# HYPERDIFFERENTIAL OPERATORS AND INTEGRAL TRANSFORMS

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ABSTRACT: As a consequence of former work on canonical transforms, we collect a set of formulae which give alternate forms to known integral transforms (Fourier, bilateral Laplace, Hankel and Bargmann) as infinite-order differential operators. We comment upon the relevance of Lie algebras of second-order differential operators in group theory and in the time evolution of quantum mechanical systems.

Transforms such as the ones associated with the names of Fourier, Laplace, Hankel, Bargmann and Barut-Girardello are usually presented as mappings between Hilbert spaces realized through Lebesgue integration with a transform kernel. It is interesting to notice that parallel mappings can also be achieved through the action of hyperdifferential operators (operators which involve arbitrarily high order derivatives), which, moreover, are sometimes more transparent for the proof of certain properties. This fact has come to our attention in studying a general class of such transforms, ***canonical transforms***, which are associated with the formulation of complex canonical transformations in Quantum Mechanics. (See Refs. 1,2.) These also admit a hyperdifferential operator realization[3] on spaces of infinitely differentiable functions $\mathcal{C}^\infty$.

A. *The Fourier transform*[4] of $f(x) \in \mathcal{S} \equiv \mathcal{C}^\infty \cap \mathcal{L}^2(R)$, ($\mathcal{L}^2(R)$ is the space of Lebesgue square-integrable functions on the real line), is

$$f^F(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} dx' f(x') \exp(-ixx'), \tag{1}$$

and its inverse follows the form (1) but has $\exp(ixx')$ for its kernel. The transform (1) can also be implemented as

$$f^F(x) = \exp(i\pi/4) \exp \{(\tfrac{1}{4} i\pi) [(d^2/dx^2) - x^2]\} f(x), \tag{2}$$

and the inverse can be likewise constructed. The proof of this fact rests in noticing that the operator exponent contains the one-dimensional quantum harmonic oscillator Hamiltonian[5] whose eigenfunctions are $\{\psi_n(x)\}_{n=0}^{\infty}$ with eigenvalues $n + \frac{1}{2}$. When acting on $\psi_n(x)$, Eq. (2) yields $\psi_n^F(x) = (-i)^n \psi_n(x)$. As the set $\{\psi_n(x)\}_{n=0}^{\infty}$ is a complete basis for $\mathcal{S}$, the equivalence of (1) and (2) follows. The restriction on $\mathcal{S}$ here and below, seems to be unnecessarily stringent.

B. *The Bilateral Laplace transform* can be obtained from (1)-(2) through a change of variable and normalization so that, for functions $f(x)$ in $\mathcal{S}$ for which the integral exists,

$$f^L(x) = \int_{-\infty}^{\infty} dx' f(x') \exp(-xx') \tag{3}$$

is represented as

$$f^L(x) = i(2\pi)^{\frac{1}{2}} \exp\{(-\tfrac{1}{4}\pi) [(d^2/dx^2) + x^2]\} f(x). \tag{4}$$

(The full proof of the equivalence of (3) and (4) and their inverses requires the results of Ref. 1.)

C. *The Hankel transform*, defined as the radial part of the $n$-dimensional version of (1), of a function $F(\mathbf{x}) = f(x) Y_L(\Omega_x)$ where $Y_L(\Omega_x)$ transforms as an $SO(n)$-irreducible tensor[2], is

$$f^H(x) = \exp(-i\pi L/2) \quad x^{1-\frac{1}{2}n} \int_0^{\infty} dx' f(x') x'^{\frac{1}{2}n} J_{\frac{1}{2}n+L-1}(xx'), \tag{5}$$

where $J_\nu$ is the Bessel function, can be similarly realized, out of (2), as

$$f^H(x) = \exp(in\pi/4)\exp\{i\tfrac{1}{4}\pi\ [(d^2/dx^2) + ((n-1)/x)(d/dx) - L(L+n-2)/x^2 - x^2]\}f(x) \tag{6}$$

D. *The Bargmann transform*[6] maps unitarily $\mathcal{L}^2(\boldsymbol{R})$ onto the Hilbert space of entire analytic functions of growth $(2, 1/2)$. In its integral version, it reads

$$f^B(x) = \pi^{\frac{1}{4}} \int_{-\infty}^{\infty} dx' f(x') \exp[-\tfrac{1}{2}(x^2 + x'^2) + 2^{\frac{1}{2}} xx'], \tag{7}$$

and maps the harmonic oscillator states $\psi_n(x)$ onto the functions $(n!)^{-1/2}x^n$. It can be realized on $\mathcal{S}$ in hyperdifferential form as

$$\begin{aligned} f^B(x) &= (2\pi)^{1/4} \exp\{\pi/8\ [(d^2/dx^2) + x^2]\} f(x) \\ &= \pi^{\frac{1}{4}} \exp[\tfrac{1}{2}(d^2/dx^2)] \exp(x^2/4)\ f(2^{-\frac{1}{2}}x). \end{aligned} \tag{8}$$

The proof of this fact lies in the use of the Baker-Campbell-Hausdorff relation

$$\begin{aligned} \exp\{-i\theta\tfrac{1}{2}\ [d^2/dx^2) + x^2]\} &= \exp[-i\tfrac{1}{2}\ \tanh\theta(d^2/dx^2)] \times \\ &\times\ \exp\{\tfrac{1}{2}\ln\cosh\theta\,[x(d/dx) + (d/dx)x]\} \times \\ &\times \exp(-i\tfrac{1}{2}\ \tanh\theta x^2) \end{aligned} \tag{9}$$

for $\theta = i\pi/4$, acting on the basis $\{\psi_n(x)\}_{n=0}^{\infty}$ of $\mathcal{S}$. The last factor cancels the $\exp(-x^2/2)$ of $\psi_n(x)$, the middle factor rescales the $x$ argument, while the first, through the little-known relation[7]

$$x^n = 2^{-n}\exp\ [\tfrac{1}{4}(\ d^2/dx^2)]\, H_n(x) \tag{10}$$

yields $(n!)^{-1/2}x^n$ for $\psi_n^B(x)$. The hyperdifferential form (6) tells us, in particular, that the Bargmann transform of the repulsive harmonic oscillator (generalized) eigenfunctions are multiples of themselves. This is true in spite of the fact that they lie outside $\mathcal{S}$.

The radial part of an $n$-dimensional version of (7)-(10) yields the Barut-Girardello transform,[2,8] but we shall not enter into this. Eq. (10) is interesting in its own right. Its inverse gives an expression for the Hermite polynomials as

$$H_n(x) = \exp\ [-\tfrac{1}{4}(d^2/dx^2)]\ (2x)^n. \tag{11}$$

The radial-part treatment of (7)-(11) yields,[2,7] for Laguerre polynomials,

$$L_n^{(\alpha-1)}(-x) = (n!)^{-1} \exp\left[x(d^2/dx^2) + \alpha(d/dx)\right] x^n \tag{12}$$

which can be independently verified to hold by series. A generalized version of Eq. (12) appears in Ref. 2. (Can similar forms be written for Jacobi polynomials?)

E. *The time evolution operator for quantum systems* obeying a Schrödinger equation with a Hamiltonian $H$ is $\exp[itH]$. This form of describing time evolution of a system also holds for classical systems.[9]

F. *The time evolution of solutions of the heat equation*[10] ($\partial_{xx}v = \partial_t u$) can be described as follows: if $u_0(x) \in C^\infty$ is the initial temperature distribution in a one-dimensional conducting rod of unit diffusivity, the temperature distribution $u(x,t)$ at any later time $t$ is described by

$$u(x,t) = [\mathcal{G}(x,t) * u_0](x) \tag{13}$$

where $\mathcal{G}(x,t)$ is the system's Green's function and $*$ is the convolution operator. Equivalently, the solution can be written as

$$u(x,t) = \exp[t(d^2/dx^2)]\, u_0(x) \tag{14}$$

which can be seen formally to satisfy the heat equation. It also yields information on the kind of temperature distribution which can be regressed in time and the amount of regression.[11] (Can a similar description be made for the wave equation?)

G. *Group transformations*. Physicists dealing with group theory are familiar with the hyperdifferential operators obtained when exponentiating the first-order opeators constituting the Lie algebra. Here we have been exponentiating second-order operators. These also stem from Lie algebras. The corresponding integrated group action, however, is of the *non-local* kind[12]

$$f(x) \xrightarrow{g} f^g(x) = \int dx'\, K_g(x,x')\, f(x'), \tag{15}$$

and seems to have been relatively little used in investigating the symmetries of a system until recently.[12]

It thus seems, that hyperdifferential operators can yield a host of special function relations and new mathematical insight, applicable in the study of integral transforms and the description of physical systems.

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## RESUMEN

Como una consecuencia de nuestro trabajo con transformadas canónicas, presentamos un conjunto de fórmulas que dan representaciones alternativas de transformadas integrales conocidas (Fourier, Laplace bilateral, Hankel y Bargmann) como operadores diferenciales de orden infinito. Comentamos sobre la relevancia de las álgebras de Lie de operadores diferenciales de segundo orden en teoría de grupos y en la evolución en el tiempo de sistemas cuánticos.