrix}
$$

$$
= \exp\left(i\pi\,[k+\tfrac{1}{4}\{\lambda+\lambda'\}]\right)\Gamma\left(\tfrac{1}{2}[\lambda+\lambda']\right)\ \left[\Gamma(\tfrac{1}{2}\lambda'-k)\Gamma(\tfrac{1}{2}\lambda'+k)\Gamma(\tfrac{1}{2}\lambda-k)\Gamma(\tfrac{1}{2}\lambda+k)\right]^{-\frac{1}{2}} \times
$$

$$
\times\ \alpha^{-\frac{1}{2}(\lambda+\lambda')}\beta^{\frac{1}{2}\lambda-k}(\beta^{*})^{\frac{1}{2}\lambda'-k}\ {}_2F_1(k-\tfrac{1}{2}\lambda',\, k-\tfrac{1}{2}\lambda;\, 1-\tfrac{1}{2}[\lambda+\lambda'];\ |\alpha|^2/|\beta|^2)\,. \tag{2.8}
$$

III. THE REPULSIVE OSCILLATOR BASIS

We can consider the generalized eigenbases of $H^r \equiv 2J_1$ or of $H^d \equiv 2J_2$, generators of $SO(1,1)$ subgroups. These are related through[12]

$$
\begin{aligned}
H^r &= \exp(i\tfrac{1}{2}\pi J_3)\, H^d \exp(-i\tfrac{1}{2}\pi J_3) \\
&= \tfrac{1}{2}\left[(-d^2/dr^2) + (\mu/r^2) - r^2\right] = \left[\exp(i\tfrac{1}{2}\pi J_3)\right] \times \\
&\times \{-(i/2)\left[r(d/dr) + (d/dr)r\right]\} \exp(-i\tfrac{1}{2}\pi J_3),
\end{aligned} \tag{3.1a}
$$

and correspondingly, the eigenfunctions as

$$
\psi^r_\lambda(r) = \left[\mathfrak{J}(1/\sqrt{2})\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \psi^d_\lambda\right](r). \tag{3.1b}
$$

The easier basis to consider is that of H^d:

$$
\psi^d_\lambda(r) = (2\pi)^{-\frac{1}{2}}\, r^{i\lambda-\frac{1}{2}}\ , \tag{3.2}
$$

with eigenvalue $\lambda \in \mathbf{R}$, whose orthonormality and completeness is given by the theory of Mellin transforms, from which we can find the eigenbasis of H^r through (1.3) and (3.2) as

$$
\begin{aligned}
\psi^r_\lambda(r) &= C_{k\lambda}\left[\exp(i\tfrac{1}{2}\pi k)\right] r^{-\frac{1}{2}}\, M_{i\frac{1}{2}\lambda,\,k-\frac{1}{2}}\{[\exp(-i\tfrac{1}{2}\pi)]\, r^2\} \\
&= C_{k\lambda}\, r^{2k-\frac{1}{2}}\left[\exp(i\tfrac{1}{2}r^2)\right]\ {}_1F_1(k-i\tfrac{1}{2}\lambda;\, 2k;\, [\exp(-i\tfrac{1}{2}\pi)]\, r^2),
\end{aligned} \tag{3.3a}
$$

where

$$
C_{k\lambda} \equiv (2\pi)^{-\frac{1}{2}} \exp i\left[\tfrac{1}{2}\pi(k-i\tfrac{1}{2}\lambda) + \tfrac{1}{2}\lambda \ln 2\right]\Gamma(k+i\tfrac{1}{2}\lambda)/\Gamma(2k)\,, \tag{3.3b}
$$

$M_{\mu\nu}$ being the Whittaker function.[11] The choice of phase of ψ_λ^r is a consequence of the simpler choice of phase for ψ_λ^d as (3.3). Since the latter is easier, we shall use it in order to calculate the ${}^dD_{\lambda'\lambda}^k$ - functions.

We can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} (1/\sqrt{2}) \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \bar{a} & 0 \\ \bar{c} & \bar{a}^{-1} \end{pmatrix} \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} (1/\sqrt{2}) \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{3.4a}$$

with $\tanh\theta = -b'/a'$,

$$\exp\theta = [(a'-b')/(a'+b')]^{\frac{1}{2}}, \tag{3.4b}$$

$$\bar{a} = (a'^2 - b'^2)^{\frac{1}{2}}, \qquad \bar{c} = (c'a' - b'd')(a'^2 - b'^2)^{-\frac{1}{2}},$$

and, in order to avoid extra phase problems, we consider first the case $|a'| > |b'|$, so that $\bar{a} > 0$. Eqs. (1.3b), (1.5a), (3.1b) and (3.4) then imply

$$[\mathfrak{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\psi_\lambda^d](r) = (a'+b')^{-\frac{1}{2}-i\frac{1}{2}\lambda}\,(a'-b')^{-\frac{1}{2}+i\frac{1}{2}\lambda} \times$$
$$\times \exp[(i/2)(c'a' - b'd')/(a'^2 - b'^2)\,r^2]\,\psi_\lambda^r[r/(a'^2 - b'^2)^{\frac{1}{2}}]. \tag{3.5}$$

The construction of the matrix element (1.2) thus involves a single integration (Ref. 11, Eq. 7.621.1) which yields

$${}^dD_{\lambda'\lambda}^k\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv (\psi_{\lambda'}^d, \mathfrak{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\psi_\lambda^d) = \Gamma(k - i\tfrac{1}{2}\lambda')\,\Gamma(k + i\tfrac{1}{2}\lambda)\,[4\pi\Gamma(2k)]^{-1} \times$$
$$\times\, a^{-k-i\frac{1}{2}\lambda}\,[2b\exp(i\tfrac{1}{2}\pi)]^{i\frac{1}{2}(\lambda-\lambda')}\,d^{-k+i\frac{1}{2}\lambda'}\,{}_2F_1(k - i\tfrac{1}{2}\lambda', k + i\tfrac{1}{2}\lambda; 2k; 1/(ad)), \tag{3.6}$$

which is valid for $ad > 1$ (i.e. $bc < 0$),[13] but whose analytic continuation to the whole ad - plane appears to present no problem. The limit $c \to 0$ ($ad \to 1$) is well defined,[14] indeed

$$ {}^{d}D^{k}_{\lambda'\lambda}\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} = \Gamma(k-i\tfrac{1}{2}\lambda')\,\Gamma(k+i\tfrac{1}{2}\lambda)\,\Gamma(i\tfrac{1}{2}[\lambda'-\lambda])\;\left[4\pi\Gamma(k+i\tfrac{1}{2}\lambda')\Gamma(k-i\tfrac{1}{2}\lambda)\right]^{-1}\times $$

$$ \times\, a^{-i\frac{1}{2}(\lambda+\lambda')}\left[2b\exp(i\tfrac{1}{2}\pi)\right]^{i\frac{1}{2}(\lambda-\lambda')} . \tag{3.7} $$

An alternative expression for (3.6) can be constructed using $b=(ad-1)/c$ and an identity for the hypergeometric function (Ref. 11, Eq. 9.131.1c) as

$$ {}^{d}D^{k}_{\lambda'\lambda}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \{\Gamma(k-i\tfrac{1}{2}\lambda')\,\Gamma(k+i\tfrac{1}{2}\lambda)[4\pi\,\Gamma(2k)]^{-1}\}\,a^{-k-i\frac{1}{2}\lambda'}\;\times $$

$$ \times\;[\tfrac{1}{2}c\exp(-i\pi/2)]^{i\frac{1}{2}(\lambda'-\lambda)}\,d^{-k+i\frac{1}{2}\lambda}\;{}_2F_1(k+i\tfrac{1}{2}\lambda',\,k-i\tfrac{1}{2}\lambda;\,2k;\,1/(ad)) . \tag{3.8} $$

In (3.8) the limit $b\to 0$ $(ad\to 1)$ is manifest:[14]

$$ {}^{d}D^{k}_{\lambda'\lambda}\begin{pmatrix} a & 0 \\ c & a^{-1}\end{pmatrix} = 1/(4\pi)\Gamma(i\tfrac{1}{2}[\lambda-\lambda'])\,a^{-i\frac{1}{2}(\lambda'+\lambda)}[\tfrac{1}{2}c\exp(-i\pi/2)]^{i\frac{1}{2}(\lambda'-\lambda)} \tag{3.9} $$

The behaviour of the ${}^{d}D^{k}_{\lambda'\lambda}(g)$ functions as $g\to 1$ provides a check on the realization (3.6) - (3.9). This can be done through the one-parameter subgroups $c\to 0$ in (3.9) or $b\to 0$ in (3.7). For this we have to evaluate (Ref. 11, Eq. 8.312.2)

$$ \underset{b\to 0}{\mathrm{l.i.m.}}\,(-ib)^{i\frac{1}{2}(\lambda'-\lambda)}\Gamma(i\tfrac{1}{2}[\lambda'-\lambda]) = \underset{b\to 0}{\mathrm{l.i.m.}}\int_0^\infty dt\,[\exp(ibt)]\,t^{i\frac{1}{2}(\lambda-\lambda')-1} = 2\pi\,\delta(\tfrac{1}{2}[\lambda-\lambda']), \tag{3.10a} $$

the last step being a consequence of the completeness of the Mellin transform kernel. Hence we can state that

$$ \underset{g\to 1}{\mathrm{l.i.m.}}\;{}^{d}D^{k}_{\lambda'\lambda}(g) = \delta(\lambda'-\lambda) . \tag{3.10b} $$

Eqs. (3.6) - (3.9) give unitary representations, as can be directly verified

$$ {}^{d}D^{k}_{\lambda'\lambda}\begin{pmatrix} d & b\exp(i\pi) \\ c\exp(i\pi) & a\end{pmatrix} = [\,{}^{d}D^{k}_{\lambda\lambda'}\begin{pmatrix} a & b \\ c & d\end{pmatrix}]^{*} \tag{3.11} $$

in all cases.

It is also of interest to determine the asymptotic properties of the representation (3.6) - (3.9); in particular from (1.5a), giving the b-element a phase $\exp(i\pi)$

$$ {}^dD^{k}_{\lambda'\lambda}\begin{pmatrix}\cosh\frac{1}{2}\alpha & -\sinh\frac{1}{2}\alpha\\ -\sinh\frac{1}{2}\alpha & \cosh\frac{1}{2}\alpha\end{pmatrix}\xrightarrow[\alpha\to\infty]{}\Gamma(k-i\tfrac{1}{2}\lambda')\Gamma(k+i\tfrac{1}{2}\lambda)\;[4\pi\Gamma(2k)]^{-1}\times $$
$$ \times[2\exp(-i\pi/2)]^{i\frac{1}{2}(\lambda-\lambda')}\quad\exp(-k\alpha)\,, \tag{3.12} $$

which is the expected behaviour.[15] In comparing with the literature, we can point out that Reference 4 is in error.[16]

The case studied so far in (3.4) applies to $|a'|>|b'|$ that is $|a-b|>|a+b|$ which for g near to 1 implies $b<0$. When $b>0$, we resort to the following argument. Consider the decomposition

$$ \begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} b\exp(i\pi) & a\\ d\exp(i\pi) & c\end{pmatrix}\begin{pmatrix}0 & \exp(-i\pi)\\ 1 & 0\end{pmatrix}. \tag{3.13} $$

Now, $\begin{pmatrix}0 & \exp(-i\pi)\\ 1 & 0\end{pmatrix}$ represents $\exp(i\pi J_3)$ which is, up to a phase, the Hankel transform. As $\exp(i\pi J_3)\,J_2\,\exp(-i\pi J_3)=-J_2$ and the same for J_1, we expect

$$ [\mathfrak{J}\begin{pmatrix}0 & \exp(-i\pi)\\ 1 & 0\end{pmatrix}\psi^d_\lambda](r)=\Phi_{k,\lambda}\;\psi^d_{-\lambda}(r)\,,\quad |\Phi_{k,\lambda}|=1\,. \tag{3.14a} $$

Indeed, direct calculation from (1.3a) yields (Ref. 11, Eq. 6.561.14)

$$ \Phi_{k,\lambda}=\exp i\,[\pi k+\lambda\ln 2+2\arg\Gamma(k+i\tfrac{1}{2}\lambda)]\,. \tag{3.14b} $$

Hence, in all cases where $|a-b|<|a+b|$ we decompose (3.6) through (3.13) and (3.14) as

$$ {}^dD^{k}_{\lambda'\lambda}\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\Phi_{k,\lambda}\;{}^dD^{k}_{\lambda',-\lambda}\begin{pmatrix} b\exp(i\pi) & a\\ d\exp(i\pi) & c\end{pmatrix}, \tag{3.15} $$

where the ${}^dD^k$ on the right does satisfy the conditions for which (3.6) is valid. Now using an identity for the hypergeometric function (Ref. 11, Eq. 9.131.1a) we can see that the general form (3.6) satisfies (3.15) with the correct phase. We claim thus that (3.6) – (3.9) give the right analytic continuation for any g.

Finally, use of (3.1) gives the related representation

$$ {}^rD^{k}_{\lambda'\lambda}\begin{pmatrix} a & b\\ c & d\end{pmatrix}=(\psi^r_{\lambda'},\mathfrak{J}\begin{pmatrix} a & b\\ c & d\end{pmatrix}\psi^r_\lambda)= $$
$$ ={}^dD^{k}_{\lambda'\lambda}\begin{pmatrix}\frac{1}{2}[a-b-c+d] & \frac{1}{2}[a+b-c-d]\\ \frac{1}{2}[a-b+c-d] & \frac{1}{2}[a+b+c+d]\end{pmatrix}, \tag{3.16} $$

in terms of the repulsive oscillator eigenfunctions.

IV. THE PURE CENTRIFUGAL BASIS

The eigenfunctions of

$$H^{f} \equiv J_1 + J_3 = \tfrac{1}{2}\left[-(d^2/dr^2) + (\mu/r^2)\right] \tag{4.1}$$

are, with an appropiate choice of phase

$$\psi^{f}_{\lambda}(r) = \left[\exp(i\pi k)\right](\lambda r)^{\frac{1}{2}}\, J_{2k-1}(\lambda r) \tag{4.2}$$

with eigenvalues $\frac{1}{2}\lambda^2$, $\lambda \in R^{+}$. A simpler operator on the same orbit[12] is

$$H^{\delta} \equiv J_3 - J_1 = \left[\exp(i\pi J_3)\right] H^{f} \left[\exp(-i\pi J_3)\right] = \tfrac{1}{2} r^2 , \tag{4.3}$$

since its eigenfunctions are

$$\psi^{\delta}_{\lambda}(r) = \delta(r-\lambda) , \tag{4.4}$$

with the same eigenvalue $\frac{1}{2}\lambda^2$, and related to (4.2) through a Hankel transform

$$\psi^{f}_{\lambda}(r) = \left[\mathfrak{J}\begin{pmatrix} 0 & \exp(i\pi) \\ 1 & 0 \end{pmatrix}\psi^{\delta}_{\lambda}\right](r) . \tag{4.5}$$

The action of a general transform g on the ψ^{δ}_{λ} basis is only the transform kernel of (1.3a) at $r' = \lambda$:

$$\left[\mathfrak{J}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\psi^{\delta}_{\lambda}\right](r) = \left[\exp(-i\pi k)\right] b^{-1}(\lambda r)^{\frac{1}{2}} \exp\left[(i/2b)(a\lambda^2 + dr^2)\right] J_{2k-1}(\lambda r/b) . \tag{4.6}$$

Hence, without further computation, we find

$$\begin{aligned} {}^{\delta}D^{k}_{\lambda'\lambda}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \left(\psi^{\delta}_{\lambda'}, \mathfrak{J}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\psi^{\delta}_{\lambda}\right) = \\ &= \left[\exp(-i\pi k)\right] b^{-1}(\lambda'\lambda)^{\frac{1}{2}} \exp\left[(i/2b)(a\lambda^2 + d\lambda'^2)\right] J_{2k-1}(\lambda'\lambda/b) . \end{aligned} \tag{4.7}$$

For $|b| \to 0$, where (4.7) appears indeterminate, we have[6,7]

$$ {}^{\delta}D^{k}_{\lambda'\lambda}\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = |a|^{-\frac{1}{2}}\ \delta(\lambda - \lambda' / |a|) \exp\left[(ic/2a)\lambda'^2\right], \tag{4.8} $$

so that the behaviour at the origin is the appropiate one. Unitarity in the sense (3.11) also holds for the ${}^{\delta}D^{k}$ functions which are, after all, only the integral kernels of (1.2).

For the eigenbasis (4.2) of the pure centrifugal Hamiltonian (4.1), we have from (4.5)

$$ {}^{f}D^{k}_{\lambda'\lambda}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \left(\psi^{f}_{\lambda'}, \mathfrak{I}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\psi^{f}_{\lambda}\right) = {}^{\delta}D^{k}_{\lambda'\lambda}\begin{pmatrix} d & c\exp(i\pi) \\ b\exp(-i\pi) & a \end{pmatrix}. \tag{4.9} $$

We wish to stress that the D-functions associated with the eigenbases of any operator in the same orbit as H^{δ} will be expressible in terms of the ${}^{\delta}D$-functions as in (4.9), i.e., as ${}^{\delta}D$-functions of related transformations. Similar remarks apply, of course, to the other two orbits analyzed in Sections II and III.

V. MIXED-BASIS MATRIX ELEMENTS

The mixed-basis matrix elements[17] can now be calculated in terms of the simplest eigenbasis corresponding to pairs of operators in two orbits. For the ψ^{δ} basis (4.4), the mixed matrix elements with either of the others are trivial to compute. Thus the centrifugal-harmonic matrix elements are precisely (2.4) or, written in terms of confluent hypergeometric functions (Ref. 11, Eq. 8.972.1),

$$ \begin{aligned} {}^{\delta b}D^{k}_{\lambda' n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &\equiv \left(\psi^{\delta}_{\lambda'}, \mathfrak{I}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\psi^{b}_{n}\right) = \exp\left[2i(n+k)\arg(a-ib)\right] \times \\ &\times \left[2\Gamma(n+2k)/n!\right]^{\frac{1}{2}} (1/\Gamma(2k))\lambda'^{-\frac{1}{2}} \left[\lambda'^2/(a^2+b^2)\right]^{k} \exp\{(-\lambda'^2/2)\left[(d-ic)/(a+ib)\right]\} \times \\ &\times {}_1F_1(-n; 2k; \lambda'^2/[a^2+b^2]). \end{aligned} \tag{5.1} $$

Similarly, the centrifugal-repulsive matrix elements are precisely (3.5) or

$$ {}^{\delta d}D^{k}_{\lambda'\lambda}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv (\psi^{\delta}_{\lambda'}, \mathfrak{I}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \psi^{d}_{\lambda}) = \exp[i\pi(k - i\tfrac{1}{2}\lambda + {}^{3}/_{8}) + i\tfrac{1}{2}\lambda \ln(2b/a)] \times $$

$$ \times \; [\Gamma(k+i\tfrac{1}{2}\lambda)/\Gamma(2k)]\,(4\pi ab)^{-\frac{1}{2}}\,[\lambda'^{2}\exp(i\,\tfrac{1}{2}\pi)/2ab]^{k-\frac{1}{4}}\,[\exp(ic\lambda'/2a)] \times $$

$$ \times \; {}_{1}F_{1}(k - i\tfrac{1}{2}\lambda;\, 2k;\, \lambda'^{2}\exp(i\,\tfrac{1}{2}\pi)/2ab)\,. \qquad (5.2) $$

Finally, from (2.4) and (3.3) we find the repulsive-harmonic matrix elements (Ref. 11, Eq. 7.414.7),

$$ {}^{db}D^{k}_{\lambda'\lambda}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv (\psi^{d}_{\lambda'}, \mathfrak{I}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \psi^{b}_{\lambda}) = \exp[2i\,(n+k)\arg(a-ib)] \times $$

$$ \times \; [\Gamma(2k+n)/4\pi n!]^{\frac{1}{2}}\,[\Gamma(k - i\tfrac{1}{2}\lambda'/\Gamma(2k)]\,(a^{2}+b^{2})^{-2k+i\frac{1}{2}\lambda'}\,[2(\bar{a}+ib)/(d-ic)]^{k-i\frac{1}{2}\lambda'} \times $$

$$ \times \; {}_{2}F_{1}(-n, k-i\tfrac{1}{2}\lambda';\, 2k;\, 2[a+ib]/[d-ic])\,. \qquad (5.3) $$

Expressions for ${}^{fb}D$, ${}^{fr}D$ and ${}^{rb}D$ can easily be found using the above formulae and the relation between the bases as given previously in the text. From (5.1)-(5.3), the completeness relations for the bases and group multiplication, a host of special function relations depending on the parameters of g can be found. These techniques have been used in References 7, 12 and 18.

VI. REPRESENTATIONS ON $\mathfrak{L}^2(\mathbf{R})$

Up to this point we have considered the space $\mathfrak{L}^2(\mathbf{R}^+)$ only. A few remarks are necessary in order to clarify the extension to $\mathfrak{L}^2(\mathbf{R})$. Clearly, the operators J_i in (1.4) are invariant under the replacement $r \to (\exp i\pi) r$ and they are even in r. The eigenfunctions of the "Hamiltonian-like" generators, ψ^{b}_{n}, ψ^{r}_{λ} and ψ^{f}_{λ} given by (2.2), (3.4) and (4.2) respectively, belonging to D^{+}_{k} exhibit a phase as $\psi_{\lambda}(r\exp i\pi) = \psi_{\lambda}(r)\exp[i\pi(2k - \tfrac{1}{2})]$ and behave as $r^{2k-\frac{1}{2}}$ when $r \to 0$. They are thus regular for $k \geqslant \tfrac{1}{4}$ and irregular but square-integrable for $k > 0$.

i) For $\mu \geqslant \tfrac{3}{4}$ there is a unique self-adjoint extension of the operators J_i, as there is one $\mathfrak{L}^2(\mathbf{R}^+)$ solution for $k = \tfrac{1}{2}(1 + [\mu + \tfrac{1}{4}]^{\frac{1}{2}}) \geqslant 1$, the eigenfunctions and spectra are given by (2.2), (3.4) and (4.2). In order to consider the system over the whole real line, we have the two degenerate eigenstates $\psi^{\pm}_{k}(r) = \{\psi_{k}(r) \text{ for } r \gtrless 0 \text{ and } 0 \text{ for } r \lessgtr 0\}$, or any linear combination of them, since we are not forced to demand continuity of the functions and its derivative at the origin.

ii) For $\frac{3}{4} > \mu \geqslant -\frac{1}{4}$, $\mu \neq 0$, both solutions ψ_{k_1}, $\frac{1}{2} \leqslant k_1 < 1$, $k_1 \neq \frac{3}{4}$ and ψ_{k_2}, $0 < k_2 \leqslant \frac{1}{2}$, $k_2 \neq \frac{1}{4}$ are $\mathcal{L}^2(R^+)$ in a neighborhood of $r = 0$, thus there is a one-parameter family of self-adjoint extensions and one must add a boundary condition to pick one of these. In the foregoing we chose the boundary condition which gives rise to the eigenbasis (2.2a). Now since the boundary condition is related to the behaviour at the origin, one can pick two different boundary conditions by choosing either of the two k values given in (1.6). These are not the only boundary conditions, however. The procedure for implementing a general boundary condition is detailed in Ref. 19. Such general conditions are apparently related to the supplementary series of representations of $\overline{SL(2,R)}$ and we will say nothing more about them. It is not difficult to see that for either of the two choices in (1.6) the spectrum is given by (2.2b), i.e. it is bounded from below and thus belongs to the discrete series, namely $D_{k_1}^+$ and $D_{k_2}^+$. The study of the system for the whole real line follows as in (i).

iii) For $\mu = 0$, corresponding to the two boundary conditions $\psi(0) = 0$ and $\psi'(0) = 0$ we have the two representations $D_{\frac{3}{4}}^+$ and $D_{\frac{1}{4}}^+$. How do we extend these to the entire line? Since for $r = 0$ there is no longer a singularity, we must demand that both the wave function and its derivative be continuous there. Furthermore, since the wavefunctions for $D_{\frac{3}{4}}^+$ are odd and those for $D_{\frac{1}{4}}^+$ are even, we need both to obtain a complete set for $\mathcal{L}^2(R)$. Thus the harmonic oscillator belongs to the reducible representation $D_{\frac{1}{4}}^+ \oplus D_{\frac{3}{4}}^+$.

iv) If the centrifugal part is more attractive than $\mu = -\frac{1}{4}$, the representations appear to belong to the principal series and the energies are not bounded from below.

ACKNOWLEDGMENT

It is a pleasure to thank Prof. Marcos Moshinsky for his interest in this work.

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13. In choosing this range for Eq. 7.621.1 of Ref. 11, one also makes use of Eq. 9.231.2.
14. Although the hypergeometric function at unit argument cannot be evaluated for the values of the parameters, the formal analytic continuation is valid and can be evaluated directly from the integral definition of (3.6) and Euler's formula for the Gamma function.
15. This is also typical of the principal series of all $SO(n,1)$ groups, see K.B. Wolf, J. Math. Phys. 12 (1971) 197, Eqs. (5.8) and (5.14a), and C.P. Boyer, J. Math. Phys. 12 (1971) 1599.
16. Equation (2.11) of Reference 4; although the absolute value of (3.12) has the same behaviour (replacing λ by $2p$), there is an extra factor 4^k which does not appear in (3.12).
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RESUMEN

Usando técnicas de transformadas canónicas, encontramos los elementos de matriz de las representaciones irreducibles unitarias, o kernels integrales, para transformaciones finitas de $SL(2,R)$ cuando el generador del subgrupo diagonal es cualquiera de los subgrupos uniparamétricos no equivalentes $SO(2)$, $SO(1,1)$ ó $E(2)$. El método reduce el problema a resolver una sola integral y los resultados se dan en términos de funciones hipergeométricas. Se dan también las matrices de representación en bases mixtas.