NOTE ON THE INVARIANCE IN FORM OF LAGRANGE'S EQUATIONS

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RESUMEN

Se demuestra que las ec. de Lagrange también conservan su forma bajo ciertas transformaciones que actuan sobre la variable independiente t. El significado físico de éstas transformaciones es muy claro.

SUMMARY

It is shown that Lagrange's equations of Classical Mechanics keep their form under a certain kind of transformations acting on the independent variable t. The physical meaning of these transformations is very clear.

Note on the invariance in form of Lagrange's equations

When the Lagrange equations for the motion of an holonomous mechanical system have been obtained[1], many teachers of Analytical Mechanics usually propose to their students the following exercise: Prove that Lagrange's equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 , \qquad (1)$$

conserve their form when new coordinates Q are introduced such that,

$$q = f(Q, t) . \qquad (2)$$

What the students have to prove is that, in the new coordinates Q, they can safely write the equations:

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{Q}}\right) - \frac{\partial \bar{L}}{\partial Q} = 0 , \tag{3}$$

$\bar{L}$ being defined by:

$$\bar{L}\ (Q;\dot{Q};t) \equiv L\ (\ f(Q,t);\ f,_{Q}\ \dot{Q} + f,_{t}\ ;\ t\ ) \tag{4}$$

But it is less known, even among teachers, that the above invariance in form of Lagrange's equations *holds* for the more general transfor-- mations,

$$\begin{aligned} q &= f(Q,t') \\ t &= g(t') \end{aligned} \tag{5}$$

provided that $g(t')$ is given by:

$$q(t') = at' + b , \tag{6}$$

a and b being real numbers.

The transformations defined by (5) and (6) include displacements in time (a=1), changes in the units used for the measure of t(a>0, b=0), and the reversal of time (a=-1, b=0).

In order to simplify the writing we use Q and q instead of $Q_1,..,Q_n$ and $q_1,..,q_n$. Analogously, we shall write $\dot{q}$; $\acute{Q}$; $f,_Q$; $f,_{t'}$; $f,_{QQ}$,... -- instead of dq/dt; dQ/dt'; $\partial f/\partial Q$; $\partial f/\partial t'$; $\partial^2 f/\partial Q^2$,...

The proof of the mentioned *extended* form-invariance is straighforward. Indeed, from eq. (5) we get:

$$\frac{dq}{dt} = \frac{f,_Q\cdot dQ + f,_{t'}.\ dt'}{g,_{t'}\ dt'} = \frac{f,_Q\cdot\acute{Q} + f,_{t'}}{g,_{t'}}$$

and consequently,

$$L(Q;\acute{Q};t') = L(\ f(Q,t');\ \frac{f,_Q\cdot\acute{Q} + f,_{t'}}{g,_{t'}};\ g(t')\ ) \tag{7}$$

Now, we have to find the restrictions that are to be imposed on -

$g(t')$ in order that the equations,

$$\frac{d}{dt'}\left(\frac{\partial\bar{L}}{\partial\acute{Q}}\right) - \frac{\partial\bar{L}}{\partial Q} = 0 \tag{8}$$

be a consequence of eq. (1).

We begin by calculating $\partial\bar{L}/\partial Q$ and $\partial L/\partial\acute{Q}$: From eq. (7) we easily get:

$$\frac{\partial\bar{L}}{\partial Q} = \frac{\partial L}{\partial q}\cdot f_{,Q} + \frac{\partial L}{\partial\dot{q}}\cdot\frac{\partial}{\partial Q}\left(\frac{f_{,Q}\cdot\acute{Q} + f_{,t'}}{g_{,t'}}\right) =$$

$$= \frac{\partial\bar{L}}{\partial q}\cdot f_{,Q} + \frac{\partial L}{\partial\dot{q}}\cdot\frac{f_{,QQ}\cdot\acute{Q} + f_{,t'Q}}{g_{,t'}} \tag{9}$$

and

$$\frac{\partial L}{\partial\acute{Q}} = \frac{\partial L}{\partial\dot{q}}\cdot\frac{\partial}{\partial\acute{Q}}\left(\frac{f_{,Q}\cdot\acute{Q} + f_{,t'}}{g_{,t'}}\right) = \frac{\partial L}{\partial\dot{q}}\cdot\frac{f_{,Q}}{g_{,t'}}$$

Therefore,

$$\frac{d}{dt'}\left(\frac{\partial L}{\partial\acute{Q}}\right) = \frac{d}{dt'}\left(\frac{\partial L}{\partial\dot{q}}\right)\cdot\frac{f_{,Q}}{g_{,t'}} + \frac{\partial L}{\partial\dot{q}}\cdot\frac{d}{dt'}\left(\frac{f_{,Q}}{g_{,t'}}\right)$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}}\right)\cdot\frac{dt}{dt'}\cdot\frac{f_{,Q}}{g_{,t'}} + \frac{\partial L}{\partial\dot{q}}\cdot\frac{g_{,t'}(f_{,QQ}\cdot Q + f_{,Qt'}) - f_{,Q}\cdot g_{,t't'}}{(g_{,t'})^2} =$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}}\right)\cdot f_{,Q} + \frac{\partial L}{\partial\dot{q}}\cdot\left(\frac{f_{,QQ}\cdot\acute{Q} + f_{,Qt'}}{g_{,t'}} - \frac{f_{,Q}\cdot g_{,t't'}}{(g_{,t'})^2}\right) \tag{10}$$

From (9) and (10) we get:

$$\frac{d}{dt'}\left(\frac{\partial\bar{L}}{\partial\acute{Q}}\right) - \frac{\partial\bar{L}}{\partial Q} = \left(\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}}\right) - \frac{\partial L}{\partial q}\right)\cdot f_{,Q} - \frac{f_{,Q}\cdot g_{,t't'}}{(g_{,t'})^2} \tag{11}$$

Therefore, taking into account eq. (1), we finally have:

$$\frac{d}{dt'}\left(\frac{\partial\bar{L}}{\partial\acute{Q}}\right) - \frac{\partial L}{\partial Q} = -\frac{f_{,Q}\cdot g_{,t't'}}{(g_{,t'})^2} \tag{12}$$

and, accordingly, in order to get to eq. (8) -since $f(Q,t')$ are arbitrary functions- we must have:

$$g,_{t't'} = 0$$

that is,

$$g(t') = at' + b ,$$

which is just what we desired to prove.

This result can be considered as the mathematical translation of the physical fact that in Classical Mechanics the absolute role played by time is *preserved* under the transformations given by eq. (6). The result is in accordance with the freedom that any physicist has of choosing at pleasure the origin, units and direction of the time variable.

REFERENCES

1. E.T. Wittaker: A treatise on the Analytical Dynamics of particles and rigid bodies, Dover Publ. N.Y. (1944) Chapter II. H. Goldstein: Clas--sical Mechanics, Addison Wesley, Reading. Mass. (1963)Chapter I. L.A. Pars: A Treatise on Analytical Dynamics, Heinemann. London (1965) pp. 88-89. L. Landau, E. Lifschitz: Mecanique, Ed. MIR, Moscou (1966) Chapter I. P. Mittelstaedt: Klassiche Mechanik, Bibliographyches Institut. Manheim 1970. Kapitel II.