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HEISENBERG-LIKE EQUATION OF MOTION FOR DIRAC ALGEBRA IN GENERAL RELATIVITY

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ABSTRACT

The Heisenberg equations of motion are applied to γ^5 and the spin tensor, $\sigma^{\mu\nu}$. In both cases, the limit for the special-relativistic situation gives a Zitterbewegung equation, the frequency being double of that of the Compton frequency. These results agree with previous ones by S.K. Wong and G. Szamosi. The use of a diagonal metric, such as the Robertson-Walker metric has an ambiguous effect on the frequency: Either we redefine the gamma or the spin tensor, and then formally the frequency is the same, or there is no redefinition, and the spectrum becomes complex.

RESUMEN

Se aplican las ecuaciones de movimiento de Heisenberg a γ^5 y al tensor de espín $\sigma^{\mu\nu}$. En ambos casos, en el límite de relatividad especial se obtiene una ecuación de Zitterbewegung, con una frecuencia igual al doble de la de Compton. Estos resultados están de acuerdo con los obtenidos por S.K. Wong y G. Szamosi. El uso de una métrica diagonal, como la de Robertson-Walker, tiene un efecto ambiguo en la frecuencia: Si redefinimos la gama o el tensor de espín, entonces formalmente la frecuencia es la misma, o, en caso contrario, si no hay redefinición, el espectro se torna complejo.

I. INTRODUCTION

Many results on this subject have been found by Wong[1]. Previously, Szamosi[2] had found the form the equations take in the special-relativistic limit. In this brief paper, we give a couple of results that, for some reason, were not taken into account in Ref. 1.

II. EQUATIONS OF MOTION FOR γ^5 AND THE SPIN TENSOR

The idea is of course to write down the Heisenberg equations of motion, valid for an arbitrary operator, and then to apply them to the elements of the Dirac Algebra. It is worth mentioning here that Wong[1] considers it important to justify his technique by showing that it applies rigorously to the mean values of the observables. Here we ensure covariance by use of the generalized derivative operator (including spin), which may be considered as the logical extension of the Schrödinger representation of momentum, $p_x \to i\hbar\,\partial_x$, etc.

Here we use the notation of Chapman and Leiter[3].

The spin covariant derivative is written ∇_ρ and acts on the various quantities according to their transformation properties, like a spinor or a gamma matrix. The Fock-Ivanenko coefficients Γ_μ appear here. It is clear that the fully covariant derivative for a tensor is just the ordinary covariant derivative[4]:

$$\nabla_\rho g^{\mu\nu} = g^{\mu\nu};\rho \quad .$$

Chapman and Leiter[3] give recipes for calculating the Fock-Ivanenko coefficients, and so does J.G. Fletcher[5]. This last author finds expressions to calculate the F-I coefficients as explicit functions of the gamma matrices.

A. *Equation of motion for* γ^5

The properties of this matrix, defined by

$$\gamma^5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4 \quad ,$$

are important because of its relation with the problem of charge conjugation (see, for example, Ref. 6), and to chirality[7].

The technique used here is very simple: the equation of motion is written down using the ordinary coordinates of general relativity and each side of it is taken to be an operator acting on a free solution of the Dirac's equation. By cancelling out constant factors in both sides, we are left with

$$\dot{\gamma}^5 \psi = \gamma^\rho (\nabla_\rho \gamma^5)\psi + [\gamma^\rho, \gamma^5] (\nabla_\rho \psi) \quad .$$

Now: the first term at righ is zero, as is always assumed, in order to preserve Ricci's theorem (see Ref. 3, for example); since γ^5 anticommutes with all the other gamma matrices, making use of the anticommutation relation and simplifying

$$\dot{\gamma}^5 \psi = -2\gamma^5 \gamma^\rho \nabla_\rho \psi \quad , \tag{1}$$

which is valid in the space of general relativity, a space endowed with curvature and spin.

B. *Time evolution of the spin tensor*

In the same way, the equation we start with is

$$\dot{\sigma}^{\mu\nu} \psi = [\gamma^\rho \nabla_\rho, \sigma^{\mu\nu}] \psi \quad .$$

The commutator is easily evaluated by using the AC relations twice, and we find

$$\dot{\sigma}^{\mu\nu} \psi = -2i (g^{\rho\mu} \gamma^\nu - g^{\rho\nu} \gamma^\mu) \nabla_\rho \psi \quad . \tag{2}$$

C. *The equations of motion in the special-relativity limit*

The preceeding results may be transformed to the form corresponding to the special-relativity limit, simply by dropping the connections (both affine and Fock-Ivanenko):

In this way, Eq. (1) gives, after some manipulation

$$\dot{\gamma}^5{}_{\text{flat}}\ \psi = -2\gamma^5\ \gamma^\rho\ \partial_\rho\ \psi = -2i\hbar^{-1}\ \gamma^5\ (-m_\circ c)\ \psi \quad , \qquad (3)$$

and for an arbitrary (free) spinor, it can be put in operator form:

$$\dot{\gamma}^5{}_{\text{flat}} = i\lambda_\circ^{-1}\ \gamma^5{}_{\text{flat}} \quad ,$$

which is the Zitterbewegung equation for γ . [See Eq. (27-c) in Ref. 2].

D. Equation for the spin tensor

The same procedure gives for $\sigma^{\mu\nu}$ the expression

$$\dot{\sigma}^{\mu\nu}\ \psi = 2i\ (\gamma^\mu\ \psi_{,\nu} - \gamma^\nu\ \psi_{,\mu}) \quad . \qquad (4)$$

Observe that in special relativity the linear momentum for a free particle is conserved; this last equation is then equivalent to the time derivative of the angular momentum, as defined in Schrödinger's representation.

Repeating the procedure gives $\ddot{\sigma}^{\mu\nu}$ and $\dddot{\sigma}^{\mu\nu}$. The relation between them is, as may be proved after a simple but lenghthy calculation,

$$\ddot{\sigma}^{\mu\nu} + i\lambda_\circ\ \dddot{\sigma}^{\mu\nu} = 0 \quad , \qquad (5)$$

which is the equation of motion (again Zitterbewegung) for the spin tensor, or rather, for its derivatives. In Ref. 2, an expression is obtained for tensors of higher order, like $\sigma_{\lambda\mu\nu}$, but the calculation is much too lenghthy, and will not be shown here. See Ref. 2, Eq. (27-a).

To end this point, note that the resulting wavelength is half that of the Compton wavelength, so the frequency is double. Szamosi has shown (Ref. 2) that all coordinates move by performing this special oscillation, and this shows that the movement of a particle should be confined to a 4-dimensional tube, of diameter equal to the Compton wavelength of the particle. The Zitterbewegung is thus an important characteristic of the motion, al least in flat 4-space. The big problem arises when one realizes that this frequency, around 10^{21} sec^{-1}, is not an observable, in the sense of quantum mechanics. In other words, expressions are obtained formally as limit form of others, more general and valid in General Relativity, but it seems as if their meaning is not yet completely clear.

III. EFFECT OF A CHANGE IN METRIC ON THE FREQUENCY OF MOTION

In Ref. 3, Eq. (31), tensors are given that allow one to go from flat 4-space matrices γ^u to others contained in a locally tangent universe, the change in metric generating the change in all those properties not depending on AC relations, since these are intrinsic, i.e., representation-independent. Let's study the Robertson-Walker case (8). Search for the time evolution of γ^5 gives the expression ($\tilde{\gamma}$ is the matrix in RW geometry)

$$\dot{\tilde{\gamma}} = i(1 - \gamma^{(4)})^{-1} \lambda_o^{-1} \gamma^5$$

and this result may be interpreted in two ways:
First: re-define the gamma matrix in Robertson-Walker geometry; then

$$\tilde{\gamma}^5 = (1 - \gamma^{(4)})^{-1} \gamma^5 \quad ,$$

and then

$$\dot{\tilde{\gamma}}^5 = i\lambda_o^{-1} \tilde{\gamma}^5 \quad ,$$

the same frequency as before.
Second: take the factor $(1 - \gamma^{(4)})$ as a frequency-splitting factor; then, after some algebra, we find a doubly degenerate spectrum, the new frequencies being

$$\nu_1 = \nu_2 = \left(\frac{1+i}{2}\lambda_o\right)^{-1} \quad ;$$

$$\nu_3 = \nu_4 = \left(\frac{1-i}{2}\lambda_o\right)^{-1} \quad .$$

The frequencies are now complex, but the order of magnitude is the same, so no new problem about observability should arise.
Finally: the study of the Robertson-Walker induced change in the frequency, if any, was suggested by Dr. G. Szamosi. This author has tried to generalize his results to non-diagonal metrics, but the amount of calculation involved apparently does not seem to be justified by the results.

The same type of calculation gives a similar result for the spin tensor: either the tensor is redefined, and the frequency is the same as before, or there is a splitting with complex-valued frequencies.

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