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A NONLOCAL EQUATION FOR THE WAVE FUNCTION OF THE HARMONIC OSCILLATOR WHEN THE POSITION SPECTRUM IS TO BE MADE DISCRETE

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ABSTRACT

A discrete spectrum for position and a nonlocal equation for the wave function with its respective solutions are obtained, in a trivial way, by replacing the usual position and momentum operators by finite matrices, obtained by truncating the infinite matrices for these operators in the energy representation. Some extremal properties of the energy eigenfunctions and of the position eigenvalues are shown.

RESUMEN

Al reemplazar las matrices representativas usuales de los operadores posición y momento en la representación de la energía por matrices finitas, se obtienen, de manera trivial, un espectro discreto para la posición y una ecuación no-local para la función de onda con sus respectivas soluciones. Se muestran algunas propiedades extremas de las eigenfunciones de la energía y de los eigenvalores de la posición.

1. INTRODUCTION

In quantum mechanics, the idea that position may have discrete values has not stopped being present through its development, and the interest to it has not been lessened by time. In spite of that, in the framework of the theory, it has not been possible to find its place yet. If such a thing would occur, some changes in the fundamental postulates of quantum mechanics are expected, because without them, it is not possible to imagine such a discontinuity in the spectrum of the position operator. Schrödinger's equation, for instance, should be transformed in another equation, which would take into account this feature; furthermore, if that spectrum would be finite, the quantum condition $[\hat{x},\hat{p}]=i\hbar\hat{1}$ should be changed too[1], causing all changes that this could bring with it.

At this point, it is convenient to make the following considerations about this subject.

As it is known, the postulates of quantum mechanics assign elements of a Hilbert space to states of a physical system, linear operators to dynamical variables, and eigenvalues of Hermitian operators to possible results of a measurement. They do not state anything about the continuous (discontinuous) nature of the eigenvalue spectrum, and therefore, in general, we must accept both possibilities except when we deal with the position case. Here, the postulates give as a matter of fact, the continuous nature of the $\hat{x}$-spectrum in the whole range $(-\infty, +\infty)$.

It has been possible to get results agreeing with experimental data from application of those postulates to physical problems. Nevertheless, suppose for a moment that they would state the existence of a discrete spectrum for position, that those problems could be worked out, and that the measurable results would agree with experimental data too; what would occur then?

One can not cease to ask oneself how quantum mechanics (the equations of motion of which were inspired by classical mechanics) lets energy, in some cases, to take discrete values while it forbids the position to do the same as those two variables have identical continuous nature in classical mechanics? Being quantum mechanics a different theory, it is evident that it does not have to preserve the continuous-continuous relationship of that pair of variables, but neither, in my opinion, it should forbid discontinuity for one of them. What distinguishes them, since one can take discrete values and the other can not? About this, the following considerations will be useful.

In the beginning of quantum mechanics, physicists were busy explaining a phenomenon that was facing them: the spectral lines. In order to explain it, it was necessary to use the Bohr's hypothesis, which has no analogue in the context of classical mechanics and can be written as

$$\int m\dot{x}dx = J = nh \quad .$$

The right-hand side of this equation represents the necessity of reproducing the discrete values of energy in the hydrogen atom. The left-hand side needs the continuity of position and does not have to represent any discrete value of the position since there was no experiment which would request it. That was the equation in which Heisenberg[2] replaced

$$x(t) = \sum_{\alpha=-\infty}^{\infty} a_\alpha(n)e^{i\alpha\omega_n t}$$

to obtain, first,

$$\hbar = m \sum_{\alpha=-\infty}^{\infty} \alpha \frac{d}{dn} (\alpha\omega_n |a_\alpha|^2) \quad ,$$

and then, the formula which Born and Jordan[1] wrote as

$$\sum_k (p_{nk}\, q_{kn} - q_{nk}\, p_{kn}) = \frac{\hbar}{i} \quad ,$$

and that, finally, became the quantum condition:

$$[\hat{x},\hat{p}] = i\hbar\hat{1} \quad .$$

Imagine now, for a moment, that an experiment requiring discrete values of position would exist. This condition and the Schrödinger equation are expected to change. What should the commutator $[\hat{x},\hat{p}]$ be equal to? What would the form of Schrödinger's equation be? One of the requisites for development of the theory is to know the answers of these questions.

The purpose of this work is to show that if a discrete spectrum for position in the harmonic oscillator is assumed, the Schrödinger equation is transformed into an nonlocal equation (the usual differential operator does not appear) compatible with certain extremal properties. We work with the linear harmonic oscillator because it is a simple problem from which we can get a lot of information.

In section 2 we state the basic hypothesis. The key idea is to substitute the representative matrix of each operator by one of its blocks. In section 3 we show that the position eigenvalues turn out to be the Hermite polynomials' roots, moreover, we obtain the wave functions; some of its algebraic properties are shown in the next section, and in 5, the momentum eigenfunctions are obtained. In section 6 we get the form of the Schrödinger equation and in section 7 we show its extremal properties. Finally, in section 8, we make some remarks about possible extensions of the work.

2. THE HYPOTHESES

Let us begin by considering the question: in which way can we induce a discrete spectrum for the position of the oscillator? This question carries its answer and the Heisenberg representation leads us to it.

The Hamiltonian of an harmonic oscillator of mass m and angular frequency ω is

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 \quad .$$

The usual treatment of the $\hat{H}$-eigenvalue problem is as follows. Let us obtain the representative matrices of $\hat{x}$ and $\hat{p}$ in the basis where $\hat{H}$ is diagonal. It is known that, as time is disregarded,

$$\langle E_k|\hat{x}|E_j\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{k}\,\delta_{k,j+1} + \sqrt{k+1}\,\delta_{k,j-1}\right]$$

and

$$\langle E_k|\hat{p}|E_j\rangle = -i\sqrt{\frac{\hbar m\omega}{2}}\left[-\sqrt{k}\,\delta_{k,j+1} + \sqrt{k+1}\,\delta_{k,j-1}\right] \quad .$$

Here E_k is the k-th energy eigenvalue and the brackets denote the expectation value. Therefore, the eigenvalue equation for $\hat{x}$ takes the form

$$\sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix} 0 & \sqrt{1} & 0 & \\ \sqrt{1} & 0 & \sqrt{2} & \cdots \\ 0 & \sqrt{2} & 0 & \\ & \vdots & & \ddots \end{pmatrix}\begin{pmatrix} H_0 \\ H_1 \\ H_2 \\ \vdots \end{pmatrix} = x\begin{pmatrix} H_0 \\ H_1 \\ H_2 \\ \vdots \end{pmatrix} ,$$

where H_i is the hamiltonian for the i-th harmonic oscillator.

To obtain a discrete spectrum, we substitute the above equation by

$$\sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix} 0 & \sqrt{1} & 0 & & & \\ \sqrt{1} & 0 & \sqrt{2} & & & \\ 0 & \sqrt{2} & 0 & & & \\ & & & 0 & \sqrt{N-1} & 0 \\ & & & \sqrt{N-1} & 0 & \sqrt{N} \\ & & & 0 & \sqrt{N} & 0 \end{pmatrix}\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix} = x\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix} , \quad (1)$$

where the p_k are the momentum eigenfunctions.

When supposing this equation to be the correspondent one to the eigenvalue problem of $\hat{x}$, we are doing two things: first, to substitute the Hilbert space by a finite-dimensional one and second, to accept that the representative matrix of $\hat{x}$ is the one given by Eq. (1). A word about this is necessary. If the ket space is finite-dimensional, the $\hat{H}$-spectrum must be finite too, and this means that an upper bound for its eigenvalues exists. Thus,

$$
(\langle E_j|\hat{H}|E_k\rangle) = \begin{pmatrix} E_0 & 0 & 0 & & & \\ 0 & E_1 & 0 & & & \\ 0 & 0 & E_2 & & & \\ & & & \ddots & & \\ & & & & E_{N-1} & 0 \\ & & & & 0 & E_N \end{pmatrix}, \tag{2}
$$

where E_k is expected to be $(k+1/2)\hbar\omega$.

Since the key idea is to substitute an infinite matrix by one of its blocks, we have an alternative: to choose $\langle E_j|\hat{H}|E_k\rangle$ as given by Eq. (2), where $E_n = (n+1/2)\hbar\omega$ (for $n = 0,1,2,\ldots,N$) as one of the fundamental representatives, or to choose

$$
(\langle E_j|\hat{x}|E_k\rangle) = \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & \sqrt{1} & 0 & & & \\ \sqrt{1} & 0 & \sqrt{2} & & & \\ 0 & \sqrt{2} & 0 & & & \\ & & & \ddots & & \\ & & & 0 & \sqrt{N-1} & 0 \\ & & & \sqrt{N-1} & 0 & \sqrt{N} \\ & & & 0 & \sqrt{N} & 0 \end{pmatrix} \tag{3}
$$

and

$$
(\langle E_j|\hat{p}|E_k\rangle) = -i\sqrt{\frac{\hbar m\omega}{2}} \begin{pmatrix} 0 & \sqrt{1} & 0 & & & \\ -\sqrt{1} & 0 & \sqrt{2} & & & \\ 0 & -\sqrt{2} & 0 & & & \\ & & & \ddots & & \\ & & & 0 & \sqrt{N-1} & 0 \\ & & & -\sqrt{N-1} & 0 & \sqrt{N} \\ & & & 0 & -\sqrt{N} & 0 \end{pmatrix}, \tag{4}
$$

as the fundamental ones, and in terms of them to find $\langle E_j|\hat{H}|E_k\rangle$.

These two possibilities are not equivalent because of the non-linearity of $\hat{H}$ in $\hat{x}$ and $\hat{p}$. If we choose the first, the matrices $(\langle E_j|\hat{x}|E_k\rangle)$ and $(\langle E_j|\hat{p}|E_k\rangle)$ are different from those given by Eqs. (3) and (4), just for the last elements, and if we choose the second, the eigenvalues of $\hat{H}$ become

$$E_n = \begin{cases} (n + \frac{1}{2})\hbar\omega , & n = 0,1,2,\ldots,N-1 , \\ \frac{N}{2}\hbar\omega , & n = N . \end{cases}$$

In spite of the value of E_N, we shall suppose valid the representatives in Eqs. (3) and (4). The symmetry obtained in all results will be considered as a justification of this procedure.

About the quantum condition $[\hat{x},\hat{p}] = i\hbar\,\hat{1}$, it should be said that, according to Eqs. (3) and (4), it becomes

$$\langle E_j|[\hat{x},\hat{p}]|E_k\rangle = i\hbar\,\delta_{jk} - i\hbar\langle E_j|\hat{\mathcal{O}}|E_k\rangle \quad , \qquad (5)$$

where

$$\langle E_j|\hat{\mathcal{O}}|E_k\rangle = \begin{cases} N + 1 , & j = k = N , \\ 0 , & \text{otherwise} . \end{cases} \qquad (6)$$

The operator $\hat{\mathcal{O}}$, enables us to answer some questions arising in the creation-annihilation operators treatment of the $\hat{H}$-eigenvalue problem. For example, the creation operator, defined in the usual manner,

$$\hat{a}^{\dagger} = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} - i\hat{p}) \quad ,$$

is now allowed to satisfy the relation

$$\hat{a}^{\dagger}|E_n\rangle = 0 \quad ,$$

in addition to the well-known one: $\hat{a}|E_0\rangle = 0$.

3. EIGENVALUE PROBLEM OF $\hat{x}$

We have already written down the diagonal form of $\hat{H}$ and the commutator of $\hat{x}$ and $\hat{p}$. Our next step is to solve the eigenvalue problem of $\hat{x}$ (Eq. (1)).

Let us define $\xi = \sqrt{2m\,\omega\hbar^{-1}}\,x$. It is not difficult to see that

$$He_{N+1}(\xi) = 0$$

is obtained from the characteristic equation associated with Eq. (1), $\det(\hat{x} - x\hat{1}) = 0$. Here, $He_{N+1}(\xi)$ is the $(N+1)$-th Hermite polynomial belonging to the generating function $e^{-\xi^2/2}$. Therefore, the eigenvalues of $\hat{x}$ are all real, distinct, and symmetrically located about the origin inside the interval $(-2\sqrt{N+1}, 2\sqrt{N+1})$. When $N \to \infty$, these points are everywhere dense in every finite segment of the real line[3,4].

Also, the recurrence equation for the eigenket components $p_k(\xi)$,

$$\begin{aligned} \sqrt{1}\,p_1(\xi) &= \xi p_0(\xi) \quad , \\ \sqrt{1}\,p_0(\xi) + \sqrt{2}\,p_2(\xi) &= \xi p_1(\xi) \quad , \\ &\;\;\vdots \\ \sqrt{N-1}\,p_{N-2}(\xi) + \sqrt{N}\,p_N(\xi) &= \xi p_{N-1}(\xi) \quad , \\ \sqrt{N}\,p_{N-1}(\xi) &= \xi p_N(\xi) \quad , \end{aligned} \tag{7}$$

can be obtained from (1).

To solve them, let us commence by the last one. Taking into account the normalization condition, we get

$$p_N(\xi) = \sqrt{\frac{N!}{(N+1)!}} \quad ,$$

$$p_{N-1}(\xi) = \sqrt{\frac{(N-1)!}{(N+1)!}}\,\xi \quad ,$$

$$p_{N-2}(\xi) = \sqrt{\frac{(N-2)!}{(N+1)!}}\,(\xi^2 - N) \quad ,$$

$$p_{N-3}(\xi) = \sqrt{\frac{(N-3)!}{(N+1)!}}\left[\ \xi^3 - (2N-1)\xi\ \right] \quad ,$$

$$p_{N-4}(\xi) = \sqrt{\frac{(N-4)!}{(N+1)!}}\left[\ \xi^4 - 3(N-1)\xi^2 + N(N-2)\ \right] \quad , \tag{8}$$

$$p_{N-5}(\xi) = \sqrt{\frac{(N-5)!}{(N+1)!}}\left[\ \xi^5 - 2(2N-3)\xi^3 + 3(N^2-3N+1)\xi\ \right] \quad ,$$

$$\vdots$$

etc.

As it will be seen below, these polynomials satisfy

$$p_{N-\ell}(\xi_j) = \sqrt{\frac{N!}{(N+1)(N-\ell)!}}\ \frac{He_{N-\ell}(\xi_j)}{He_N(\xi_j)} \quad , \tag{9}$$

for all values of ℓ and j. ξ_j is a root of $He_{N+1}(\xi)$. This result was necessary in the view of Eq. (7), because it is what we would have obtained if, to solve Eq. (7), we had begun with the first of them.

Since $|\xi_j\rangle$ is an eigenket of a Hermitian operator in a finite-dimensional space, its components satisfy the following ortonormality relations:

$$\sum_{\ell=0}^{N} p_\ell(\xi_j)p_\ell(\xi_k) = \delta_{jk} \quad , \tag{10}$$

$$\sum_{j=1}^{N+1} p_\ell(\xi_j)p_k(\xi_j) = \delta_{\ell k} \quad . \tag{11}$$

However, it is instructive, from the algebraic point of view, to give a more "direct" proof of those relations, because more properties of the polynomials $p_\ell(\xi)$ are shown. This is what we shall do in the following section.

4. ALGEBRAIC DIGRESSION

First, let us show that Eq. (9) is satisfied by the polynomials defined in Eq. (8). To do this, let us write down

$$p_{N-\ell}(\xi) = \sqrt{\frac{(N-\ell)!}{(N+1)!}} \sum_{\mu=0}^{[\frac{\ell}{2}]} c_\mu^\ell(N)\,\xi^{\ell-2\mu} \quad , \qquad (12)$$

where $[\ \ell/2\ ]$ means the highest integer contained in $[\ 0,\ell/2\ ]$. Substituting Eq. (12) in Eq. (7) we obtain the difference equation

$$c_\mu^\ell(N) = c_\mu^{\ell-1}(N) - (N-\ell+2)c_{\mu-1}^{\ell-2}(N) \qquad (1 \leqslant \mu \leqslant [\ \ell/2\ ] \quad ,$$

for the coefficients of the polynomials (8). Here, $c_0^\ell(N) = 1$ and, if ℓ is even, $c_{[\ell/2]}^{\ell-1} = 0$. A particular solution of that equation is

$$c_\mu^\ell(N) = \frac{(-1)^\mu}{2^\mu \mu!} \sum_{k=0}^{\mu} (-1)^k \binom{\mu}{k} (N+1)_{2(\mu-k)} (N-\ell+2k)_{2k} \quad ,$$

where $(k)_\ell$ is the factorial of k of grade ℓ, i.e., $(k)_\ell = k(k-1)(k-2)\ldots(k-\ell+1)$.

Observe that

$$c_\mu^\ell\ (\ell-1) = \frac{(-1)^\mu\ \ell!}{2^\mu \mu!\,(\ell-2\mu)!} = a_\mu^\ell \quad ,$$

where a_μ^ℓ is the μ-th coefficient of He_ℓ when it is written as

$$He_\ell(\xi) = \sum_{\mu=0}^{[\frac{\ell}{2}]} a_\mu^\ell\ \xi^{\ell-2\mu} \quad .$$

Therefore, $c_\mu^{N+1}(N) = a_\mu^{N+1}$ and

$$\xi p_0(\xi) - p_1(\xi) \equiv p_{-1}(\xi) = p_{N-(N+1)}(\xi) = \frac{1}{\sqrt{(N+1)!}} He_{N+1}(\xi) \quad .$$

By using Eq. (7) and the fact that $He_{N+1}(\xi_j) = 0$, one can come to Eq. (9).

Now, by using this result, the relation

$$\sum_{\ell=0}^{N} p_\ell(\xi_j)p_\ell(\xi_k) = \frac{N!}{(N+1)He_N(\xi_j)He_N(\xi_k)} \sum_{\ell=0}^{N} \frac{He_\ell(\xi_j)He_\ell(\xi_k)}{\ell!}$$

can be obtained, and with the Christoffel-Darboux formula for Hermite polynomials, Eq. (10) can be proved.

Let us go to the other ortogonality relation, Eq. (11). In order to give a proof of it, let N be an even number. Note that the coefficients $C_\mu^\ell(N)$ can be rewritten as a matrix, to obtain

$$\begin{pmatrix} p_N(\xi) \\ p_{N-1}(\xi) \\ p_{N-2}(\xi) \\ \vdots \\ p_0(\xi) \end{pmatrix} = RA_1A_2 \begin{pmatrix} 1 \\ \xi \\ \xi^2 \\ \vdots \\ \xi^N \end{pmatrix} , \qquad (13)$$

where R, A_1 and A_2 are matrices:

$$R = (R_{jk}) \quad , \quad R_{jk} = \sqrt{\frac{(N-j)!}{(N+1)!}}\,\delta_{jk} \quad ,$$

$$A_1 = \begin{pmatrix} a_0^N & 0 & 0 & 0 & & 0 \\ 0 & a_0^{N-1} & 0 & 0 & \cdots & 0 \\ -a_1^N & 0 & a_0^{N-2} & 0 & & 0 \\ 0 & -a_0^{N-1} & 0 & a_0^{N-3} & & 0 \\ \vdots & & & & \ddots & \\ (-1)^{N/2}a_{N/2}^N & 0 & & & & a_0^0 \end{pmatrix} ,$$

and

$$A_2 = \begin{pmatrix} a_0^{N+1} & 0 & 0 & 0 & & 0 \\ 0 & a_0^{N+1} & 0 & 0 & \cdots & 0 \\ a_1^{N+1} & 0 & a_0^{N+1} & 0 & & 0 \\ 0 & a_1^{N+1} & 0 & a_0^{N+1} & & 0 \\ & \vdots & & & \ddots & \\ a_{N/2}^{N+1} & 0 & & & & a_0^{N+1} \end{pmatrix} .$$

If we agree to denote the sum $\sum_{j=1}^{N+1} \xi_j^{\ell}$ as $S^{(\ell)}$, we shall have

$$\sum_j A_2 \begin{pmatrix} 1 \\ \xi_j \\ \xi_j^2 \\ \vdots \\ \xi_j^N \end{pmatrix} = A_2 \begin{pmatrix} S^{(0)} \\ S^{(1)} \\ S^{(2)} \\ \vdots \\ S^{(N)} \end{pmatrix} = \begin{pmatrix} a_0^{N+1} S^{(0)} \\ 0 \\ a_1^{N+1} S^{(0)} - 2a_1^{N+1} \\ \vdots \\ a_{N/2}^{N+1} S^{(0)} - N a_{N/2}^{N+1} \end{pmatrix} = \begin{pmatrix} (N+1) a_0^{N+1} \\ 0 \\ (N-1) a_1^{N+1} \\ \vdots \\ a_{N/2}^{N+1} \end{pmatrix} ,$$

where we have used Newton's formulas for the sums of powers of polynomial roots[5]. Hence, it follows that

$$A_1 A_2 \begin{pmatrix} S^{(0)} \\ S^{(1)} \\ \vdots \\ S^{(2\mu)} \\ S^{(2\mu+1)} \\ \vdots \end{pmatrix} = \begin{pmatrix} N+1 \\ 0 \\ \vdots \\ \sum_{k=0}^{\mu} (-1)^{k+\mu} (N+1-2k) a_{\mu-k}^{N-2k} a_k^{N+1} \\ 0 \\ \vdots \end{pmatrix} ,$$

but

$$\sum_{k=0}^{\mu} (-1)^{k+\mu}(N+1-2k)a_{\mu-k}^{N-2k}a_{k}^{N+1} = (-1)^{\mu}(N+1)a_{\mu}^{N}\delta_{\mu 0} \quad ,$$

so

$$A_1A_2 \begin{pmatrix} S^{(0)} \\ S^{(1)} \\ S^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ S^{(N)} \end{pmatrix} = \begin{pmatrix} N+1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \quad .$$

Remembering now Eq. (13), we can see that

$$\sum_{j=1}^{N+1} p_N(\xi_j)p_{N-k}(\xi_j) = \delta_{k0} \qquad (0 \leqslant k \leqslant N) \quad .$$

In the same way, it can be shown that

$$\sum_{j=1}^{N+1} p_{N-1}(\xi_j)p_{N-k}(\xi_j) = \delta_{k1} \qquad (0 \leqslant k \leqslant N) \quad ,$$

and, by induction [using the recurrence equation of $p_{N-\ell}(\xi)$], one can prove that Eq. (11) holds.

The proof follows the same steps for N odd.

Note that if we replace Eq. (9) into Eq. (11), the latter becomes

$$\sum_{j=1}^{N+1} \frac{N!}{(N+1)\ [He_N(\xi_j)]^2} = 1$$

for $\ell = k = 0$. It agrees with the result obtained for the sum of Christoffel numbers associated to the He polynomials[3,4].

5. EIGENVALUE PROBLEM OF $\hat{p}$

Having obtained the explicit form and some properties of the $\hat{x}$-eigenket components in the $\hat{H}$-basis, let us proceed now to solve the eigenvalue problem of $\hat{p}$.

Defining $\eta = \sqrt{2/(\hbar m\omega)}\, p$, and using the representative Eq. (4), the eigenvalue equation can be cast into the form

$$\begin{pmatrix} 0 & \sqrt{1} & 0 & & & \\ -\sqrt{1} & 0 & \sqrt{2} & & & \\ 0 & -\sqrt{2} & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & \sqrt{N} \\ & & & & -\sqrt{N} & 0 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_N \end{pmatrix} = i\eta \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_N \end{pmatrix} .$$

Hence, we get the characteristic equation

$$He_{N+1}(\eta) = 0 \quad ,$$

showing that the $\hat{p}$-spectrum coincides with the $\hat{x}$-spectrum, excepting a constant. Also, from the corresponding recurrence equation, it is obtained that

$$g_{N-\ell}(i\eta) = (-i)^{\ell}\, p_{N-\ell}(\eta) \quad ,$$

and therefore, these polynomials satisfy the relations

$$\sum_{\ell=0}^{N} g_{\ell}^{*}(i\eta_j) g_{\ell}(i\eta_k) = \delta_{jk} \quad ,$$

$$\sum_{j=1}^{N+1} g_{\ell}^{*}(i\eta_j) g_{k}(i\eta_j) = \delta_{\ell k} \quad ,$$

where η_j is the j-th root of $He_{N+1}(\eta)$ and * means complex conjugate.

6. REPRESENTATIVES OF $\hat{p}$ AND $\hat{H}$ IN THE $\hat{x}$-BASIS

In order to give an answer to one question formulated in the beginning of this work (to know the form of the eigenvalue equation of $\hat{H}$ in the $\hat{x}$-representation), it is required to find $\langle x_j|\hat{p}|x_k\rangle$. The most direct way of doing this, is replacing the orthogonal matrix

$$(\langle E_\ell|x_j\rangle) = \begin{pmatrix} p_0(\xi_1) & p_0(\xi_2) & \cdots & p_0(\xi_{N+1}) \\ p_1(\xi_1) & p_1(\xi_2) & \cdots & p_1(\xi_{N+1}) \\ \cdot & \cdot & \cdots & \cdot \\ p_N(\xi_1) & p_N(\xi_2) & \cdots & p_N(\xi_{N+1}) \end{pmatrix}$$

and Eq. (4) in

$$\langle x_j|\hat{p}|x_k\rangle = \sum_{\ell,n} \langle x_j|E_\ell\rangle\langle E_\ell|\hat{p}|E_n\rangle\langle E_n|x_k\rangle \quad ,$$

to obtain

$$\langle x_j|\hat{p}|x_k\rangle = i\sqrt{\frac{\hbar m\omega}{2}} \sum_{\ell=0}^{N-1} \sqrt{\ell+1}\, [p_{\ell+1}(\xi_j)p_\ell(\xi_k) - p_{\ell+1}(\xi_k)p_\ell(\xi_j)] \quad . \quad (14)$$

Let us calculate $\langle x_j|\hat{p}|x_k\rangle x_j$. By using the recurrence equation of $p_\ell(\xi)$, the explicit form of $p_N(\xi_j)$ and Eq. (10), one can see that

$$\langle x_j|\hat{p}|x_k\rangle x_j = i\,\frac{\hbar}{2} \left\{\delta_{jk} - 1 + \sum_{\ell=1}^{N-1} \sqrt{\ell(\ell+1)}\, [p_{\ell+1}(\xi_j)p_{\ell-1}(\xi_k) - p_{\ell+1}(\xi_k)p_{\ell-1}(\xi_j)]\right\}.$$

In the same way,

$$\langle x_j|\hat{p}|x_k\rangle x_k = i\,\frac{\hbar}{2} \left\{1 - \delta_{jk} + \sum_{\ell=1}^{N-1} \sqrt{\ell(\ell+1)}\, [p_{\ell+1}(\xi_j)p_{\ell-1}(\xi_k) - p_{\ell+1}(\xi_k)p_{\ell-1}(\xi_j)]\right\}.$$

So, subtracting these two equations, we can find that

$$\langle x_j|\hat{p}|x_k\rangle = \begin{cases} (-i\hbar)\dfrac{1}{x_j - x_k} , & x_j \neq x_k , \\ 0 , & x_j = x_k . \end{cases} \tag{15}$$

This is the representative of $\hat{p}$ in the $\hat{x}$-basis. It is interesting to notice the superficial resemblance of this result with the well-known relationship

$$\hat{p} \longleftrightarrow (-i\hbar)\frac{d}{dx} ,$$

when $\hat{x}$ is let to have a continuous spectrum. Do not forget that the difference $x_j - x_k$ can be close to zero, or as great as $\sqrt{(N\hbar)/(2m\omega)}$.

With Eq. (15), the form of the $\hat{H}$-eigenvalue equation

$$(\frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2)|E_n\rangle = E_n|E_n\rangle$$

in the $\hat{x}$-basis can be found immediately. It transforms in

$$-\frac{\hbar^2}{2m}\sum_{k,\ell}{}' \frac{p_n(\xi_k)}{(x_j - x_\ell)(x_\ell - x_k)} + \frac{1}{2}m\omega^2 x_j^2 p_n(\xi_j) = E_n p_n(\xi_j) , \tag{16}$$

where (Σ') means to sum over values making a non-null denominator. $p_n(\xi_k) = p_n(\sqrt{2m\omega\hbar^{-1}}\, x_k)$ is the wave function defined in x_k, corresponding to the eigenvalue E_n. Obviously, it is a solution of Eq. (16).

With this equation, we have finished answering the questions made in the beginning of this work. It is not a differential equation, as Schrödinger's, nor a differences one. An interesting feature, is its nonlocality: the "differential" operator that appears there, takes into account almost all points belonging to the domain of definition and not only those contained in the neighbourhood of some point, as it occurs in the Schrödinger equation. It is interesting too, to ask oneself if its nonlocality is preserved in its limit form when $N \to \infty$ (is such a form exists), because of the place this may occupy in the discussion about the divergences in local field theories[6].

7. EXTREMAL PROPERTIES

It is known that in a finite linear space an eigenvalue equation as $\hat{H}|E_n\rangle = E_n|E_n\rangle$ has certain extremal properties as its eigenvectors are concerned: the Rayleigh quotient,

$$R[x] = \frac{\langle x|\hat{H}|x\rangle}{\langle x|x\rangle} \quad ,$$

defined for any vector $|x\rangle$ such that $\langle x|x\rangle \neq 0$, takes a stationary value in $|x\rangle = |E_n\rangle$. This value is a minimum if $|E_n\rangle = |E_0\rangle$ and maximum if $|E_n\rangle$ is the eigenvector with the greatest eigenvalue (Rayleigh's Principle). In our case the minimum and maximum values of $R[x]$ are $E_0 = (1/2)\hbar\omega$ and $E_{N-1} = [(2N-1)/2]\hbar\omega$, respectively.

What we are going to show here, is that in addition to these properties, the eigenvalues of position make Tr $\hat{H}$ to take a stationary value in $Z_k = x_k$ as Tr $\hat{H}$ is allowed to be a function of (N+1) continuous variables Z_k:

$$\mathrm{Tr}\ \hat{H}(Z_1,\ldots,Z_{N+1}) = \hbar\omega \left[\sum_{k,\ell=1}^{N+1}{}' \frac{1}{(Z_k - Z_\ell)^2} + \frac{1}{4}\sum_{k=1}^{N+1} Z_k^2 \right] \quad .$$

The condition for a stationary point,

$$\frac{\partial}{\partial Z_k} \mathrm{Tr}\ \hat{H}(Z_1,\ldots,Z_{N+1})\Big|_{Z_i=\xi_i} = 0 \quad , \qquad k = 1,2,\ldots,N+1 \quad ,$$

requires the points ξ_j to satisfy

$$\sum_k{}' \frac{1}{(\xi_j - \xi_k)^3} = \frac{1}{8}\xi_j \quad , \tag{17}$$

and, in order to demonstrate this equation, let us regard the following.

From the Christoffel-Darboux formula for the polynomials $p_\ell(\xi)$, one can see that

$$\sum_{\ell=0}^{N-1} \sqrt{N-\ell}\ \left[p_{N-\ell-1}(\xi_j)p_{N-\ell}(\xi_k) - p_{N-\ell}(\xi_j)p_{N-\ell-1}(\xi_k) \right] =$$

$$= (\xi_j - \xi_k) \sum_{\ell=0}^{N-1} \sum_{m=0}^{\ell} p_{N-m}(\xi_j)p_{N-m}(\xi_k) \quad .$$

If we notice that

$$\sum_{\ell=0}^{N-1} \sum_{m=0}^{\ell} p_{N-m}(\xi_j)p_{N-m}(\xi_k) = \sum_{\ell=1}^{N} \ell\ p_\ell(\xi_j)p_\ell(\xi_k) \quad ,$$

and make use of Eqs. (14) and (15), we can obtain

$$\frac{1}{(\xi_j - \xi_k)^2} = \frac{1}{2} \sum_{\ell=1}^{N} \ell\ p_\ell(\xi_j)p_\ell(\xi_k) \quad , \qquad \xi_j \neq \xi_k \quad ;$$

and hence

$$\sum_k{}' \frac{1}{(\xi_j - \xi_k)^3} = \frac{1}{2} \sum_{\ell=1}^{N} \ell\ p_\ell(\xi_j) \left[\sum_k{}' \frac{p_\ell(\xi_k)}{(\xi_j - \xi_k)} \right] =$$

$$= \frac{1}{2} \sum_{\ell=1}^{N} \ell\ p_\ell(\xi_j) \left[\mathcal{D}\ p_\ell(\xi_j) \right] \quad ,$$

where we have defined

$$\mathcal{D}\ p_\ell(\xi_j) \equiv \sum_k{}' \frac{p_\ell(\xi_k)}{(\xi_j - \xi_k)} \quad .$$

To obtain $\mathcal{D}\ p_\ell(\xi_j)$, let us make use of Eqs. (14), (15) and (11) once more. This gives us

$$\mathcal{D}\ p_\ell(\xi_j) = \begin{cases} -\frac{1}{2}\ p_1(\xi_j) \quad , & \ell = 0 \quad , \\ \frac{1}{2}\ [\sqrt{\ell}\ p_{\ell-1}(\xi_j) - \sqrt{\ell+1}\ p_{\ell+1}(\xi_j)] \ , & 1 \leqslant \ell \leqslant N-1 \quad , \\ \frac{1}{2}\sqrt{N}\ p_{N-1}(\xi_j) \quad , & \ell = N \quad . \end{cases}$$

And now, if we replace this result in the equation for $\sum_k{}' \frac{1}{(\xi_j - \xi_k)^3}$ we obtain

$$\sum_k{}' \frac{1}{(\xi_j - \xi_k)^3} = \frac{1}{4} \sum_{\ell=1}^{N} \sqrt{\ell}\, p_{\ell-1}(\xi_j) p_\ell(\xi_j)$$

after some algebra. The right-hand side sum is just the diagonal element $\langle x_j | \hat{a}^\dagger | x_j \rangle$, where $\hat{a}^\dagger$ is the creation operator; consequently

$$\langle x_j | \hat{a}^\dagger | x_j \rangle = \frac{1}{\sqrt{2m\hbar\omega}} \langle x_j | (m\omega\hat{x} - i\hat{p}) | x_j \rangle = \frac{1}{2} \xi_j \quad .$$

Therefore, we have that Eq. (17), condition for a stationary value of $\mathrm{Tr}\, \hat{H}(Z_1, Z_2, \ldots, Z_{N+1})$ in $(\xi_1, \xi_2, \ldots, \xi_{N+1})$, holds, giving thus an additional property to the variational principle above mentioned, with the following difference: in Rayleigh's Principle the variations are made on the states $|x\rangle$; in our case on the values Z_k.

8. CONCLUDING REMARKS

To conclude the present work we shall remark some points that one should take into account in order to, if possible, make an extension of this work to other problems with a discrete spectrum for position. The displayed results have not been obtained from a general systematic method, but in spite of this, one can expect that:

1) The commutator $[\hat{x}, \hat{p}]$, the representative of which is

$$(\langle x_j | [\hat{x}, \hat{p}] | x_k \rangle) = -i\hbar \begin{pmatrix} 0 & 1 & 1 & \cdot & & 1 \\ 1 & 0 & 1 & \cdot & & 1 \\ 1 & 1 & 0 & \cdot & & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & 0 & 1 \\ 1 & 1 & & \cdot & 1 & 0 \end{pmatrix} = (\langle p_j | [\hat{x}, \hat{p}] | p_k \rangle) \quad ,$$

is valid for the finite case, since, suggestively, it becomes the same