GRAVITATIONAL FREQUENCY SHIFT EFFECT IN THE SOLAR SYSTEM

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ABSTRACT

An extension of the Parameterized Post-Newtonian (PPN) formalism to third order in the expansion parameter m/r (where $m = GM/c^2$ denotes the mass of the source of the field and r the distance to its center) is used to derive analytical expressions accurate to the same order for the prediction of the experimental measurements of the frequency shift effect on electromagnetic signals traveling within the solar system. An experimental situation is considered for which it is seen that the consequences of including higher order terms are undetectable by present day observations or experiments. Some deliberation on issues in the historic context within which the development of the relevant ideas took place is considered necessary to round this work out and is presented in an introductory section.

RESUMEN

Una extensión del formalismo post-newtoniano parametrizado a tercer orden en el parámetro m/r (donde $m = GM/c^2$ denota la masa de la fuente del campo y r la distancia a su centro) se utiliza para derivar expresiones analíticas exactas al mismo orden para la predicción de las mediciones del efecto de corrimiento en frecuencia experimentado por señales electromagnéticas viajando dentro del sistema solar. Se considera una situación experimental para la cual se encuentra que las variaciones debidas a la inclusión de términos de orden mayor no pueden ser detectadas con las observaciones o los experimentos actuales. Para redondear este trabajo se considera necesaria cierta deliberación sobre algunos hechos dentro del contexto histórico en el que se desarrollaron las ideas relevantes y ello se presenta en la sección introductoria.

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1. INTRODUCTION

In 1907 Einstein published a memoir introducing what he later called the principle of equivalence and deriving for the first time the gravitational frequency shift effect(1). This derivation was wrong and it was not until 1935 that, after several unsuccessful attempts, a satisfactory derivation was given by Synge(2). For a detailed examination of the many falacious formal derivations of the effect and of the reactions of the theorists to the observational tests covering the period from 1907 through the early 1920's, when the opinions of both the theorists and the experimentalists tended to stabilize, the reader is referred to the study by Earman and Glymour(3) where an analysis is given on the attempts to base the spectral shift directly on the principle of equivalence or on some combination of the quantum principle and the conservation of energy.

Accurate measurements of the gravitational frequency shift represent a test for theories that partially violate the equivalence principle. That gravitation does not affect the rate of clocks in the sense that in a freely falling frame of reference the period of the clocks does not depend on the acceleration but is determined solely by the values of the atomic constants is a consequence of the strong equivalence principle: "The laws of nature are universal in any local Lorentzian frame of reference". Lightman and Lee(4) and also Will(5) have shown that it is possible to construct theories which violate the strong equivalence principle, but not the weak equivalence principle: "Identical laws of free fall of test bodies"; i.e., in these theories frequency standards based on the hyperfine structure of atomic levels can change their frequency in a gravitational field.

It is also important to note that the effect has not always been interpreted theoretically in the same way. Some studies treat the gravitational frequency shift as a consequence of general relativity (Einstein himself(6) proposed this experiment as one of the three first possible tests of general relativity); while in other more recent work, the effect is considered as one of the experimental bases of general relativity. Textbooks like those of Landau and Lifschitz(7) and Weinberg(8) consider that the gravitational red shift is contained in general relativity as one of the forms of expression of the weak equivalence principle. In the monograph of Misner, Thorne and Wheeler[9], the red shift serves as experimental evidence in favour of the weak equivalence principle in a very important formulation for general relativity: "The paths of test bodies are geodesics of spacetime".

In what follows the application of a frequency shift effect formula[11] is exemplified considering and idealized experimental situation within the solar system (section 2), and the results are graphically presented (section 3). The use of experimental data on the frequency shift to investigate the structure of the sun and to set further restrictions on still viable theories of gravitation, constitutes part of present research programs at ESA, NASA and JPL[10].

2. EXPERIMENTAL CONFIGURATION

Considering the solar field as possesing spherical symmetry, the planets and satellites as test particles moving in its presence, and under the usual assumptions for the general framework within which metric gravitational theories are tested (space-time being a Riemannian manifold, atomic clocks keeping proper time S, the existence of a metric field g_{ij} defining an interval dS and the equations of motion for test particles and photons being derivable from a variational principle) the difference between proper time S and coordinate time t for an observer situated at a distance r from the center of the field's source and moving in a circular planar orbit, is only a factor which depends on the distance r, i.e., in spherical polar coordinates[11] ($x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$):

$$dS = \pm \left[g_{oo}(r) + g_{33}(r)\omega^2\right]^{1/2} \quad dt = \pm h(r)dt \quad , \tag{2.1}$$

where ω, the angular speed of the observer at r, is given by

$$\omega = \frac{d\phi}{dt} = \pm \left[- \frac{d\, g_{oo}(r)}{dr} \bigg/ \frac{d\, g_{33}(r)}{dr}\right]^{1/2}_{r=const} \quad . \tag{2.2}$$

The previous equations are exact and valid for any metric theory of gravity. Assuming now that the field equations of whatever theory one considers admit asymptotically flat vacuum solutions, it is possible to introduce a variant of the parameterized Post-Post-Newtonian (PP^2N) formalism, consisting of a series expansion for the metric components accurate to third order in the parameter m/r, where $m = GM/c^2$ represents the mass of the field's source and r the distance to its center[12].

These expressions are given by

$$g_{oo}(r) = 1 - 2\alpha\frac{m}{r} + 2\beta\frac{m^2}{r^2} - 2\eta\frac{m^3}{r^3} \quad , \tag{2.3a}$$

$$g_{11}(r) = -\left(1 + 2\gamma\frac{m}{r} + 4\delta\frac{m^2}{r^2} + 8\lambda\frac{m^3}{r^3} \right) \quad , \tag{2.3b}$$

$$g_{33}(r) = - r^2 \quad . \tag{2.3c}$$

In fact, $g_{33}(r)$ is not being expanded, rather it is defined in such a way as to identify r, θ and ϕ with spherical polar coordinates and to interpret r as the distance from the source. The results will then be coordinate independent[13].

The coefficients $\Sigma = \{\alpha,\beta,\gamma,\delta,\eta,\lambda\}$ in these expressions, introduced in the spirit of Eddigton's analysis[14], serve as markers throughout the remaining calculations and indicate how the various parts of the metric contribute to the final results. Thus they supply a partial answer to the questions: what part of the total theoretical framework could a particular observation or experiment test? and, to what extent are the gravitational effects non-linear, if any? Karlhede[15] has discussed the second problem within the Post-Newtonian formalism concluding that, to such approximation, it is not possible to measure any non-linear effects since the Post-Newtonian metric can be made linear by a suitable choice of coordinates.

The frequency shift effect on a wave signal defined by

$$Z \equiv \frac{\nu_e - \nu_r}{\nu_e} \quad , \tag{2.4}$$

may be expressed as[11]

$$Z = 1 - \frac{dS_e}{dS_r} \quad , \tag{2.5}$$

where $dS_{e(r)}$ denotes the proper time during which the signal is emitted (received).

This expression means that the measurement of the effect consists of comparing how a clock on the receiver runs with respect to an identical clock based on the emitter, i.e., a comparison of the elapsed proper times during emission and reception when the world line of the emitter is mapped onto the world line of the receiver by means of null geodesics.

Using Eq. (2.1), Eq. (2.5) may now be rewritten as

$$Z = 1 - \frac{h_e}{h_r}\frac{dt_e}{dt_r} \quad . \tag{2.6}$$

The quotient dt_e/dt_r may be obtained using the relation

$$t_r = t_e \pm f(m,\ r_o,\ r_e,\ r_r,\Sigma) \quad , \tag{2.7}$$

where $f(m,\ r_o,\ r_e,\ r_r,\Sigma)$, representing the coordinate time it takes for the wave signal to go from the emitter to the receiver when both follow coplanar orbits, is given by[11]

$$f(m,r_o,\ r_e,\ r_r,\Sigma) = \Bigg\{ \left(1 + \frac{Cm^3}{r\, r_o^2} \right) \sqrt{r - r_o^2} + (\alpha+\gamma)m \,\mathrm{Ln}\, \frac{r + \sqrt{r^2 - r_o^2}}{r_o} +$$

$$+ m\left[\alpha - (3\alpha + 2\gamma)\frac{\alpha m}{2r_o} + \left(\frac{F}{r_o} + \frac{G}{r}\right)\frac{m^2}{2r_o}\right]\left(\frac{r - r_o}{r + r_o}\right)^{1/2} +$$

$$+ \frac{Em^2}{2r_o}\left(1 + \frac{\alpha m}{r_o}\right)\left[\frac{\pi}{2} - \arcsin\left(\frac{r_o}{r}\right)\right] + \frac{\alpha^2 m^2 r}{2r_o}\left[1 - \frac{(3\alpha+\gamma)m}{r_o}\right]\frac{(r - r_o)^{1/2}}{(r + r_o)^{3/2}}$$

$$+ \frac{\alpha^3 m^2 r^2}{2r_o^2}\frac{(r - r_o)^{1/2}}{(r + r_o)^{5/2}}\Bigg\}_{r_e}^{r_r} \quad , \tag{2.8}$$

where

$$C = 4\lambda + 2\alpha\delta - 2\beta\gamma + \frac{3\gamma\alpha^2}{2} + \eta - 4\alpha\beta + \frac{5\alpha^3}{2} - 2\delta\gamma - \frac{\alpha\gamma^2}{2} + \frac{\gamma^3}{2} \quad , \tag{2.9a}$$

$$E = 8\alpha^2 + 4\delta + 4\alpha\gamma - 4\beta - \gamma^2 \quad , \tag{2.9b}$$

$$F = 4\eta - 20\alpha\beta + 30\alpha^3 + 11\alpha^2\gamma + 8\alpha\delta - 2\,\alpha\gamma^2 \quad , \tag{2.9c}$$

$$G = 4\alpha\delta + 5\alpha^2\gamma - 8\alpha\beta + 11\alpha^3 - \alpha\gamma^2 + 2\eta \quad . \tag{2.9d}$$

The first term in Eq. (2.8) represents the Newtonian contribution, the other terms are gravitational corrections of first, second and third order in m/r.

The distance of closest approach of the wave signal to the field's source, denoted by r_o in Eq. (2.7,8), plays an important role in determining the sign of Eq. (2.7). According to Fig. 1, the existence of r_o depends on whether the angular separation between emitter and receiver, when the signal is emitted, is greater than a certain critical value or not. This critical value corresponds to the situation in which the wave signal arrives tangential to the orbit of the receiver (solid line trajectory in Fig. 1). For values of the angular distance less than the critical value, the wave signal is received before it passes near to the field's source (dotted line trajectory in same figure). Thus the travel coordinate time in going from r_e to r_r is given by

$$t_r = t_e + \begin{cases} f(m,\, r_o,\, r_e,\, r_o,\Sigma) - f(m,\, r_o,\, r_o,\, r_r,\Sigma) & (2.10a) \\ f(m, r_o,\, r_e,\, r_r,\Sigma) & (2.10b) \\ f(m, r_o,\, r_e,\, r_o,\Sigma) + f(m,\, r_o,\, r_o,\, r_r,\Sigma)\ , & (2.10c) \end{cases}$$

according to whether the angular separation between emitter and receiver is less than, equal to or greater than the critical value, respectively. Inserting Eq. (2.7) in Eq. (2.6) one obtains

$$Z = 1 - \frac{h_e}{h_r}\left\{1 - \frac{df}{dr_o}\bigg/\frac{dt_r}{dt_o}\right\} \quad , \qquad (2.11)$$

since, as mentioned above, the function f is most sensitive to changes in r_o due to the motion of the emitter and receiver. In order for the wave signal to reach the receiver it is necessary that the angular positions of both the receiver and the signal at the coordinate time of reception be equal. The angular position of the receiver is obtained integrating Eq. (2.2) to give

$$\phi_r = \phi_{ir} \pm \omega_r\,(t_r - t_i) \quad , \qquad (2.12)$$

where the subscript i denotes the initial value of the variables showing it.

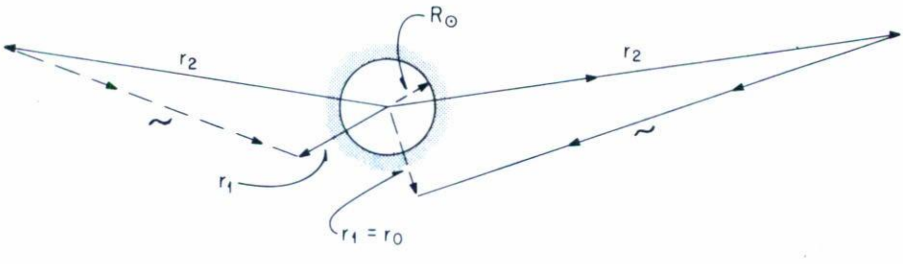

Fig. 1. Schematic diagram showing the role of the distance of closest approach of the wave signal (∿) to the field's source.

The angular displacement of the wave signal is, similarly to Eqs. (2.10), given by

$$\phi = \phi_i + \begin{cases} q(m,r_o,\ r_e,\ r_o,\Sigma) - q(m,\ r_o,\ r_o,\ r_r,\Sigma) & (2.13a) \\ q(m,r_o,\ r_e,\ r_r,\Sigma) & (2.13b) \\ q(m,r_o,\ r_e,\ r_o,\Sigma) + q(m,r_o,\ r_o,\ r_r,\Sigma)\ , & (2.13c) \end{cases}$$

according to whether $\Delta\phi = \phi_r - \phi_e$ is less than, equal to or greater than the critical value, respectively.

The function $q(m,r_o,\ r_e,\ r_r,\Sigma)$ is, for the case of coplanar orbits, given by[11]

$$\begin{aligned} q(m,r_o,\ r_e,\ r_r,\Sigma) = \Bigg\{ & \left[1 + \left(\frac{1}{2} - \frac{\alpha m}{r_o}\right)\frac{Em^2}{2r_o^2}\right]\left[\frac{\pi}{2} - \arcsin\left(\frac{r_o}{r}\right)\right] + \\ & + \frac{m}{r_o r}\left[\gamma + \frac{(4\delta - \gamma^2)m}{4r} + \left(J\left(1 + \frac{1}{2r^2}\right) - \frac{(2\alpha + \gamma)\beta}{r_o^2}\right)m^2\right]\sqrt{r^2 - r_o^2} + \\ & + \frac{m}{r_o}\left[\alpha - \frac{(5\alpha + 4\gamma)\alpha m}{4r_o} + \left(\frac{2D - 3\alpha\beta}{r_o} + \frac{D}{r}\right)\frac{m^2}{r_o}\right]\left(\frac{r - r_o}{r + r_o}\right)^{1/2} + \\ & + \frac{\alpha^2 m^2}{4r_o^2}\left(\frac{r - r_o}{r + r_o}\right)^{3/2} + \frac{\alpha^2 m^3}{2r_o^3}\left[\frac{(19\alpha + 3\gamma)r^2}{3} + (14\alpha + 3\gamma)r\,r_o + \right. \\ & \left. + (9\alpha + 2\gamma)r_o^2\right]\frac{(r - r_o)^{1/2}}{(r + r_o)^{5/2}}\Bigg\}_{r_e}^{r_r}\ , \end{aligned} \qquad (2.14)$$

where E is given by Eq. (2.9b) and

$$D = 4\alpha^3 + 2\alpha^2\gamma - 2\alpha\beta + 2\alpha\delta - \frac{\alpha\gamma^2}{2} + \eta \quad , \tag{2.15a}$$

$$J = \frac{1}{3}(\gamma^3 - 4\gamma\delta + 8\lambda) \quad . \tag{2.15b}$$

Equating Eqs. (2.12) and (2.13) one obtains

$$\phi_{ie} - \phi_{ir} + q(m, r_o, r_e, r_r, \Sigma) - \omega_r(t_r - t_i) = 0 \quad ,$$

which, differentiating with respect to r_o, gives

$$\frac{dt_r}{dr_o} = \frac{1}{\omega_r}\frac{dg}{dr_o} \quad .$$

Inserting this expression in Eq. (2.11) one finally obtains for the one-way frequency shift that

$$Z = 1 - \frac{h_e}{h_r}\left(1 - \omega_r \frac{df}{dr_o} \bigg/ \frac{dg}{dr_o}\right) \quad . \tag{2.16}$$

Analogously, for the round trip frequency shift effect given by

$$Z = 1 - \frac{dS_e}{dS_b}\frac{dS_b}{dS_r} \quad , \tag{2.17}$$

where dS_b is the proper time during which the signal is reflectled back to the emitter, one obtains

$$Z = \omega_b\left(\frac{df_0}{dr_o} \bigg/ \frac{dq_0}{dr_o}\right) + \omega_r\left(\frac{df_i}{dr_o} \bigg/ \frac{dq_i}{dr_o}\right) - \omega_b\,\omega_r\left(\frac{df_0}{dr_o} \bigg/ \frac{dq_0}{dr_o}\right)\left(\frac{df_i}{dr_o} \bigg/ \frac{dq_i}{dr_o}\right) \quad , \tag{2.18}$$

where the subscripts 0 and i indicate the first (outgoing signal) and second (incoming signal) parts of the trip, correspondingly; and ω_b and ω_r ($=\omega_e$) represent, in that order, the angular velocities of the reflector and receiver (= emitter).

3. RESULTS AND DISCUSSION

Given the large number of still-viable theories of gravitation, only the predictions corresponding to general relativity theory ($\Sigma = \{1,0,1,1,0,1\}$) and those corresponding to a set $\Sigma = \{1,0,1,1,1,2\}$ are calculated. The range covered, taking these two sets of values as extremes, includes most of the still viable metric theories. Both sets differ only in the values for η and λ so that an assessment on the influence of the m^3/r^3 terms on the results is possible. The results presented in graphs showed in Figs. 2-6 correspond to the special geometrical configuration in which the earth and satellite follow circular coplanar orbits around the sun at 1.495985×10^{13} cm (= 1 AU) and at 2.78392×10^{11} cm (= 4 $R_\odot$), respectively.

These graphs show the different contributions from the different powers of m/r to the Newtonian calculation of the frequency shift during a whole orbit of the satellite around the sun. Fig. 2 contains the total calculations (including all terms), Fig. 3 contains the contribution due to terms of first order in m/r, the contribution of second order terms in m/r is drawn in Fig. 4 and finally, the contribution of m^3/r^3 terms is in Fig. 5 for general relativity theory and in Fig. 6 for the case in which $\eta = 1$ and $\lambda = 2$ (Figs 2, 3 and 4 are identical for the two cases and also in both of them the Newtonian calculations differ from the total calculations in a way which makes the two coresponding graphs to look identical and therefore a separate graph with these calculations is not presented). A blackout period due to the eclipsing of the satellite by the field's source is marked in each of the mentioned figures.

The results presented in Fig. 2 agree, to the corresponding significant figures, with the expected values from the Newtonian calculations. The maximum value for the frequency shift is given, to first approximation , by the maximum value of the dominant Doppler shift. The latter occurs when the relative velocity between earth and satellite is greater, i.e., when the wave signal arrives tangential to the motion of the satellite. In this case the Doppler shift is given by

$$\frac{i + v \cos \theta}{(i - v^2)^{1/2}} \simeq \left(\frac{1 + v}{1 - v}\right)^{1/2} \simeq 1 + v = 1 + (m/r)^{1/2} .$$

A second calculation refers to the transverse Doppler shift, which occurs when the earth and satellite move almost parallel to each other and the signal arrives perpendicular to the motion of the satellite. For this case:

$$\frac{1 + v \cos \theta}{(1 - v^2)^{1/2}} = \frac{1 + \sin (90^\circ-\theta)}{(1 - v^2)^{1/2}} \simeq 1 + v\ (90^\circ-\theta) = 1 + \left(\frac{m}{r}\right)^{1/2} (90^\circ-\theta) \ .$$

From Figs. 2-5 it is apparent that the first relativistic correction essentially is m/r orders of magnitude less than the Newtonian value, that the second correction is in turn m/r orders of magnitude less than the first one and that the same relation exhibits the third correction with respect to the second one.

The predicted values reported here would then be m/r orders of magnitude more accurate than the values calculated with the Post-Newtonian formalism. In particular, if one expects to detect variations due to terms of order m^3/r^3, the required stability for the experimental clocks (the only devices involved in a frequency shift experiment) should be of one part in 10^{18}. This improvement is far greater than the obtainable accuracy with present experiments and observations(10).

Advances in technology as well as the gathering of more data are expected to provide us in the near future with more accurate experiments and more precise observational results for the measurements involved in this test of metric theories of gravitation.

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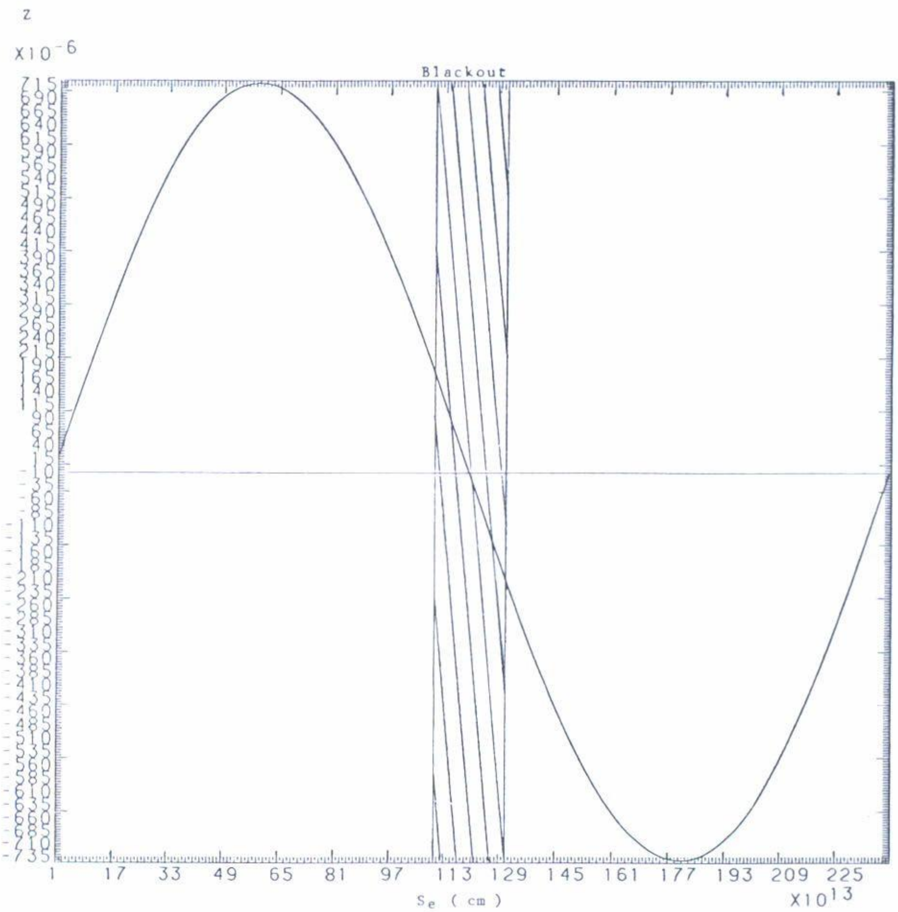

Fig. 2. Frequency shift as a function of the proper emission time.

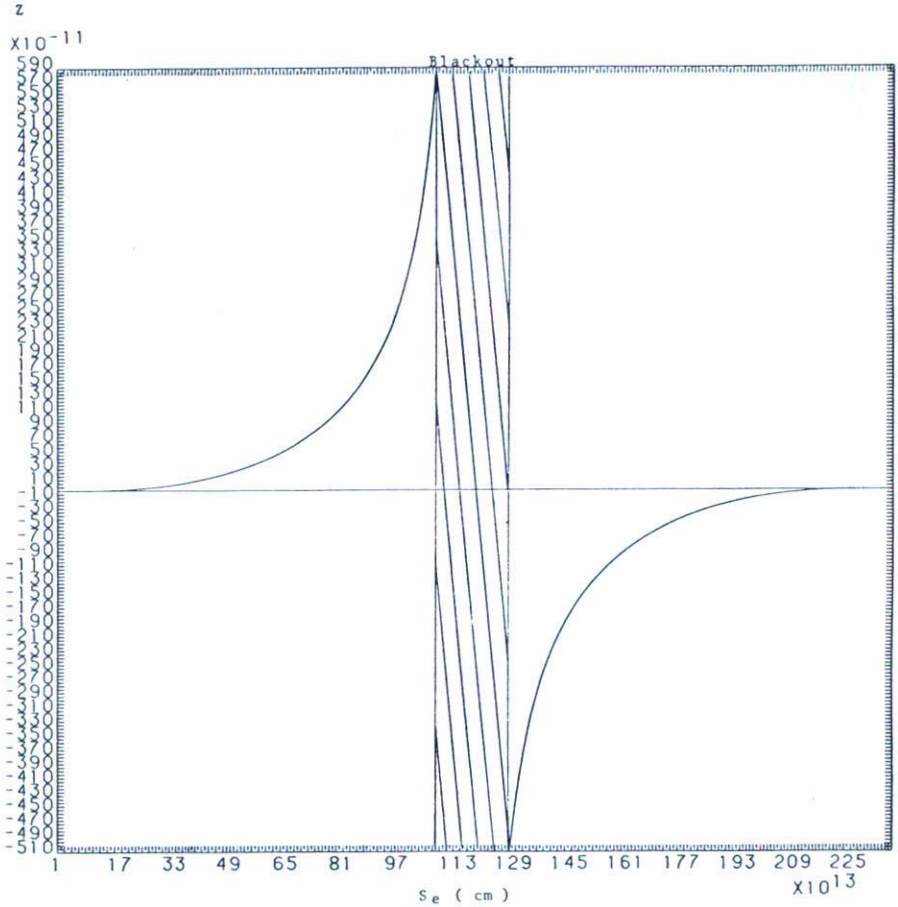

Fig. 3. First correction to the frequency shift.

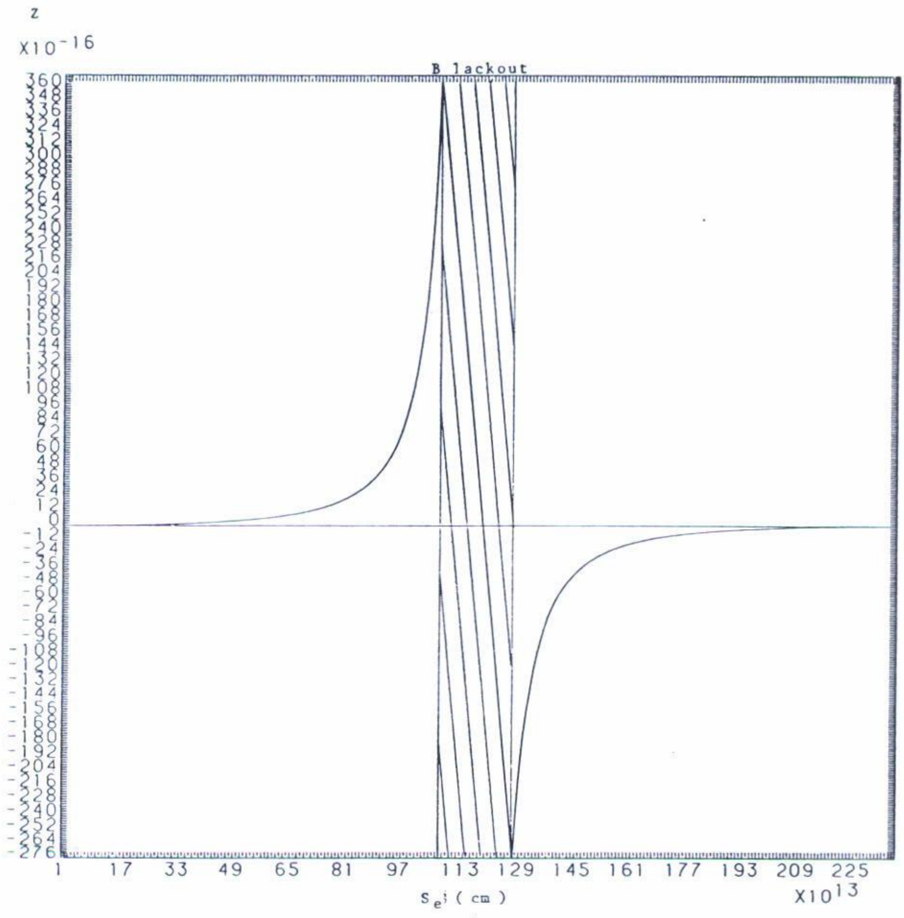

Fig. 4. Second correction to the frequency shift.

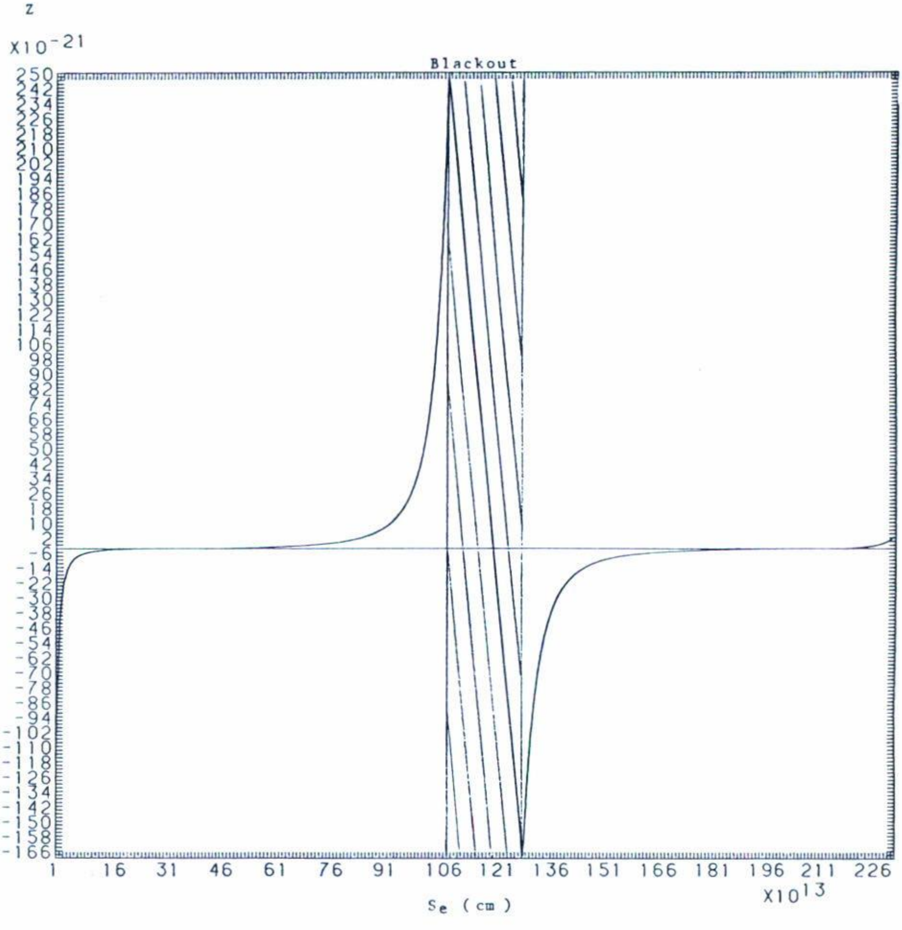

Fig. 5. Third correction to the frequency shift.

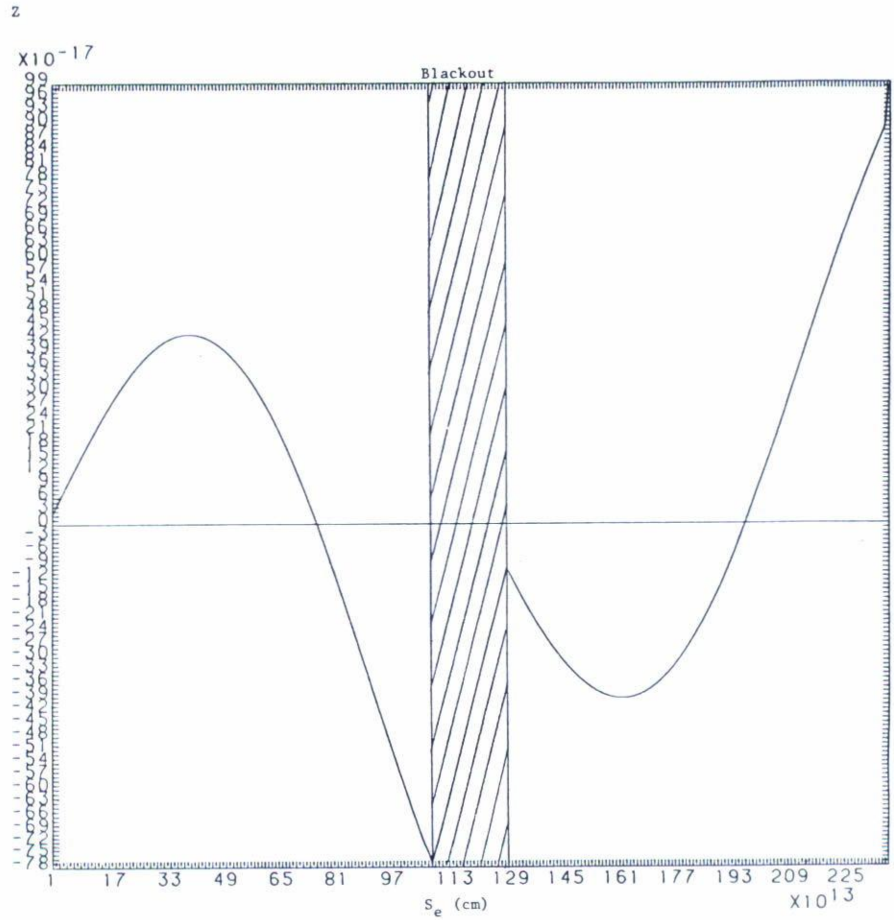

Fig. 6. Analogue of Fig. 5 with $\eta = 1$, $\lambda = 2$