THE AIRY FUNCTION AS A NON-SUBGROUP BASIS FOR THE OSCILLATOR REPRESENTATION ($D^+_{1/4} + D^+_{3/4}$) OF SL (2,R)

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ABSTRACT

The linear potential Schrödinger Hamiltonian can be used to classify the rows of the oscillator representation ($D^+_{1/4} + D^+_{3/4}$) of SL(2,ℝ). The transformation properties of the generalized eigenfunctions of this Hamiltonian -Airy functions- are seem to be particularly simple, although the representation matrix elements do not reduce to known special functions. The Airy basis, provides also a generalized orthonormal basis for Bargamann's Hilbert space of analytic functions.

1. INTRODUCTION

In Reference 1, we found the matrix elements of all D^+_k series representations of SL(2,ℝ) for all subgroup reductions using the techniques of canonical transforms in order to reduce the problem to the evaluation of a single integral. Here we shall do the same for a basis of displaced -- Airy functions.

The *oscillator* representation of SL(2,ℝ) ≃ SP(2,ℝ) ≃ SU(1,1) ≃ -- SO(2,1) is spanned by

$$I_1 = \frac{1}{4}(P^2-Q^2),\ I_2 = \frac{1}{4}(QP+PQ),\ I_3 = \frac{1}{4}(P^2+Q^2) \tag{1.1}$$

in the usual quantum mechanical realization where Q and P are the coordinate and momentum operators, self-adjoint on the real line ℝ. Their commutation relations are well known. On calculating the Casimir operator we see that $I^2 = I_1^2 + I_2^2 - I_3^2 = -3/16$, i.e. $k(1-k) = -3/16$, so that the representation generated by (1.1) is the direct sum $D^+_{osc} = D^+_{1/4} + D^+_{3/4}$.

This is four-fold valued on SO(2,1) and double valued on SU(1,1) $\simeq$SL(2,R). As the inversion operator $\mathbf{P}$ commutes with (1.1), it can be used to distinguish between the two representations present in D^{+}_{osc}: from the I_3-reduction it is known that the basis functions for the $D^{+}_{1/4}$ representation have even parity, while those of $D^{+}_{3/4}$ have odd parity.

In classifying the rows of the oscillator representation, we can make use of any of its subgroups: these belong to one of three conjugation classes whose representatives are usually chosen as: (a) SO(2) generated by I_3 giving rise to the quantum harmonic oscillator eigenstates $\Psi^{h}_{n}(q)$, $n = 0,1,2...$ with spectrum $\frac{1}{2}(n + \frac{1}{2})$ where even n corresponds to $D^{+}_{1/4}$ and odd n to $D^{+}_{3/4}$; (b) the non-compact SO(1,1) subgroup generated by I_1 (or I_2) whose spectrum covers twice the real line, and (c) the non-compact E(1) subgroup generated by $I_1 + I_3 = \frac{1}{2}P^2$ and whose spectrum covers twice the positive half-axis. Winternitz and collaborators(3) have shown that second-order operators in the enveloping algebra of SL(2,$\mathbb{R}$) can be used to provide complete and orthonormal sets of functions on the group or its coset manifolds which are closely related to the separable coordinate systems on a (2+1)-dimensional hyperboloid. This has been studies in detail by Kalnins and Miller(4) for the nine orbits into which the second-order operator space splits under the adjoint action of the group, three of these being related to the subgroup decompositions.

In this paper we will give some results for the oscillator representation using a new type of basis provided by the generalized eigenfunctions of the operator

$$L = \frac{1}{2}P^2 + Q \tag{1.2}$$

which are the displaced Airy functions. The spectrum of (1.2) is continuous and covers the real line(4,5,6). The reasons for regarding the problem as interesting are the following: Although any self-adjoint Schrödinger Hamiltonian can provide a complete orthonormal (possibly generalized) eigenbasis which can serve as a unitary representation basis for the algebra (1.1), the eigenfunctions of the free-fall or linear potential Schrödinger Hamiltonian (1.2) are one of the relatively few solutions which can be written in terms of known special functions. Secondly, the operator (1.2) is not an element nor belongs to the enveloping algebra of (1.1). It is neither a subgroup nor a non-subgroup operator in the sense described above. It can be though of as $I_1 + I_3 + [2(I_3-I_1)]^{1/2}$. We - are able to present our results because of the fact that the realization (1.1) can be embeded into a $W \otimes SL(2,\mathbb{R})$ algebra, where w is the Heigsenberg-Weyl algebra[6] with generators Q, P and $\mathbb{1}$, with the well known commutation relations. Then (1.2) becomes a subgroup generator, not conjugate to any subgroup generator in $SL(2,\mathbb{R})$ or in w alone. This will be a feature common to any group whose algebra can be similarly embeded, in particular the $SP(2n,\mathbb{R})$ algebra of linear canonical transformations in n-dimensional Quantum Mechanics[7,8]. It should be noted, though, that we are able to work only within the oscillator representation of $SL(2,\mathbb{R})$ because only there does $I_1+I_3+[2(I_3-I_1)]^{1/2}$ have a simple form (1.2) in terms of differential operators.

In Ref. 5, Kalnins and Miller considered the eigenfunctions of (1.2) as a one-variable realization of a subgroup basis in $W \otimes SL(2,\mathbb{R})$. However, working within the algebra rather than within the group, the finite transformation properties of this basis were not fully explored. These are - quite simple and easy to calculate with the techniques of Ref. 6, which we sketch in Section 2. Moreover, Airy functions constitute a generalized basis for the Bargmann Hilbert space of analytic functions. In - Section 3 we build the mixed-basis matrix elements with the harmonic oscillator eigenbasis, generalizing one result by Boyer[10]. Lastly, in Section 4 we add some remarks on the decomposition of the oscillator - representation into its irreducible components.

2. AIRY FUNCTIONS AND THEIR SL(2,ℝ) TRANSFORMATION

The regular solutions of the equation $L\Psi = \lambda\Psi$, where L is the operator (1.2) can be found quite easily[5,6] if we subject this to a Fourier transform, whereupon we have $(\frac{1}{2}Q^2 - P)\,\tilde{\Psi} = \lambda\tilde{\Psi}$ where the normalized solutions are

$$\tilde{\Psi}_\lambda^{\ell}(q) = e^{-i\Pi/4}(2\Pi)^{-1/2}\ \exp i(-\lambda q+\frac{1}{6}q^3),\ \lambda \in \mathbb{R} \tag{2.1}$$

which are orthonormal in the Dirac sense and complete for $L^2(\mathbb{R})$. The inverse Fourier transform (times $e^{i\pi/4}$) of (2.1) yields[11], through Airy's integral

$$\Psi_\lambda^{\ell}(q) = 2^{1/3}\ \mathrm{Ai}(2^{1/3}[q-\lambda]) = \Psi_{\lambda+x}^{\ell}(q+x) \tag{2.2}$$

which; being unitary transforms of a generalized basis for $L^2(\mathbb{R})$, are - also generalized basis for this space.

The Airy function Ai(Z) can be represented in terms of Macdonald functions[12] or hypergeometric series

$$\mathrm{Ai}(Z) = \frac{1}{\pi}\sqrt{Z/3}\,K_{1/3}\,(\tfrac{2}{3}Z^{3/2})$$

$$= \frac{3^{-2/3}}{\Gamma(2/3)}\sum_{n=0}^{\infty}\frac{3^n(1/3)_n}{(3n)!}Z^{3n} + \frac{3^{-1/3}}{\Gamma(1/3)}\sum_{n=0}^{\infty}\frac{3^n(2/3)_n}{(3n+1)!}Z^{3n+1} \qquad (2.3)$$

$$= \frac{2\cdot 3^{-1/2}\pi}{\Gamma(1/3)\Gamma(2/3)}\left[\frac{3^{-2/3}}{\Gamma(2/3)}\;{}_0F_1(\tfrac{2}{3},\tfrac{1}{9}Z^3) + Z\frac{3^{-1/3}}{\Gamma(1/3)}\;{}_0F_1(\tfrac{4}{3},\tfrac{1}{9}Z^3)\right],$$

where $(a)_n$ is the Pochhammer symbol. The Airy function is thus an entire analytic function. It has zeros on the negative real axis only. Asymptotically, it behaves as $\frac{1}{2}\pi^{-1/2}Z^{-1/4}\exp(-\frac{2}{3}Z^{3/2})$ for $|\arg z| < \pi$ and as $\mathrm{Ai}(-Z) \sim \pi^{-1/2}Z^{-1/4}\sin(\frac{2}{3}Z^{-3/2} + \frac{1}{2}\pi)$ for $|\arg z| < 2\pi/3$, so it is of growth (3/2,2/3).

Now, the application of a transformation generated by the algebra - (1.1) can be described through a complex linear canonical transform[6,8]

$$\left[C\begin{pmatrix}a&b\\c&d\end{pmatrix}f\right](q) = (2\pi b)^{-1/2}e^{-i\pi/4}\int_{\mathbb{R}}dq'\,\exp\{\tfrac{i}{2b}[aq'^2-2qq'+dq^2]\}f(q'), \qquad (2.4)$$

$$\left[\exp(\tfrac{ic}{2a}\{I_3-I_1\})\,\exp(i\ell n a^{-2}I_2)\,\exp(-\tfrac{ib}{2a}\{I_3+I_1\})f\right](q)$$

where ad - bc = 1. The last term in (2.4) decomposes the matrix $M\equiv\begin{pmatrix}a&b\\c&d\end{pmatrix}$ into a product of a lower-triangular, a diagonal and an upper-triangular matrix. This is possible for all M except those with a = 0. The Fourier transform[11] is one of the cases where this decomposition fails. - Some well-known transforms can be seen in Ref. 9. Adjoining now the -- Heisenberg-Weyl group action

$$[\exp i(xQ+yP+z\mathbb{1})f](q) = [T(x,y,z)f](q) = \exp i(xq+\tfrac{1}{2}xy+z)\,f(q+y) \qquad (2.5)$$

we have the product of transforms

$$F\{\begin{pmatrix}a&b\\c&d\end{pmatrix}(xyz)\} = C\begin{pmatrix}a&b\\c&d\end{pmatrix}T(xyz) \qquad (2.6)$$

whose composition, defining $\xi = (x,y)$, $\Omega \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and ξ^T is the transpose of ξ, is

$$\{M_1,\ \xi_1,\ z_1\}\ \{M_2,\ \xi_2,\ z_2\} = \{M_1M_2,\ \xi_1M_2 + \xi_2,\ z_1+z_2+\tfrac{1}{2}\ \xi_1M_2\Omega\xi_2^T\}, \quad (2.7)$$

defines that of the elements of the WSL(2,ℝ) ≡ W ⊗ SL(2,ℝ) group. The - subgroup generated by the operator (1.2) in WSL(2,ℝ) is

$$[\exp(itL)f]\ (q) = [F\ \{(\begin{smallmatrix} 1 & -t \\ 0 & 1 \end{smallmatrix})\ (-t, \tfrac{1}{2}\ t^2,\ -\tfrac{1}{6}\ t^3)\}f]\ (q) \qquad (2.8)$$

Lastly, composing (2.5), (2.6) through (2.7) and taking the limit $b \to 0$ we see that the subgroup of WSL(2,ℝ),

$$[F\{(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix})\ (xyz)\}\ f]\ (q) = a^{-1/2}\ \exp i(\frac{cq^2}{2a} + \frac{xq}{a} + \frac{1}{2}xy+z) f(\frac{q}{a} + y),$$

defines the *geometric* transform subgroup.

The basic argument for finding the transformation of the Airy functions Ψ_λ^ℓ (q) under SL(2,ℝ) can now be given: we decompose the general $M = (\begin{smallmatrix} a b \\ c d \end{smallmatrix})$ into a transform (2.8) in the subgroup generated by L, times a geometric transform (2.9). The former will only multiply Ψ_λ^ℓ (q) by a factor $e^{i\lambda t}$ (below, $t = -b/a$), while the latter one has its action given explicitly by (2.9):

$$[C(\begin{smallmatrix} ab \\ cd \end{smallmatrix})\Psi_\lambda^\ell](q) = [F(\begin{smallmatrix} a0 \\ ca^{-1} \end{smallmatrix})(-\frac{b}{a}, \frac{1}{2}\frac{b^2}{a^2}, \frac{1}{6}\frac{b^3}{a^3})\}\ F\{(\begin{smallmatrix} 1 & b/a \\ 0 & 1 \end{smallmatrix})(\frac{b}{a}, \frac{1}{2}\frac{b^2}{a^2}, \frac{1}{6}\frac{b^3}{a^3})\}\Psi_\lambda^\ell](q)$$

$$= e^{-i\lambda b/a}[F\{(\begin{smallmatrix} a0 \\ ca^{-1} \end{smallmatrix})(-\frac{b}{a}, \frac{1}{2}\frac{b^2}{a^2}, \frac{1}{6}\frac{b^3}{a^3})\}\Psi_\lambda^\ell](q) \qquad (2.9a)$$

$$= a^{-1/2}\ \exp i(\frac{cq^2}{2a} - \frac{bq}{a^2} - \frac{5}{12}\frac{b^3}{a^3} - \frac{b}{a}\lambda)\Psi_\lambda^\ell\ (\frac{q}{a} + \frac{b^2}{2a^2}),$$

which shows that the function Ψ_λ^ℓ (q) is self-reproducing[9] under canonical transforms.

The action (2.9) is well defined on Ψ_λ^ℓ (q) not only for real matrices M but for complex ones as well as long as the kernel is a decreasing Gaussian or at most oscillating (as for M real), the condition being - $\mathrm{Im}\ b^*a \geq 0$. When $a = 0$, unimodularity implies $b = -1/c$ and a different decomposition of $M = \begin{pmatrix} 0 & -1/c \\ c & d \end{pmatrix}$ is needed. Using (2.9) and (2.1)[11], we find

$$\left[C\begin{pmatrix} 0 & -1/c \\ c & d \end{pmatrix}\Psi_\lambda^{\ell}\right](q) = \left[C\begin{pmatrix} -1/c & 0 \\ d & -c \end{pmatrix} C\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Psi_\lambda^{\ell}\right](q) =$$

$$= \left[C\begin{pmatrix} -1/c & 0 \\ d & -c \end{pmatrix}\tilde{\Psi}_\lambda^{\ell}\right](q) = (-c)^{1/2}\exp\left[-\frac{1}{2}\, icdq^2\right]\tilde{\Psi}_\lambda^{\ell}(-cq) \tag{2.9b}$$

Eqs. (2.9) give the transformation of the Ψ_λ-basis under SL(2,$\mathbb{R}$). The operator (2.4) is still unitary if seen as a mapping between $f \varepsilon L^2(\mathbb{R})$ and $C(M) f \varepsilon F_M$, Hilbert spaces of analytic functions a la Bargmann described in Ref. 8. In particular, for the Bargmann transform[13] where $B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$,

$$\left[C(B)\Psi_\lambda^{\ell}\right](q) \equiv \bar{\Psi}_\lambda^{\ell}(q) = 2^{1/4}\exp\left(\frac{1}{2}q^2 - \sqrt{2}\,q + \frac{5}{12} - \lambda\right)\Psi_\lambda^{\ell}\left(\sqrt{2}\,q - \frac{1}{2}\right)$$

$$= 2^{7/12}\exp\left[\frac{1}{2}q^2 - \sqrt{2}\,q + \frac{5}{12} - \lambda\right]\mathrm{Ai}\left(2^{5/6}q - 2^{-2/3} - 2^{1/3}\lambda\right) \tag{2.10}$$

This is an entire analytic function over the complex plane $\mathbb{C}$. The factor $\exp \frac{1}{2} q^2$, however, places it just outside the Bargmann Hilbert space F_B which consists of functions of growth (2,1/2). Nevetheless, (2.10) provide a generalized orthonormal basis for as in the corresponding scalar products[13]

$$\left(\bar{\Psi}_{\lambda'}^{\ell}, \bar{\Psi}_{\lambda}^{\ell}\right)_B = \left(\Psi_{\lambda'}^{\ell}, \Psi_{\lambda}^{\ell}\right) = \left(\tilde{\Psi}_{\lambda'}^{\ell}, \tilde{\Psi}_{\lambda}^{\ell}\right) = \delta(\lambda - \lambda') \tag{2.11}$$

3. SL(2,$\mathbb{R}$) MATRIX ELEMENTS IN THE AIRY AND MIXED BASES

We want to calculate the SL(2,$\mathbb{R}$) representation between the Airy - basis states $\Psi_\lambda^{\ell}(q)$ given by (2.2), that is

$$\mathcal{D}_{\lambda'\lambda}^{\ell}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \left(\Psi_{\lambda'}^{\ell}, C\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Psi_\lambda^{\ell}\right) = 2^{2/3}a^{-1/2}\exp i\left(-\frac{5}{12}\frac{b^3}{a^3} - \frac{b}{a}\lambda\right) \times$$

$$\times \int_R dq\, \mathrm{Ai}\left(2^{1/3}[q - \lambda']\right) \exp i\left(\frac{cq^2}{2a} - \frac{bq}{a^2}\right)\mathrm{Ai}\left(2^{1/3}\left[\frac{q}{a} + \frac{b^2}{2a^2} - \lambda\right]\right) = \tag{3.1}$$

$$= \left(C\begin{pmatrix}0&-1\\1&0\end{pmatrix}\Psi_{\lambda'}^{\ell},\ C\begin{pmatrix}0&-1\\1&0\end{pmatrix} C\begin{pmatrix}a&b\\c&d\end{pmatrix}\Psi_{\lambda}^{\ell}\right) = \left(P\tilde{\Psi}_{\lambda'}^{\ell},\ C\begin{pmatrix}c&d\\-a&-b\end{pmatrix}\Psi_{\lambda}^{\ell}\right) =$$

$$= 2^{1/3}(2\pi c)^{-1/2}\exp i\left(\frac{5}{12}\frac{d^3}{c^3} - \frac{d\lambda}{c} + \frac{\pi}{4}\right)\int_{\mathbb{R}} dq\, \exp i\left(\frac{q^3}{6} - \frac{aq^2}{2c} - \left[\frac{d}{c^2} + \lambda'\right]q\right)$$

$$\mathrm{Ai}\left(2^{1/3}\left[\frac{q}{c} + \frac{d^2}{2c^2} - \lambda\right]\right),$$

where in the last three terms we have used the transform(11) $C\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ and the fact that $\left|C\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right|^2 = \mathbf{P}$ is the inversion operator. Although the last integral appears somewhat simpler than the first it does not seem possible to evaluate it in terms of known special functions with the aid of (2.3). We will search for its evaluation in terms of series using mixed basis matrix elements through calculating the overlap of (2.9) with the orthonormal harmonic oscillator wavefunctions $\Psi_n^h(q)$ in terms of their Bargmann transforms(8,11) $\bar{\Psi}_n^h(q) = (2\pi)^{-1/4}(n!)^{-1/2}\, q^n$ i.e.

$$\left(\Psi_n^h\ ,\ C\begin{pmatrix}a&b\\c&d\end{pmatrix}\Psi_{\lambda}^{\ell}\right) =$$

$$= \left(\bar{\Psi}_n^h\ ,\ C\frac{1}{\sqrt{2}}\begin{pmatrix}1&-i\\-i&1\end{pmatrix}\ C\begin{pmatrix}a&b\\c&d\end{pmatrix}\Psi_{\lambda}^{\ell}\right)_B \qquad (3.2)$$

$$= \left(\bar{\Psi}_n^h\ ,\ C\frac{1}{\sqrt{2}}\begin{pmatrix}a-ic & b-id\\c-ia & b-ib\end{pmatrix}\Psi_{\lambda}^{\ell}\right)_B$$

We must thus find the Taylor expansion of

$$\left|C\frac{1}{\sqrt{2}}\begin{pmatrix}a-ic & b-id\\c-ia & d-ib\end{pmatrix}\Psi_{\lambda}^{\ell}\right|(q) = C_M^{\lambda}\exp|\alpha q^2 + \beta q|\mathrm{Ai}(\gamma q+\delta) \equiv C_M^{\lambda}\sum_{n=0}^{\infty} A_n(\alpha,\beta,\gamma,\delta)q^n$$

with $M = \begin{pmatrix}a&b\\c&d\end{pmatrix}$ and (3.3)

$$C_M^{\lambda} = 2^{7/12}(a-ic)^{-1/2}\exp\left[\frac{5}{12}\frac{(d+ib)^3}{(a-ic)^3} - \frac{d+ib}{a-ic}\lambda\right]$$

$$\alpha = (a+ic)/2(a-ic),\ \beta = -\sqrt{2}\,(d+ib)/(a-ic)^2 \qquad (3.4)$$

$$\gamma = 2^{5/6}/(a-ic),\ \delta = -2^{1/3}\left[\lambda + (d+ib)^2/2(a-ic)^2\right],$$

where α, β, γ and δ are thus independent quantitites.

We expand first

$$\mathrm{Ai}\,(\gamma q + \delta) = \sum_{n=0}^{\infty} \frac{(\gamma q)^n}{n!} \frac{d^n}{dz^n} \mathrm{Ai}(z) \Bigg|_{z=\delta} \equiv \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} A_n(\delta)\, q^n \qquad (3.5)$$

Exchanging double summations we find

$$A_n(\delta) = \sum_{k=|(n+2)/3|}^{\infty} |U_R + \delta V_R| \frac{\delta^{3k-n}}{(3k-n)!} + n \sum_{k=|(n+1)/3|}^{\infty} V_R \frac{\delta^{3k-n+1}}{(3k-n+1)!} \qquad (3.6)$$

where $[r]$ is the integer part of r and

$$U_k = 3^{k-2/3} (1/3)_k / \Gamma(2/3), \; V_k = 3^{k-1/3} (2/3)_k / (3k+1)\Gamma(1/3) \qquad (3.7)$$

thus providing a series representation for the n^{th} derivative of Airy function. Of course $A_o(\delta) = \mathrm{Ai}(\delta)$, while $A_n(0)$ involves two summands, one is nonzero when $n \equiv 0 \bmod 3$, and another when $n \equiv 1 \bmod 3$, which - reconstitute the series (2.3). Making now use of the known series expansion of the exponential functions, (3.6) and a triple-summation exchange, we find for the coefficients of (3.3)

$$A_n(\alpha,\beta,\gamma,\delta) = \sum_{\ell=0}^{|n/2|} \sum_{m=0}^{n-2\ell} \frac{\alpha^{\ell} \beta^{m} \gamma^{n-2\ell-m}}{\ell!\, m!\, (n-2\ell-m)!} A_{n-2\ell-m}(\delta), \qquad (3.8)$$

where the relation between the Greek and Latin entries is given by (3.5). In reviewing the literature, we can point out that Boyer[10] has treated a particular case of the expansion coefficients (3.8). As these are related to several other overlap coefficients, generating functions and integral relations, we establish the precise connection in the Appendix.

The mixed basis matrix element (3.2) is thus

$$(\Psi_n^h,\, C\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Psi_\lambda^{\ell}) = (2\pi)^{1/4} C_M^{\lambda} (n!)^{1/2} A_n\,(\alpha,\beta,\gamma,\delta), \qquad (3.9)$$

with the constant C_M^{λ} defined in (3.4) and, noting that $C_{\mathbb{1}}^{\lambda}$ and $A_n(\mathbb{1})$ are real,

$$\mathcal{D}_{\lambda'\lambda}^{\ell} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (2\pi)^{1/2} C_{\mathbb{1}}^{\lambda'} C_M^{\lambda} \sum_{n=0} n!\, A_n(\tfrac{1}{2} - \sqrt{2},\; 2^{5/6} - 2^{1/3}[\lambda' + \tfrac{1}{2}]) A_n(\alpha,\beta,\gamma,\delta) \qquad (3.10)$$

These D-matrices do not become diagonal for any subgroup. Their unitary

$$D^{\ell}_{\lambda'\lambda}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = D^{\ell}_{\lambda\lambda'}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}^{*} \tag{3.11}$$

is a consequence of (3.1), but is not manifest in the form (3.10). We - can use the property of the Ψ^{ℓ}_{λ} functions given by the extreme members of (2.2) and (2.9) to show that

$$\left[C\begin{pmatrix}a & b\\ c & d\end{pmatrix}\Psi^{\ell}_{\lambda+x}\right](q) = e^{-kx/a}\left[C\begin{pmatrix}a & b\\ c & d\end{pmatrix}\Psi^{\ell}_{\lambda}\right](q-ax), \tag{3.12}$$

and use this in proving the relation

$$\mathcal{D}^{\ell}_{\lambda'+ax,\lambda+x}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = e^{-bx/a}\,\mathcal{D}^{\ell}_{\lambda'\lambda}\begin{pmatrix}a & b\\ c & d\end{pmatrix}, \tag{3.13}$$

between the matrix elements (3.1).

The mixed matrix elements between the Airy basis and coherent states are given by (3.3) directly, while the mixed matrix elements with the eigenstates[5] of $I_3 \mp I_1$ (δ's and oscillating exponentials) are given by (2.9) and its Fourier transform, which can be readily implemented through matrix algebra and amounts to a transformation $(a,b,c,d) \to (c,d,-a,-b)$. The mixed matrix elements with the repulsive oscillator eigenfuctions of I_1 are essentially the Mellin transform of a function related to (2.9). It does also not seem to lead to known special functions.

4. THE IRREDUCIBLE PIECES OF THE OSCILLATOR REPRESENTATION

The $D^{+}_{1/4}$ and $D^{+}_{1/3}$ irreducible components of the oscillator representation[2] can be seen to correspond to the matrix (3.1) evaluated between the even and odd parts of the basis functions, i. e.

$$\Psi^{1/2\,\mp\,1/4,\ell}_{\lambda}(q) \equiv \frac{1}{\sqrt{2}}\left[\Psi_{\lambda}{}^{\ell}(q) \pm \Psi_{\lambda}{}^{\ell}(-q)\right] \tag{4.1}$$

Now, as $|L,\mathbf{P}| \neq 0$ the functions (4.1) are not eigenfunctions of L in - (2.1). The situation is akin to considering the even and odd parts of e^{imq}, cosine and sine functions, none of which are eigenfunctions of -- -id/dq. The row-label λ still runs over the real line.

APPENDIX

The overlap coefficients $C_n(y_B, a_B)$ (Ref. 10, Eqs. (5.22)-(5.25) by C.P. Boyer, see also Ref. 5) can be related to ours in (3.8) through comparison of the Gaussian, exponential and Airy coefficients of our q and Boyer's z_B : The former one implies $z_B = \pm i\sqrt{\alpha}\, q$ while the latter ones, $a_B = \mp \gamma/2\sqrt{2\beta}$ and $a_B = \mp \alpha^3(2\beta)^{-3/2}$. (Variables from Ref. 10 appear the subscript B, the rest refers to our (3.5)). These last two relations imply $\alpha\beta = \gamma^3$, so that we have the restriction

$$ab + cd = 9i, \tag{A.1}$$

in addition to the unimodularity ad-bc = 1: the $C_n(y_B, a_B)$ has two free parameters while we have three. All of Ref. 10 parameters can thus be written in terms of, say, a and c as

$$a_B = \mp \alpha^3 (2\beta)^{-3/2} = \pm 2^{5/2}(a^2 + c^2)^{-3/2}, \tag{A.2}$$

$$y_B = a_B^{2/3}\delta - a_B^2 = -4\lambda/(a^2 + c^2) - 160/(a^2 + c^2) \tag{A.3}$$

$$z_B = \pm i\sqrt{\alpha}\, q = \pm i \left[(a+ic)/(a-ic)\right]^{1/2} q. \tag{A.4}$$

Substituting now (A.1)-(A.4) into the expansion (3.4) we can relate

$$A_n(\alpha,\beta,\gamma,\delta) = (4\pi)^{-1} a_B^{-1/3} \exp\left(-\tfrac{2}{3} a_B^2 - y_B\right)(\pm i\alpha^{1/2})^n C_n(y_B, a_B) \tag{A.5}$$

when (A.1) holds. Use of Eq. (5.24) of Ref. 10 now allows us to express a subset of the $A_n(\alpha,\beta,\gamma,\delta)$ in terms of a single integral involving the Fourier transform of the Airy function with a harmonic oscillator function. This last relation can also be obtained from (3) when we remember that the latter are self-reproducing[9] under canonical transforms.

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