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THE ANGULAR VELOCITY OF A RIGID BODY

Domingo Prato and Carlos Budde
Instituto de Matemática, Astronomía y Física
Universidad Nacional de Córdoba
Laprida 854, 5000 Córdoba, Argentina

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ABSTRACT

We present a direct derivation of the expression of the angular velocity vector of a rigid body without using a geometrical construction, which, at least for the authors, is not self-evident. We use only the orthogonal transformation properties and the geometrical interpretation is apparent from the present formalism.

RESUMEN

Presentamos una derivación directa de la expresión para el vector de velocidad angular de un cuerpo rígido sin utilizar una construcción geométrica, la cual, al menos para los autores, no es evidente por sí misma. Usamos únicamente las propiedades de la transformación ortogonal, y la interpretación geométrica es evidente a partir del presente formalismo.

In the following we shall consider only a pure rotation of the rigid body. Let $\{\vec{e}_i\}$ and $\{\vec{a}_i\}$ be two orthonormal bases; the first set defines an inertial frame and the other one a comoving frame with the body. The position of any point in the rigid body is given by

$$x_i \vec{a}_i = y_i \vec{e}_i \quad , \tag{1}$$

where x_i and y_i are the coordinates in the comoving and inertial frames respectively.

The coordinates x_i and y_i are related by

$$x_i = A_{ij}\, y_j \quad , \tag{2}$$

where the orthogonal matrix A is given by

$$A = \begin{pmatrix} \cos\psi\cos\phi-\cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi+\cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi-\cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi+\cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} . \tag{3}$$

Here ϕ, θ and ψ are the usual Euler's angles, and $(A^{-1})_{ij} = A_{ji}$ (see, for example, Ref. 1).

Taking the derivative with respect to time of Eq. (1) and remembering that x_i are the coordinates of a fixed point in the rigid body we obtain

$$x_i \frac{d\vec{a}_i}{dt} = \vec{e}_i \frac{dy_i}{dt} \quad . \tag{4}$$

We can express $\frac{d\vec{a}_i}{dt}$ as a linear combination of the base vectors $\vec{a}_i$ in the form

$$\frac{d\vec{a}_i}{dt} = C_{ij}\, \vec{a}_j \quad . \tag{5}$$

Since $\{\vec{a}_i\}$ is an orthonormal base the matrix C is antisymmetric.

The angular velocity vector $\vec{w}$ is defined in the comoving frame by

$$w_i = \tfrac{1}{2}\varepsilon_{ijk}\, C_{jk} \quad , \tag{6}$$

where ε_{ijk} is the Levy-Civita symbol. The matrix elements C_{jk} can be obtained from the w_i through

$$C_{jk} = \varepsilon_{ijk} w_i \quad . \tag{7}$$

Replacing Eq. (5) into Eq. (4) and then Eq. (7) in the resulting expression we get

$$x_i \varepsilon_{kij} w_k \vec{a}_j = \vec{e}_s x_i \frac{dA_{is}}{dt} \quad , \tag{8}$$

where we have expressed the y_s in terms of x_i using the inverse of the matrix A, and, since the x_i are arbitrary,

$$\varepsilon_{kij} w_k \vec{a}_j = \vec{e}_s \frac{dA_{is}}{dt}$$

$$= \vec{a}_j A_{js} \frac{dA_{is}}{dt} \quad , \tag{9}$$

therefore,

$$\varepsilon_{kij} w_k = A_{js} \frac{dA_{is}}{dt} \quad , \tag{10}$$

and finally

$$w_k = \tfrac{1}{2} \varepsilon_{kij} A_{js} \frac{dA_{is}}{dt} \quad . \tag{11}$$

The components of the angular velocity vector in the comoving frame are given explicitly by

$$\begin{aligned} w_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \ , \\ w_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \ , \\ w_3 &= \dot{\phi}\cos\theta + \dot{\psi} \ , \end{aligned} \tag{12}$$

which are , of course , the same as those given by Landau & Lifshitz[2].

A geometrical interpretation of Eq. (12) is shown in Fig. 1.

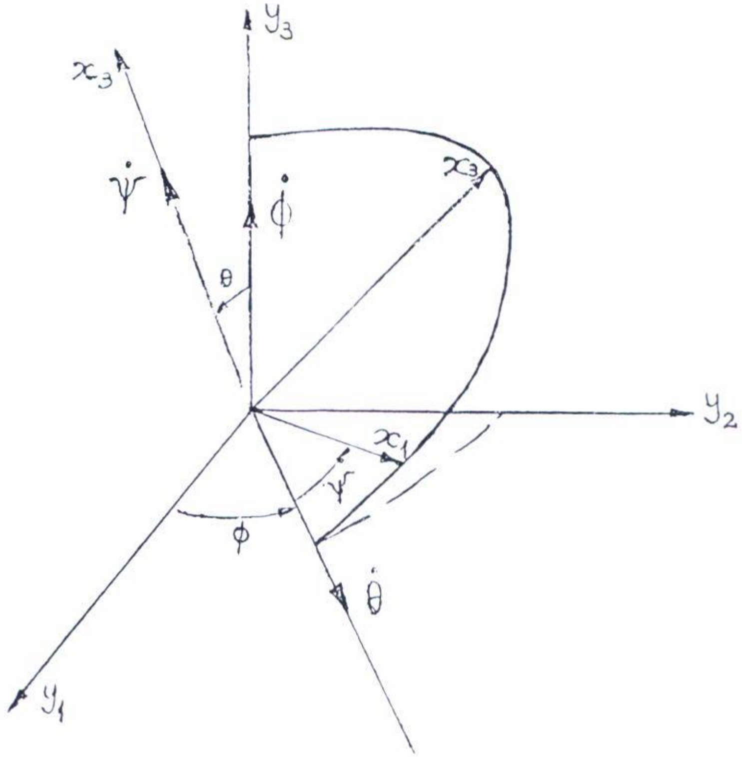

Fig. 1. Euler's angles.

A direct application of this method in the case of Lorentz transformation leads us to the obtention of the Thomas precession. Following the same line of reasoning we consider an inertial tetrad $\{e_\alpha\}$, $\alpha = 1, 2, 3, 4$, and $\{\underset{\sim}{a}_\alpha\}$ a comoving tetrad with a body. We make the usual convention: greek indices run from 1 to 4 and latin indices from 1 to 3.

A 4-vector $\underset{\sim}{X}$ can be expressed in any of the two tetrads (reference frames):

$$\underset{\sim}{X} = \underset{\sim}{e}_\alpha x_\alpha = \underset{\sim}{a}_\alpha y_\alpha \quad , \tag{13}$$

defining in this way the coordinates of an event x_α and y_α in the inertial and comoving tetrads respectively.

Let $\vec{v}(t)$ be the 3-dimensional velocity of the moving body as seen by an observer in the inertial reference frame. The instantaneous rest tetrad $\{\underset{\sim}{a}_\alpha\}$ is related to the $\{\underset{\sim}{e}_\alpha\}$ by a pure Lorentz transformation which can be parametrized by the three components v_i of $\vec{v}(t)$,

$$\underset{\sim}{e}_\alpha = \underset{\sim}{a}_\beta \, L_{\beta\alpha}(\vec{v}) \quad , \tag{14}$$

with

$$L_{ij}(\vec{v}) = \delta_{ij} + (\gamma-1)\,\frac{v_i v_j}{v^2} \; ; \; L_{i4}(\vec{v}) = L_{4i}(\vec{v}) = \frac{\gamma v_i}{c}$$

$$L_{44}(\vec{v}) = \gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}} \; . \tag{15}$$

From Eq. (14) we obtain

$$\underset{\sim}{a}_\beta = \underset{\sim}{e}_\alpha L_{\alpha\beta}^{-1}(\vec{v}) = \underset{\sim}{e}_\alpha L_{\alpha\beta}(-\vec{v}). \tag{16}$$

Taking the derivative with respect to the proper time τ of the last equation we have

$$\frac{d}{d\tau}\underset{\sim}{a}_\beta = \underset{\sim}{e}_\alpha \frac{d}{d\tau} L_{\alpha\beta}(-\vec{v}) = \underset{\sim}{a}_\mu L_{\mu\alpha}(\vec{v}) \,\frac{d}{d\tau} L_{\alpha\beta}(-\vec{v}) \; . \tag{17}$$

We are interested in the variation with respect to τ of the spatial components of the tetrad $\{\underset{\sim}{a}_\alpha\}$:

$$\frac{d}{d\tau}\,\underset{\sim}{a}_i = \underset{\sim}{a}_\mu \, L_{\mu\alpha}(\vec{v}) \,\frac{d}{d\tau}\, L_{\alpha i}(-\vec{v}) \quad . \tag{18}$$

Taking into account that $\underset{\sim}{a}_\mu \cdot \underset{\sim}{a}_\nu = g_{\mu\nu} = \text{diag}\,(1,1,1,-1)$, the spatial components in Eq. (18) are

$$\left(\frac{d}{d\tau}\underset{\sim}{a}_i\right) \cdot \underset{\sim}{a}_j = L_{j\alpha}(\vec{v}) \,\frac{d}{d\tau} L_{\alpha i}(-\vec{v}) \quad ,$$

therefore

$$\left(\frac{d}{d\tau}\underset{\sim}{a}_i\right) \cdot \underset{\sim}{a}_j = L_{j\alpha}(\vec{v}) \,\frac{d}{d\tau} L_{\alpha i}(-\vec{v}) = (\gamma-1)\,\frac{1}{v^2}\,(v_i A_j - v_j A_i), \tag{19}$$

where $A_i \equiv \frac{d}{d\tau} v_i(t)$.

Writing the spatial components of $\frac{d}{d\tau}\,\mathfrak{a}_i$ in the usual way $\overrightarrow{\frac{d}{d\tau}\,\mathfrak{a}_i}$, Eq. (19) can be expressed in the following form:

$$\left(\overrightarrow{\frac{d}{d\tau}\,\mathfrak{a}_i}\right)\cdot\vec{a}_j = (\gamma-1)\,\frac{1}{v^2}\,(v_iA_j - v_jA_i) \quad , \tag{20}$$

that suggests the definition of $\vec{w}$:

$$\left(\overrightarrow{\frac{d}{d\tau}\,\mathfrak{a}_i}\right) \equiv \vec{w} \times \vec{a}_i \quad . \tag{21}$$

Replacing Eq. (21) in (20) we obtain

$$(\vec{w} \times \vec{a}_i)\cdot\vec{a}_j = \varepsilon_{jki}\, w_R \quad , \tag{22}$$

and finally

$$\vec{w} = (\gamma-1)\,\frac{1}{v^2}\,(\vec{v} \times \vec{A}) \quad . \tag{23}$$

The vector $\vec{w}$ in Eq. (23) is the angular velocity of the well known Thomas precession (see, for example, Ref. 3).

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