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THE PRECESSION OF MERCURY'S PERIHELION VIA PERTURBATION THEORY

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ABSTRACT

Perturbation theory is used to solve the problem of the precession of Mercury's perihelion, this phenomenon being a relativistic effect. The expansion parameter appears naturally when the orbit equation is written in an appropriate form and it completely justifies the use of the first order approximation.

RESUMEN

Se aplica la teoría de perturbaciones al problema de la precesión de origen relativista del perihelio de mercurio. El parámetro del desarrollo aparece naturalmente al reescribir la ecuación de la órbita y justifica completamente la aproximación a primer orden.

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It is well known[1,2] that, in order to describe the precession of the bounded noncircular orbit of a given planet, the best example being Mercury, it is first necessary to take into account the influence of the rest of the planets besides the interaction between the sun and Mercury. Secondly, it is also necessary to treat sun-Mercury interaction beyond Newton's gravitational theory and use the linear approximation of the General Theory of Relativity, i.e., weak fields and small velocities, since in Newtonian theory only closed orbits are allowed. The first correction predicts for Mercury a precession of 531 seconds of arc per century, therefore existing a difference of about 43 seconds of arc between the theoretical and observed values.

The second correction accounts for such a difference and the perihelion shift is calculated through the solution of the corrected differential equation for the orbit[2]:

$$d^2u/d\Theta^2 + u = Gm^2M/J^2 + (3GM/c^2)u^2 \quad , \tag{1}$$

where m and J are the mass and the angular momentum of the planet, M is the mass of the sun, G the gravitational constant, c the speed of light, u is the reciprocal of the distance between the planet and the sun and the last term is the general relativistic correction.

Eq. (1) is obviously nonlinear and its approximate solution, as found in standard text books[2], corresponds to a method of succesive approximations.

On the other hand, straightforward application of Perturbation theory[3] produces both the correct value for the advance of the perihelion and the equation for the precessing orbit, thus providing a natural and systematic way to obtain approximate solutions of Eq. (1).

In order for Perturbation theory to be useful, it has to be shown that the contribution from the nonlinear term in Eq. (1) is small[3,4] and this can be achieved by defining the lengths

$$\ell_1 = J/Gm\,M$$

and

$$\ell_2 = 3GM/c^2 \quad .$$

Here $2\ell_1$ corresponds to the "Latus rectum" of the orbit and ℓ_2 is related to the "gravitational radius" of the sun. We also set

$$\varepsilon = \ell_2/\ell_1 < 10^{-7} \quad ,$$

and write Eq. (1) in the form

$$\eta'' + \eta = 1 + \varepsilon\eta^2 \quad , \tag{2}$$

where $\eta = \ell_1 u$ is a dimensionless variable. Eq. (2) shows that the nonlinear, or perturbative, term is indeed small.

If ε in Eq. (2) is made to vanish, the motion described by the resulting equation corresponds to a closed orbit of period 2π but, since $\varepsilon \neq 0$, the period must change[4] and the new period can be written as

$$P(\varepsilon) = 2\pi + \Delta(\varepsilon) \quad , \tag{3}$$

where $\Delta(\varepsilon)$ is a function assumed to be analytic, such that $\Delta(0) = 0$, hence for ε small

$$P(\varepsilon) = 2\pi(1 + a_1\varepsilon + a_2\varepsilon^2 + \ldots) \quad ,$$

the a's being so far undetermined constants related to the derivatives of $\Delta(\varepsilon)$. In order to find their values we introduce the new variable Ψ through the expression

$$\Theta = \Psi(1 + a_1\varepsilon + a_2\varepsilon^2 + \ldots) \quad ,$$

and realize that, after a complete revolution of the planet, Θ goes from 0 to P, while Ψ increases from 0 to 2π. Hence as a function of Ψ, η has a period of 2π.

Since ε is small, the equation for $\eta(\Psi)$ up to first order in ε,

$$\eta'' + (1 + 2a_1\varepsilon) = (1 + 2a_1\varepsilon)\ (1 + \varepsilon\eta^2) \quad , \tag{5}$$

is a good approximation and, to solve this equation, we assume η to be of the form

$$\eta(\Psi) = \eta_0(\Psi) + \varepsilon\eta_1(\Psi) \quad . \tag{6}$$

After substituting this into Eq. (5) and equating the coefficients of the same powers of ε in both sides, we obtain two linear differential equations:

$$\eta_0'' + \eta_0 = 1 \; ; \qquad \eta_0(0) = 1 + e \; ; \qquad \eta_0'(0) = 0$$

and

$$\eta_1'' + \eta_1 = -2a_1\eta_0 + 2a_1 + \eta^2{}_0 \; ; \qquad \eta_1(0) = 0 \; ; \qquad \eta_1'(0) = 0 \quad , \tag{7}$$

where e stands for the eccentricity of the orbit and the initial conditions have been chosen as to start from the position of the planet at the perihelion of the orbit.

The first of Eqs. (7) is immediately solved to yield the zero-order solution,

$$\eta_0 = 1 + e \cos \Psi \quad , \tag{8}$$

which is now introduced into the right hand side of the second equation to give

$$\eta_1'' + \eta_1 = (-2a_1 + 2)e \cos \Psi + (e^2/2) \; (1 + \cos 2\Psi) + 1 \tag{9}$$

and the necessary and sufficient condition for periodic solutions to exist is the elimination of the secular term, it implies that the coefficient of $\cos\Psi$ must vanish, or $a_1 = 1$.

The above result is better understood if we think of Eq. (9) as representing an harmonic motion of unit frequency acted on by periodic forces of frequencies 1 and 2. If the force of unit frequency does not vanish, the oscillations amplitude would increase indefinitely as a result of the resonance. Such a case is not allowed in our problem and, therefore, the term containing $\cos\Psi$ must vanish.

From the value of a, and Eqs. (3) and (4), we have that the period shift introduced by the perturbation is

$$\Delta(\varepsilon) = 2\pi\varepsilon \quad ,$$

and the solution to Eq. (9) satisfying the initial conditions given in

Eqs. (7) is given by

$$\eta_1(\Psi) = 1 + e^2/2 - (e^2/6)\cos 2\Psi - (1 + e^2/3)\cos\Psi \quad , \tag{10}$$

so that the complete solution to Eq. (2) is then obtained from (6), (8) and (10), and it is written as

$$u(\Theta) = \frac{1}{\ell_1}\left\{ e\cos\frac{\Theta}{1+\varepsilon} + 1 + \varepsilon\left[1 + \frac{e^2}{2} - \frac{e^2}{6}\cos\frac{2\Theta}{1+\varepsilon} + \left(1 - \frac{e^2}{3}\right)\cos\frac{\Theta}{1+\varepsilon}\right]\right\},$$

where it is clear that the period of the motion is now $2\pi(1+\varepsilon)$ in such a way that the perihelion, defined by the equation

$$\ell_1/r_{min} = \ell_1\, u(\Theta = 0) = 1 + e \quad ,$$

advances an amount

$$\Delta = 2\pi\varepsilon = 2\pi\ell_2/\ell_1 = 6\pi(6mM/J)^2$$

after a complete revolution of the planet. In terms of the geometrical parameters of the orbit this shift is

$$\Delta = 6\pi GM/ac^2\,(1 - e^2) \quad , \tag{11}$$

where a is the semimajor axis of the orbit.

Eq. (11) is the standard expression for the perihelion shift and, when appropriate numerical data are used to compute Mercury's precession, it yields the value of 43.03 sec. of arc per century, while the observed value has been reported(5) to be 41.4 ± 0.9 sec. of arc per century, in reasonable agreement with the theory.

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