Revista Mexicana de Física 25 (1976) 69-87

FLOW OF A LIQUID FILM OVER A ROTATING DISK

Naum Fraidenraich

Comisión Nacional de Estudios Geo-Heliofísicos

Observatorio Nacional de Física Cósmica de San Miguel

San Miguel, Argentina

(Recibido: marzo 17, 1975)

ABSTRACT: The flow problem of a thin layer of liquid flowing over a rotating disk is formulated. Two regions have been considered, corresponding to the limiting cases where the viscous forces are much larger than the centrifugal or convective forces, respectively. The first (center region), extends from the center up to a radius r_c and the second (external region) from a radius r_e up to infinity. The center region leads to a boundary layer problem. The layer thickness varies as r^2 and the radial velocity as r^{-3}. The external region has been solved through an asymptotic series expansion in a small parameter ϵ which represents approximately the ratio between convective and viscous forces. Solutions are obtained up to first order in ϵ and results show that the layer thickness decreases as $r^{-2/3}$ and the radial velocity as $r^{-1/3}$. A numerical estimate shows that r_e can be rather small for a reasonable set of parameters, from the point of view of the applications of this work. In the region lying between r_c and r_e a patching solution is proposed based on the center and external limiting cases.

INTRODUCTION

The problem which motivated this work is related to the production of sprays by means of spinning-disk type sprayers. Finely controlled sprays may be obtained when liquid is fed into the center of a rotating disk and centrifuged off the edge.[1] More recently this method has been used to produce electrostatic atomization through the action of electric forces at the edge of the disk.[2] In all these cases it is of interest to know the flow conditions at the edge of the rotating disk where, through the combined action of mechanical and electric forces, the liquid becomes unstable and divides into a fine spray of tiny drops.

The present work considers the steady axisymmetric flow of a thin layer of fluid over a rotating disk. The liquid is fed from a point source at a constant rate Q at the center of the disk. The radius of the disk is considered infinitely large. The existence of the free surface at a constant pressure allows us to neglect the pressure gradients throughout the thin layer of fluid, as they are very small compared with the other forces present.

The flow analysis considers two regions, corresponding to the limiting cases in which viscous forces are much larger than the centrifugal and the radial component of the inertial forces (convective forces), respectively. Correspondingly, two radii r_c and r_e may be defined which determine the outer and inner limits of the center and external regions, respectively.

A third, intermediate region, where the inertial and convective forces are equally important, may be considered between r_c and r_e. This intermediate region is the most difficult to analyze, as the complete problem must be solved there; only the two extreme cases have been solved here where a patching solution is proposed in this region.

The flow in the center is governed by the radial component of the Navier-Stokes equations and the coupling with the azimuthal component is very weak. As will be shown later, the equation in this case turn out to be of the boundary layer type.

In the external region the flow is mainly determined by the interplay of viscous and centrifugal forces. Thus the radial and azimuthal components are strongly coupled and an asymptotic expansion procedure is developed in a small functional parameter defined as

$$\epsilon = Q/Q_m = \langle u \rangle r \delta / (\omega r^2 \delta_m)$$

where Q is the volumetric flow carried by the thin layer of thickness δ and Q_m is the flow carried by one side of an immersed disk of radius r and boundary layer of thickness δ_m rotating with angular speed ω. δ_m is equal to $(\nu/\omega)^{1/2}$, ν is the kinematic viscosity of the liquid and $\langle u \rangle$ is an average over Z.[3] The flow in this case corresponds to the so called creeping motion and the second part of this work is devoted to finding the solution to this problem.

In the first part of this paper, the Navier-Stokes equation are simplified by making use of the fact that the thickness of the fluid layer is much smaller than the radial coordinate r.

FORMULATION OF THE PROBLEM AND BOUNDARY CONDITIONS

The flow will be referred to a set of cylindrical coordinates (r, ϕ, z). Since the fluid forms a thin layer over the rotating disk, the following approximation has been made

$$\partial/\partial z >> \partial/\partial r, \qquad w << u, \qquad w << v, \tag{1}$$

where u, v and w are the radial, azimuthal and axial components of the velocity.

Taking into account the inequalities (1) as well as rotational symmetry, the Navier-Stokes and continuity equations for the case of an incompressible fluid may be written as

$$u(\partial u/\partial r) - (v^2/r) + w(\partial u/\partial z) = -(1/\rho)(\partial p/\partial r) + \nu(\partial^2 u/\partial z^2) \tag{2}$$

$$u(\partial v/\partial r) + (uv/r) + w(\partial v/\partial z) = \nu(\partial^2 v/\partial z^2) \tag{3}$$

$$u(\partial w/\partial r) + w(\partial w/\partial r) = -(1/\rho)(\partial p/\partial z) + \nu(\partial^2 w/\partial z^2) \tag{4}$$

$$(\partial u/\partial r) + (u/r) + (\partial w/\partial z) = 0 \tag{5}$$

where p and ρ are the fluid pressure and density. In writing Eqs. (2)-(4) terms of the order $(\delta/r)^2$ have been neglected.

The boundary conditions on the disk surface are

$$\left.\begin{aligned} u &= 0 \\ v &= \omega r \\ w &= 0 \end{aligned}\right\} \qquad \text{at } z = 0. \tag{6a-c}$$

On the free surface, the following relations must be satisfied[4]:

$$\left.\begin{array}{l} e_{ij}t_in_j = 0 \\ p - 2\mu e_{ij}n_in_j = p_0 \end{array}\right\} \quad \text{at } z = \delta . \tag{7a,b}$$

n_i and t_i are the components of the normal and tangential vectors to the free surface and p_0 is the uniform pressure over it. e_{ij} is defined as

$$e_{ij} = \tfrac{1}{2}[(\partial u_i/\partial x_j) + (\partial u_j/\partial x_i)] . \tag{8}$$

Equations (7a, b) follow from equating the stresses on both sides of the interface at the free surface. The surface tension has been neglected in writing Eq. (7b). The solutions to be found will show this approximation to be valid. When Eq. (8) is introduced in (7a), its two components lead to the following conditions:

$$\left.\begin{array}{l} \partial u/\partial z = 0 \\ \partial v/\partial z = 0 \end{array}\right\} \quad \text{at } z = \delta , \tag{9a,b}$$

where inequalities (1) have been used, implying also that $(\delta/r)^2 << 1$.

With identical approximations, equation (7b) transforms to

$$p - 2\mu(\partial w/\partial z) = p_0. \tag{10}$$

Taking the derivative of (10) and through the use of the continuity equation, we obtain

$$-(1/\rho)(\partial p/\partial r) \doteq 2\nu(\partial/\partial r)[(\partial u/\partial r) + (u/r)] . \tag{11}$$

The right hand side is much smaller than $\nu(\partial^2 u/\partial z^2)$, and thus at the free surface $\partial p/\partial r$ may be taken equal to zero:

$$\partial p/\partial r = 0 \qquad \text{at } z = \delta . \tag{12}$$

Making a series expansion of $\partial p/\partial r$ around the point $z = \delta$ and using Eq. (12) we can write $(1/\rho)(\partial p/\partial r)_{z<\delta}$ as

$$(1/\rho)(\partial p/\partial r)_{z<\delta} = (1/\rho)(\partial/\partial r)(\partial p/\partial z)_{z=\delta}\, \delta z .$$

From Eq. (4) it can be seen that $(1/\rho)(\partial p/\partial z)$ is of the order of $\nu(\partial^2 w/\partial z^2)$ which, through the use of the continuity equation, can be shown to be of the order of $(\delta/r)\,\nu\,(\partial^2 u/\partial z^2)$. Therefore, the term $(1/\rho)(\partial p/\partial r)$ can be neglected throughout the fluid in the approximation $(\delta/r)^2 \ll 1$. Thus Eq. (2), without the term $(1/\rho)(\partial p/\partial r)$, and Eqs. (3) and (5) together with the boundary conditions (6 a–c) and (9 a,b) become the system of equations to be solved.

In order to integrate this system, it is often convenient to introduce a stream function $\psi(r,z)$ defined by

$$ur = \partial\psi/\partial z \tag{13a}$$

$$wr = -\partial\psi/\partial r \tag{13b}$$

The equations are now rewritten in dimensionless form by referring the axial coordinate z to the layer thickness $\delta(r)$ through the expression

$$\eta(r,z) = z/\delta(r), \tag{14}$$

and introducing the following relations:

$$g(r) = \delta(r)/\delta_m \tag{15}$$

$$f(r,\eta) = \psi(r,z)/Q \tag{16}$$

$$h(r,\eta) = v/\omega r. \tag{17}$$

When changing variables from r, z to r, η it follows, using Eq. (14), that the derivatives $(\partial/\partial r)$ and $(\partial/\partial z)$ transform as

$$(\partial/\partial r) \rightarrow (\partial/\partial r) - (\eta/\delta)(d\delta/dr)(\partial/\partial\eta) \tag{18a}$$

$$(\partial/\partial z) \rightarrow (1/\delta)(\partial/\partial\eta) \quad , \tag{18b}$$

Thus, using Eqs. (13) and (16), the velocities u and w can be expressed as

$$u = (Q/r\delta)f' \quad \text{and} \quad w = -(Q/r)(\partial f/\partial r) + (Q\eta/r\delta)(d\delta/dr)f', \tag{19a,b}$$

where f' stands for $\partial f/\partial\eta$. Introducing expressions (19a,b) and (17) as well as its derivatives into Eqs. (2) and (3) we obtain

$$-(Q\delta/r^2\nu)\{f'^2[1+(r/\delta)(d\delta/dr)] - rf'(\partial f'/\partial r) + rf''(\partial f/\partial r)\} - [\omega^2 r^2\delta^3/\nu Q]h^2 = f''' \tag{20}$$

$$(Q\delta/r^2\nu)[2hf' + rf'(\partial h/\partial r) - rh'(\partial f/\partial r)] = h'' \tag{21}$$

The boundary conditions (6 a–c) and (9 a,b) can be written as

$$\left.\begin{aligned} f' &= 0 \\ \partial f/\partial r &= 0 \\ h &= 1 \end{aligned}\right\} \quad \text{at } \eta = 0 \qquad (22\text{a–c})$$

and

$$\left.\begin{aligned} f'' &= 0 \\ h' &= 0 \end{aligned}\right\} \quad \text{at } \eta = 1. \qquad (23\text{a,b})$$

The problem has now been reduced to solving Eqs. (20) and (21) with f and h as unknowns and with boundary conditions (22) and (23). The thickness δ of the boundary layer is also unknown but, at least formally, f and h can be obtained as a function of δ and, since the solutions must satisfy the constancy of the volumetric flow,

$$Q = \langle u \rangle r \delta \qquad (24)$$

$\delta(r)$ follows; $\langle u \rangle$ is given by the integral $\int_0^1 u d\eta$.

Looking at Eqs. (20) and (21) one sees that two different types of solutions can be generated according to the magnitude of the factors $(\omega^2 r^2 \delta^3/\nu Q)$ and $(Q\delta/r^2\nu)$. In the first case it becomes obvious that a region around the center must exist where the relation between centrifugal and viscous forces, $(\omega^2 r^2 \delta^3/\nu Q)$, is much smaller than unity. In this case Eqs. (20) and (21) are weakly coupled and when the term containing h in Eq. (20) is neglected we are left with a boundary layer type equation. A different situation arises when the ratio between the convective and viscous forces $(Q\delta/r^2\nu)$, is much smaller than one. Eqs. (20) and (21) are strongly coupled and in order to obtain a solution, an asymptotic expansion procedure is developed using the small functional expansion parameter ϵ.

SOLUTIONS FOR THE CENTER REGION

In this region we consider that the convective forces are much larger than the centrifugal forces. Thus neglecting the term $(\omega^2 r^2 \delta^3 h^2/\nu Q)$ in Eq. (20) we are left with a differential equation in f only. This equation is of the conventional type appearing in boundary layer theory; for the solutions to be similar, i.e., $(\partial f/\partial r) = 0$, it can be seen that the following conditions must be satisfied:

$$Q\delta/r^2\nu = A \tag{25a}$$

$$(r/\delta)(d\delta/dr) = B \tag{25b}$$

where A and B are constants. Equation (25a) leads to the relation

$$\delta = (A\nu/Q)r^2 \tag{26}$$

which automatically satisfies Eq. (25b); thus B is equal to 2.

The differential equation to be solved can now be written as

$$-Cf'^2 = f''' \tag{27}$$

where $C = 3A$, with the following boundary conditions:

$$f' = 0, \qquad \eta = 0 \tag{28a}$$

$$f'' = 0, \qquad \eta = 1. \tag{28b}$$

Upon a first integration Eq. (27) gives

$$f''^2/(f'^3(1) - f'^3(\eta)) = (2/3)\,C \tag{29}$$

where the boundary condition on f'' has been used. Next, integration leads to

$$1 - \eta = [(2/3)\,Cf'(1)]^{-\frac{1}{2}} \int_x^1 (1-t^3)^{-\frac{1}{2}}\,dt \tag{30}$$

where $x = (f'(\eta)/f'(1))$. The integral can be recognized as the incomplete elliptic integral $F(\phi\backslash\alpha)/3^{\frac{1}{4}}$ where $\alpha = 75^\circ$ and $\cos\phi = (\sqrt{3}-1+x)/(\sqrt{3}+1-x)$; when the condition $f'(0) = 0$ is introduced, Eq. (30) becomes

$$\eta = 1 - [F(\phi\backslash 75^\circ)/1.86] \tag{31}$$

where $F(74{,}4^\circ\backslash 75^\circ) = 1.86$ and $74{,}4^\circ = \cos^{-1}[(\sqrt{3}-1)/(\sqrt{3}+1)]$ $(x = 0)$; thus $[(2/3)\,Cf'(1)]^{\frac{1}{2}}$ becomes equal to $1.86/3^{\frac{1}{4}}$. Expression (31) is represented in Fig. 1 and the results have been obtained numerically from tables.[5]

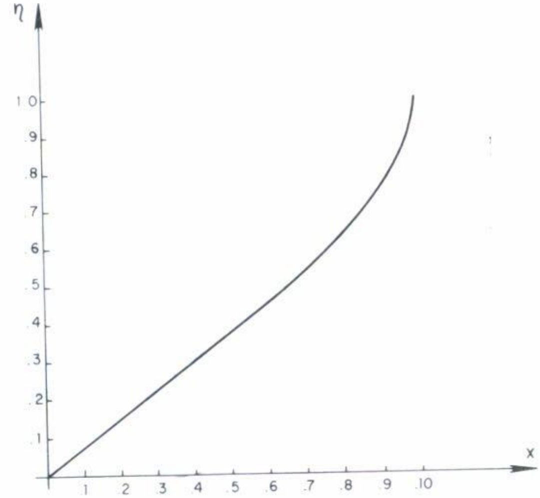

Fig. 1. Radial velocity profile in the center region $x = (f'(\eta)/f'(1))$.

The solution must still satisfy Eq. (24) which leads to the condition

$$\int_0^1 f'(\eta)\,d\eta = 1\,,$$

which, after writing $d\eta$ in terms of df' from Eq. (29), leads to

$$[f'(1)]^{\frac{1}{2}}\,[(\tfrac{2}{3})C]^{-\frac{1}{2}} \int_0^1 t\,(1-t^3)^{-\frac{1}{2}}\,dt = 1\,.$$

Also

$$[(\tfrac{2}{3})Cf'(1)]^{\frac{1}{2}} = \int_0^1 (1-t^3)^{-\frac{1}{2}}\,dt = 1.86/3^{\frac{1}{4}};$$

therefore

$$f'(1) = \int_0^1 (1-t^3)^{-\frac{1}{2}}\,dt \Big/ \int_0^1 t\,(1-t^3)^{-\frac{1}{2}}\,dt = 1.63\,, \qquad (32)$$

where $\int_0^1 t\,(1-t^3)^{-\frac{1}{2}}\,dt$ has been obtained through the use of the expression:

$$\int_x^1 t\,(1-t^3)^{-\frac{1}{2}}\,dt = (3^{-\frac{1}{4}} - 3^{\frac{1}{4}})\,F(\phi\backslash 75^\circ) + 2(3)^{\frac{1}{4}}\,E(\phi\backslash 75^\circ) - [(2\sqrt{1-x^3})/(\sqrt{3}+1-x)],$$

where E is the incomplete elliptic integral of the second kind.[6] When $x = 0$, $\phi = 74.4^\circ$ and

$$\int_0^1 t(1-t^3)^{-1/2}\,dt = 0.87$$

Finally we obtain $C \simeq 1.87$, $A \simeq 0.62$ and

$$\delta \doteq 0.62\,(\gamma r^2/Q). \tag{33}$$

It can be seen that although this approximation is only valid for small r, the relation $(\delta/r)^2$ is still much smaller than one, thus satisfying the assumption made in deriving these equations.

SOLUTIONS FOR THE EXTERNAL REGION

In this region the factor $(Q\delta/r^2\nu)$ is considered to be much smaller than unity; in that case an examination of Eqs. (20) and (21) suggests the possibility of obtaining a series solution by using the small functional parameter $(Q\delta/r^2\nu)$. Because this relation contains δ which is unknown, we will make use of a slightly different coefficient $\epsilon = (Q\delta_m/r^2\nu) = Q/Q_m$ which depends only upon known physical parameters and the radial coordinate r. Therefore if we postulate that the dependent variables are expressible asymptotically (as $\epsilon \to 0$) as power series in ϵ, we can write:

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \ldots \tag{34a}$$

$$h = h_0 + \epsilon h_1 + \epsilon^2 h_2 + \ldots \tag{34b}$$

$$g = g_0 + \epsilon g_1 + \epsilon^2 g_2 + \ldots \tag{34c}$$

Upon substitution of the series (34 a–c) into Eqs. (20) and (21) one obtains the following set of equations of zero and first order:

ϵ^0
$$(-g_0^3/\epsilon)\,h_0^2 = f_0''' \tag{35a}$$

$$h_0'' = 0 \tag{35b}$$

ϵ^1
$$g_0\left[rf_0'(\partial f_0'/\partial r) - rf_0''(\partial f_0/\partial r) - f_0'^2\left(1+(r/\delta_0)(d\delta_0/dr)\right)\right] -$$

$$-[(3g_0^2 g_1 h_0^2)/\epsilon] - [(2g_0^3 h_0 h_1)/\epsilon] = f_1''' \tag{36a}$$

$$g_0\left[2h_0 f_0' + r f_0(\partial h_0/\partial r) - r h_0'(\partial f_0/\partial r)\right] = h_1''. \qquad (36b)$$

The corresponding boundary conditions are obtained from Eqs. (22) and (23).

ϵ^0	$f_0' = 0,$	$\partial f_0/\partial r = 0,$	$h_0 = 1$	$\eta = 0$	(37a)
	$f_0'' = 0,$		$h_0' = 0$	$\eta = 1$	(37b)
ϵ^1	$f_1' = 0,$	$\partial f_1/\partial r = 0,$	$h_1 = 0$	$\eta = 0$	(38a)
	$f_1'' = 0,$		$h_1' = 0$	$\eta = 1$	(38b)

The value of δ, and therefore of g, is obtained only when the solution for the radial velocity is made to satisfy the relation (24). When this is done one obtains (see Appendix)

$$\epsilon^0 \qquad \langle u_0\rangle r\,\delta_0 = Q \qquad (39a)$$

$$\epsilon^1 \qquad \langle u_1\rangle\,\delta_0 + \langle u_0\rangle\,\delta_1 = 0 \qquad (39b)$$

where the $\langle u_i\rangle$ are defined in the Appendix and δ_i is equal to $g_i\,\delta_m$; the subscript corresponds to the different terms of Eqs. (34). In the following, solutions up to the first order are evaluated; higher order terms may easily be obtained by the method outlined.

ZERO ORDER SOLUTION

From Eq. (35b) and the corresponding boundary conditions, the function h_0 is obtained as

$$h_0 = 1, \qquad (40)$$

and introducing this result in the Eq. (35a) gives

$$f_0' = (g_0^3/\epsilon)\,(\eta - \tfrac{1}{2}\eta^2). \qquad (41)$$

Equation (41) also satisfies boundary conditions (37). Thus u_0 can be written as

$$u_0 = \omega r g_0^2 (\eta - \tfrac{1}{2}\eta^2) \tag{42}$$

and

$$\langle u_0 \rangle = \int_0^1 u_0 d\eta = (\omega r g_0^2)/3 . \tag{43}$$

Using Eq. (39a), δ_0 becomes

$$\delta_0^3 = 3Q\nu/\omega^2 r^2 . \tag{44}$$

Expression (44) is the liquid surface equation and satisfies the law $\delta_0 \sim r^{-2/3}$; this law is the result of the fact that the centrifugal forces $-\omega^2 r$ are balanced out by the viscous forces in the radial direction, as could be shown through a simple analysis which takes only these forces into account.

After a slight rearrangement, Eq. (44) can be written as

$$g_0^3 = 3\epsilon . \tag{45}$$

The fact that it is the third power of g_0 which must be smaller than unity permits one to satisfy this relation with values of g_0 slightly smaller than one. On the other hand, Eq. (41) shows that f_0' is a function of η only, because (g_0^3/ϵ) is a constant equal to 3.

The expression (44) can now be introduced in the equation giving u_0 and we obtain

$$u_0 = (9Q^2\omega^2/\nu)^{1/3} r^{-1/3} (\eta - \tfrac{1}{2}\eta^2) . \tag{46}$$

Equation (46) gives the radial velocity as a function of r and the relevant parameters, and shows that u_0 decreases very slowly in this region (as the inverse cubic root of r). The value of v_0 is ωr, as b_0 is equal to one.

The axial component of velocity w can also be expanded in terms of ϵ and the zero order term results (see Appendix)

$$w_0 = u_0 \eta (\partial \delta_0 / \partial r) \tag{47}$$

and when the expressions of δ_0 and u_0 (Eqs. (44) and (46)) are used it leads to

$$w_0 = -2Qr^{-2}(\eta^2 - \tfrac{1}{3}\eta^3) . \tag{48}$$

For $\eta = 1$, w_0 becomes equal to $u_0(d\delta_0/dr)$ as can be seen from expression (47) and it can be recognized as the condition of mass conservation at the free surface.

FIRST ORDER SOLUTION

First we calculate h from Eq. (36b) and considering that $(\partial h_0/\partial r) = 0$ and $(\partial f_0/\partial r) = 0$ (see Appendix), the differential equation which gives h_1 reads

$$h_1'' = 2h_0 f_0' g_0 \tag{49}$$

which after replacing h_0, f_0' and g_0 leads to the equation

$$h_1 = g_0(-2\eta + \eta^3 - \tfrac{1}{4}\eta^4). \tag{50}$$

Taking into account that $h_0 = 1$, $\partial f_0/\partial r = 0$ and $\partial f_0'/\partial r = 0$ the equation for f_1''', Eq. (36a), becomes

$$-g_0 f_0'^2 [1 + (r/\delta_0)(d\delta_0/dr)] - [(3g_0^2 g_1)/\epsilon] - [(2g_0^3 h_1)/\epsilon] = f_1'''. \tag{51}$$

The bracket in the first term can be calculated from Eq. (44) and turns out to be equal to ($\tfrac{1}{3}$). Using the relation $(g_0^3/\epsilon) = 3$, Eqs. (41) and (50), and the condition (VIIb) for obtaining g_1 as a function of g_0, Eq. (51) integrates to

$$f_1' = g_0[0.91\eta - 2.65\eta^2 + 2\eta^3 - (\tfrac{1}{4})\eta^4 - (\tfrac{3}{20})\eta^5 + (\tfrac{1}{40})\eta^6]. \tag{52}$$

We can now calculate u_1. From equation (IIb) it follows, after some rearrangements, that

$$u_1 = -u_0 g_0\{0.59 - [0.91\eta - 2.65\eta^2 + 2\eta^3 - (\tfrac{1}{4})\eta^4 - (\tfrac{3}{20})\eta^5 + (\tfrac{1}{40})\eta^6][3(\eta - \tfrac{1}{2}\eta^2)]^{-1}\}. \tag{53}$$

The component v_1 can be obtained from Eq. (50) and reads

$$v_1 = \omega r g_0(-2\eta + \eta^3 - \tfrac{1}{4}\eta^4). \tag{54}$$

Finally, w_1 can be obtained in a straightforward manner from Eq. (VIIIb) but as no additional feature concerning the method outlined is involved, it will not be calculated.

RESULTS AND DISCUSSION

The shearing stress and turning moment can be derived from the equations obtained. In the center region the radial component of the shearing stress, τ_{zr} is

$$\tau_{zr} = \mu\,(\partial u/\partial z)\big|_{z=0} = (\mu Q/r\delta^2)\, f''(0)$$

where $f''(0)$ is given by Eq. (29),

$$f''(0) = [(^2/_3)\, Cf'^3(1)]^{1/2} \simeq 2.3.$$

Using Eq. (33) τ_{zr} becomes

$$\tau_{zr} = (6\rho Q^3/\nu r^5). \tag{55}$$

The circunferential component $\tau_{z\phi}$ is zero within the approximation of the solutions obtained. In the external region $\tau_{z\phi}$ becomes

$$\tau_{z\phi} = \mu(\partial v/\partial z)\big|_{z=0} = -(2\omega\rho Q/r), \tag{56}$$

where v is given by

$$v = \omega r + \omega r \epsilon b_1 \tag{57}$$

and b_1 is given by Eq. (50). The radial component of the shearing stress, τ_{zr} is given by

$$\tau_{zr} = \rho(3Q\nu\omega^4)^{1/3} r^{1/3}, \tag{58}$$

and the angle between the radial and tangential components is

$$tg\alpha = -[(\partial u/\partial z)/(\partial v/\partial z)]\big|_{z=0} = (3\nu\omega/8Q^2)^{1/3}\, r^{4/3} \tag{59}$$

which increases rapidly with r.

Concerning the radii r_c and r_e which define the upper and lower limit of the center and external region, respectively, they can be estimated starting from the inequalities

$\omega^2 r^2 \delta^3/\nu Q \ll 1$ center region

$Q\,\delta/r^2\nu \ll 1$. external region

As δ is of the order of δ_m, we are left with the following expressions, (γ is a number much smaller than one)

$$r_c \simeq (Q\,\gamma/(\omega\,\nu)^{\frac{1}{2}})^{\frac{1}{2}}$$

and

$$r_e \simeq (Q/(\omega\,\nu)^{\frac{1}{2}}\gamma)^{\frac{1}{2}}.$$

The relation $(r_c/r_e) \simeq \gamma$ and γ can be taken equal to 0.1 for all practical purposes.

An estimation of r_e is of interest; thus, assuming the following physical parameters for a typical liquid

$$\rho = 1\text{g/cm}^3, \qquad \nu = 10\ \text{cm}^2/\text{seg}$$

and

$$Q = 10\ \text{cm}^3/\text{seg}, \qquad \omega = 100/\text{seg}$$

r_e becomes

$$r_e = 1.8\ \text{cm} \qquad \text{for} \qquad \gamma = 0.1\,.$$

It can be seen that for relatively small mass flows, high viscosity liquids and high spinning speed, the results obtained for the external region apply well, except for a small region near the disk center. These conditions are usually satisfied by the type of applications sought here.

The flow in the region lying between r_c and r_e poses a rather difficult problem, as the convective and centrifugal terms in equation (20) are equally important. In this region the layer thickness must achieve a maximum, as can be easily understood from the fact that before and after it increases and decreases, respectively. In the vicinity of the maximum, u must vary as $(1/r)$. A solution which corresponds to this behaviour may be obtained by adding the center and external solutions for the radial velocity

$$u = u_c + u_e\,, \tag{60}$$

where the subscripts c and e correspond to the center and external regions

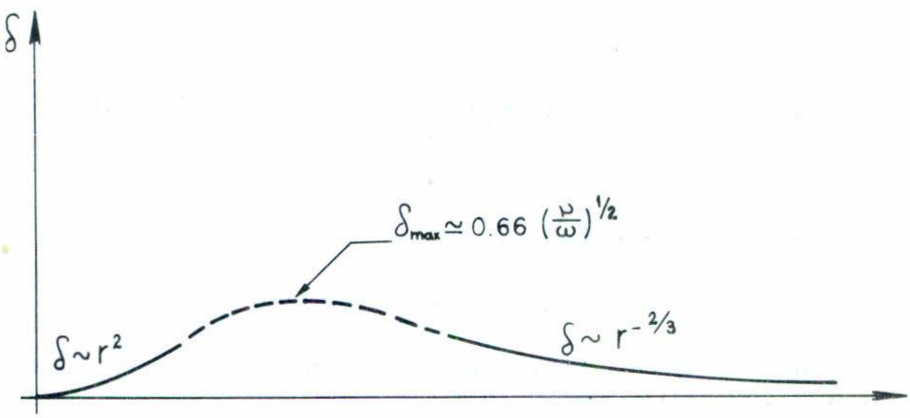

Fig. 2. Sketch of the variation of the layer thickness vs r. The dashed curve corresponds to the intermediate region.

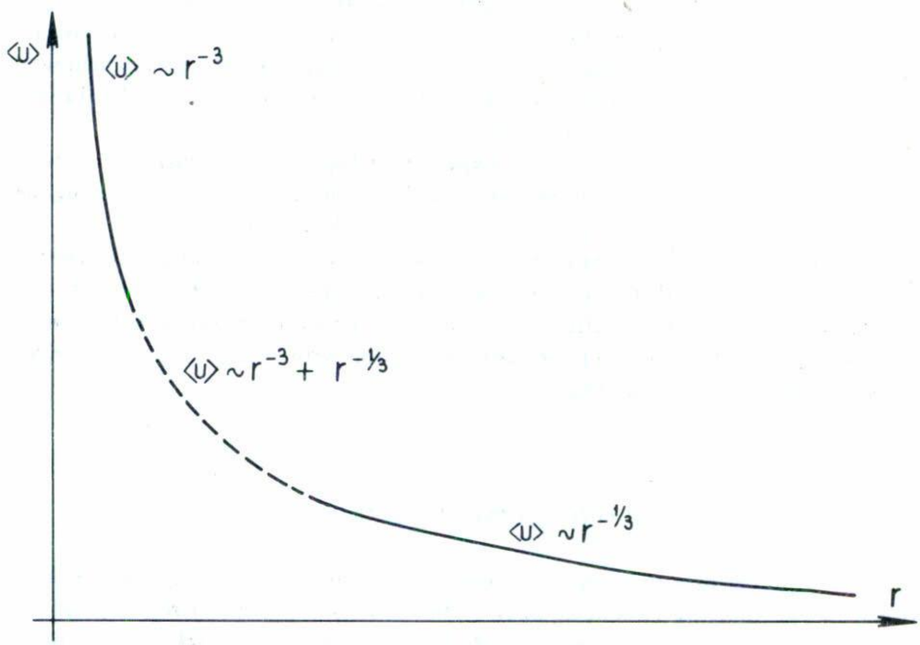

Fig. 3. Sketch of the variation of the radial velocity vs r. The dashed curve corresponds to the intermediate region.

respectively (Eqs. (31) and (42)).

$\langle u \rangle$ also satisfies

$$\langle u \rangle = Q/r\delta = (Q/r\delta_c) + (Q/r\delta_e); \tag{61}$$

thus $\delta = \delta_e \delta_c/(\delta_e + \delta_c)$. (62)

The maximum for δ can then be found,

$$(-1/\delta^2)(d\delta/dr) = -(1/\delta_c^2)(d\delta_c/dr) - (1/\delta_e^2)(d\delta_e/dr) = 0. \tag{63}$$

δ_c and δ_e are given by Eqs. (33) and (44) respectively and introducing their derivatives in Eq. (63) we obtain

$$\delta_{\text{maximum}} = 0.66\ \delta_m .$$

In justifying this solution we can say that the radial velocity u becomes u_c and u_e for small and large values of r, respectively. At the same time and as a result of the equation of continuity the layer thickness δ (Eq. (62)), apart from reproducing the corresponding laws for the center and external regions, achieves a maximum in the intermediate region. As was mentioned above, the layer thickness must satisfy this condition.

On the other hand we can not expect that Eq. (60) will satisfy the differential equations for this problem. It is only a a patching solution, joining the solutions of the two limiting cases considered in this work.

Finally the results obtained for the change of the liquid layer thickness vs r are consistent with the assumption of making the surface tension forces negligible. This may be verified by calculating the radii of curvature and comparing the radial derivative of the surface tension with the term $2\nu(\partial/\partial r)\times \times[(\partial u/\partial r) + (u/r)]$ in Eq. (11).

CONCLUDING REMARKS

The flow problem of a thin layer of liquid flowing over a rotating disk has been solved in two limiting cases: a) when $(\omega^2 r^2 \delta^3/\nu Q) \ll 1$ (center region) and b) when $(Q\delta/r^2\nu) \ll 1$ (external region). In the center region the thickness δ varies as r^2 and in the external region as $r^{-2/3}$. The average radial velocity as r^{-3} and $r^{-1/3}$ respectively. In the center region the problem admits similar solutions and $u(\eta)$ is given as an incomplete elliptic integral. In the external region the solutions are obtained through an asymtotic expansion in the parameter ϵ. The profile of the radial velocity is parabolic and the tangential velocity is equal to ωr in the zeroth order. A patching solution is found which joins the solutions of the two limiting cases considered.

APPENDIX

Values of the velocity components. When the series (34) are introduced into Eq. (19a) the following equation is obtained

$$u = u_0 + \epsilon u_1 + \dots \qquad \text{(I)}$$

where

$$\epsilon^0 \qquad u_0 = (Q/r\delta_0)\, f_0' \qquad \text{(IIa)}$$

$$\epsilon^1 \qquad u_1 = -(Q/r\delta_0)\,[(\delta_1/\delta_0)\, f_0' - f_1'] \,. \qquad \text{(IIb)}$$

Also $\langle u \rangle$ can be written as

$$\langle u \rangle = \langle u_0 \rangle + \epsilon \langle u_1 \rangle + \dots \qquad \text{(III)}$$

where

$$\langle u_0 \rangle = (Q/r\delta_0) \int_0^1 f_0'\, d\eta \qquad \text{(IVa)}$$

$$\langle u_1 \rangle = -(Q/r\delta_0) \int_0^1 [(\delta_1/\delta_0)\, f_0' - f_1']\, d\eta . \qquad \text{(IVb)}$$

At the same time Eq. (24) can be split up into diferent terms of increasing order

$$Q = \langle u_0 \rangle r\delta_0 + \epsilon(\langle u_1 \rangle r\delta_0 + \langle u_0 \rangle r\delta_1) + \dots \qquad \text{(V)}$$

which, after using Eqs. (IV) becomes

$$Q = \langle u_0 \rangle r\delta_0 + \epsilon Q \int_0^1 f_1'\, d\eta + \dots \qquad \text{(VI)}$$

Thus the following set of relations is valid

$\epsilon^0 \qquad Q = \langle u_0 \rangle r \delta_0$ (VIIa)

$\epsilon^1 \qquad 0 = \int_0^1 f_1' \, d\eta.$ (VIIb)

In the case of the axial component of velocity w a similar procedure leads to the following set of equations:

$\epsilon^0 \qquad w_0 = u_0 \eta \, (d\delta_0 / dr)$ (VIII)

$\epsilon^1 \qquad w_1 = (Q/r)(\partial f_1 / \partial r) + u_1 \eta \, (d\delta_0/dr) + u_0 \eta \, (d\delta_1/dr) - 2u_0 \eta \, (\delta_1/r),$

where Eqs. (34a) and (34c) together with (II) have been introduced into Eq. (19b). In deriving equations(VIII) the condition that $(\partial f_0 / \partial r) = 0$ at every point has been used. This conditions is obtained integrating (41) and using the fact that $(\partial f_0 / \partial r) = 0$ at $\eta = 0$ (37a).

REFERENCES

1. W.H. Walton and W.C. Prewett, Proc. Phys. Soc. (London) B 62 (1949) 341–350.
2. R.L. Hines, J. Appl. Phys. 37 (1966) 2730–2736.
3. H. Schliching, *Boundary Layer Theory*. (Mc Graw Hill Book Co. Inc., New York, 1960),p. 85.
4. G.K. Batchelor, *Introduction to Fluid Dynamics*. (Cambridge University Press, Cambridge, 1967).
5. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. (Dover Publications, Inc., New York, 1968).
6. I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*. (Academic Press, New York, 1965).

RESUMEN

Presentamos el problema del flujo de una película líquida delgada sobre un disco en rotación. Se consideran dos regiones correspondientes a los casos límites en que las fuerzas viscosas son mucho mayores que la fuerza centrífuga o la convectiva respectivamente. La primera (region central) se

extiende del centro hasta un radio r_c y la segunda (región externa) del radio r_e a infinito. La región central conduce a un problema de películas con condiciones a la frontera. El espesor de la película varía como r^2 y la velocidad radial como r^{-3}. El problema en la región externa se resuelve mediante un desarrollo en serie en el parametro pequeño ϵ, que representa aproximadamente la razón entre la fuerza convectiva y la viscosa. Se obtienen soluciones a primer orden en ϵ y se presentan resultados en los que el espesor de la película decrece como $r^{-2/3}$ y la velocidad radial como $r^{-1/3}$. Una estimación numérica muestra que r_e puede ser pequeño para un conjunto razonable de parámetros. Se propone una solución en la región entre r_c y r_e basada en los resultados de los casos límites central y externo.