

CLASSICAL EXACT SOLUTIONS OF A TWO-DIMENSIONAL SUPERSYMMETRIC MODEL

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ABSTRACT

Classical finite-energy exact solutions of a "spin" $0-1/2$ two dimensional supersymmetric model are investigated. The static case can be completely integrated in terms of any solution of the purely bosonic sector,

for an arbitrary potential. The localized fermionic part of the solution can always be rotated to zero by a supersymmetry transformation. This is not the case for the Grassmannian part of the scalar field which can be localized also. Stationary solutions for the fermionic field with potentials corresponding to the ϕ^4 and sine-Gordon theories, for specific choices of the purely bosonic sector, are also considered. In the former case we find a single localized fermionic solution which can not be rotated to zero by a supersymmetry transformation. In the latter case no such solution exists.

RESUMEN

Se investigan soluciones exactas clásicas de energía finita de un modelo supersimétrico en dos dimensiones de espín 0-1/2. El caso estático se puede integrar completamente en términos de cualquier solución del sector puramente bosónico, para un potencial arbitrario. La parte fermiónica localizada de la solución siempre puede ser rotada a cero por medio de una transformación de supersimetría. Este no es el caso de la parte Grassmanniana del campo escalar que también puede ser localizada. También se consideran soluciones estacionarias para el campo fermiónico con potenciales que corresponden a las teorías ϕ^4 y sine-Gordon, para selecciones específicas del sector puramente bosónico. En el primer caso encontramos una solución fermiónica localizada que no puede ser rotada a cero por una transformación de supersimetría. En el segundo caso no existe tal solución.

1. INTRODUCTION

Even at the level of non-operator fields, supersymmetric theories require an extension of the usual commuting number concept to include anticommuting objects (Grassmann numbers) which are related to the fields that describe fermions and to the parameters of the supersymmetry transformations. These Grassmann numbers seem to be necessary in order to have a Lagrangian with a finite number of terms, invariant under transformations that mix fermionic and bosonic fields. We will refer to this non-operator formulation of the theory as to its classical version, even though we are not dealing with commuting numbers only.

In order for this classical theory to be supersymmetric it is necessary to take seriously into account the Grassmannian character of the fermionic fields involved. The equations of motion for the fields will thus contain both kind of numbers and it will be interesting to look for

exact solutions to them. In analogy with what is done in purely bosonic systems, these solutions could lead to useful insights with respect to the corresponding supersymmetric quantum field theory.

One possible way of dealing with such a system of coupled equations, suggested in Refs. 1 and 2, consists in introducing a basis for the Grassmann algebra and expanding every field of the system as an even (commuting) or odd (anticommuting) element of the algebra. Once these expansions are substituted in the equations of motion, these can be further split because now we have to set equal each coefficient of the linearly independent elements of the basis. Usually such a decoupled system is more tractable than the original one and provides a natural algorithm to solve for the unknown commuting components of the fields. It is then possible to obtain a solution to the original system in terms of such components and the independent elements of the basis for the algebra.

Now comes the problem of interpreting such solutions because all physical quantities calculated from them will, in general, carry over some dependence in the Grassmann algebra basis, whose elements are not directly interpretable in physical terms. The ultimate resolution to this problem might be that what really makes sense is the full quantum theory where all physical quantities are non-Grassmann commuting numbers arising from matrix elements of the relevant operators. Nevertheless, it is very difficult to obtain such a theory, especially in a non-perturbative way and when nonlinearities are present. Because of this, one might expect that the route suggested in Ref. 3, would lead to useful insights regarding the quantum theory, as it is in the case of purely bosonic systems. This approach consists of performing a quantization around interesting non-perturbative classical solutions which are required to have finite classical energy and be classically stable as a minimal condition. Such an approximation can be carried out in the path integral formulation for the generating functional of the theory. Essentially, it amounts to an expansion of the action around a relevant set of classical solutions followed by a gaussian functional integration over the remaining field variables. As suggested in Ref. 4, one could try to apply similar techniques for systems involving fermions.

This is essentially our main motivation for studying exact

classical solutions of supersymmetric theories, which will be the subject of the present work. Recently there has been some interest in looking for exact classical solutions to supergravity^(2,5) but we feel that in order to test these ideas it is convenient to start from a simpler system. To this end we consider the following globally supersymmetric system in two dimensions given by the Lagrangian⁽⁶⁾

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} V^2(\phi) + \frac{i}{2} \bar{\psi} \not{\partial} \psi - \frac{1}{2} \bar{\psi} \psi V'(\phi) . \quad (1)$$

Here ϕ is a real scalar field, $V'(\phi) = \frac{d}{d\phi} V(\phi)$; ψ is an anticommuting Majorana spinor ($\bar{\psi} = \psi^\dagger \gamma^0 = \psi C$); the metric is $g^{\mu\nu} = \text{diag} (1, -1)$; and the γ -matrices are given by

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} . \quad (2)$$

The charge conjugation matrix C , which is defined by $C \gamma_\mu C^{-1} = -\gamma_\mu$ can be taken as $C = \gamma^0$ in this representation, and then the Majorana condition simply says that the spinor ψ is real. As it is well known, the action for such a system is invariant under the following supersymmetry transformations:

$$\begin{aligned} \delta\phi &= \bar{\epsilon} \psi , \\ \delta\psi &= -[i\not{\partial}\phi + V(\phi)]\epsilon , \end{aligned} \quad (3)$$

where the parameter ϵ is a constant anticommuting Majorana spinor.

The equations of motion arising from the Lagrangian (1) are

$$\square\phi + V(\phi) V'(\phi) + \frac{1}{2} \bar{\psi} \psi V'(\phi) = 0 , \quad (4a)$$

$$[i\not{\partial} - V'(\phi)]\psi = 0 . \quad (4b)$$

Equations (4) can be considered as the supersymmetric generalization of those purely bosonic systems described by a potential $\frac{1}{2} V^2(\phi)$ which include such interesting cases as the ϕ^4 and sine-Gordon theories. In fact, setting $\psi = 0$ we recover the purely bosonic sector for which there are well

known finite-energy classically-stable solutions.

Classical solutions to the coupled system (4) were found in Ref. 7, by using the invariance of the system under finite supersymmetry transformations. In fact, as suggested in Ref. 8, it is always possible to construct a solution to (4) by applying a supersymmetry transformation to a solution corresponding to the purely bosonic sector $\phi_0 \neq 0, \psi = 0$, which is already known for some class of potentials. The quantum theory around such supertranslated solutions was constructed in Ref. 7 by using Dirac's method for constrained systems and was later generalized to arbitrary (non-supersymmetric) systems with classical solutions parametrized by anticommuting numbers in Ref. 9.

The problem was considered again in Ref. 1 paying attention, among other things, to the question whether or not all solutions to the system (4) are supertranslations from the purely bosonic sector. It was found there that this is not always the case and that the bosonic part of the solution which does not come from a supertranslation is non-localized. The semiclassical quantization of the theory around such classical solutions was also briefly considered in Ref. 1 and it was argued that a naive application of the methods of Ref. 3 leads to inconsistencies.

In this work we study only the classical aspect of the problem and reexamine the coupled system (4). Section 2 contains a brief discussion of the field equations (4) along the same lines of Ref. 1, where an expansion of all the fields in terms of the simplest non-trivial basis of the Grassmann algebra is made. We also write there the most general supertranslated solution obtained from the purely bosonic sector⁽⁷⁾ and the expression for the energy associated with any solution of the field equations. In Section 3 we show that for the static case and a general potential $V(\phi)$ the system can be explicitly integrated in terms of a solution of the purely bosonic sector. We prove then that the fermionic part of the solution can always be rotated to zero by a supersymmetry transformation. The rotated bosonic part contains an extra piece of the type suggested in Ref. 1, which cannot be made zero by the supertranslation. In Section 4 we consider static solutions for the bosonic field but stationary solutions for the fermionic field. We examine the cases corresponding to the sine-Gordon and ϕ^4 potentials, for a specific choice

of the purely bosonic sector. In the former situation we find that it is not possible to have a localized fermionic solution, while in the latter case there is a single frequency such that a well behaved solution exists. This solution has finite energy, which nevertheless is an even number in the Grassmann algebra, and cannot be rotated to zero by a supersymmetry transformation. Finally, Section 5 contains a summary of the results obtained in this work.

2. EQUATIONS OF MOTION

Following Ref. 1, we rewrite the system (4) introducing a Grassmann algebra spanned by two real constant anticommuting numbers, λ_1 , λ_2 ($\lambda_1\lambda_2 = -\lambda_2\lambda_1$), in such a way that a basis for our space is provided by the four real quantities 1, λ_1 , λ_2 and $i\lambda_1\lambda_2$ (recall that $(ab)^* = b^*a^*$ for Grassmann numbers).

The two component fermionic field $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is a real (Majorana) odd element of the algebra and can be expanded as

$$\psi = \chi \lambda \quad , \quad (5)$$

where $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ and χ is a 2x2 matrix whose elements χ_{ij} ($i, j = 1, 2$) are four real non-Grassmann commuting functions of the space-time variables. Because reality is the only condition imposed by the Majorana condition on ψ , we disagree with the general parametrization

$$\chi = i\cancel{A} + B \quad , \quad (6)$$

proposed in Ref. 1, in terms of only two real functions A and B. In particular, Eq. (6) would imply the relation $\chi_{12} = -\chi_{21}$ which does not need to hold in general, as we will see in some examples considered later. Nevertheless it is still conceivable to perform a supersymmetry transformation on the fields ($\chi \rightarrow \chi'$ in particular) in order to require $\chi'_{12} = -\chi'_{21}$. Having in mind that we are dealing with global supersymmetry, which means that the arbitrary parameters of the transformation are constants (instead of space-time dependent) Majorana spinors we can see that it is impossible the fullfill such requirement in general just by changing the gauge.

The bosonic field is an even element of the algebra, and consequently can be expanded as

$$\phi = \phi_0 + i\lambda_1\lambda_2\phi_2, \quad (7)$$

where ϕ_0, ϕ_2 are non-Grassmann real commuting functions.

Substituting Eqs. (6) and (7) in the system (4) and comparing the coefficients of the elements of the basis of the Grassmann algebra we recover Eqs. (8) of Ref. 1 which, in our notation, are

$$\square\phi_0 + V_0V_0' = 0, \quad (8a)$$

$$(i\not{\partial} - V_0')\chi = 0, \quad (8b)$$

$$(\square + V_0'^2 + V_0V_0')\phi_2 = (\det. \chi)V_0', \quad (8c)$$

where $V_0 = V(\phi_0)$ and $V_0' = V'(\phi)|_{\phi=\phi_0}$. In the following we will look for solutions to Eqs. (8) which are generated by localized solutions of the non-linear equation (8a).

The purely bosonic sector corresponding to system (8) is defined by taking ϕ_0 as any solution of Eq. (8a) together with $\psi = \phi_2 = 0$. From this sector we can obtain a full set of solutions to Eqs. (8) (with $\psi \neq 0, \phi_2 \neq 0$) just by making a supersymmetry transformation on $\phi_0, \psi = \phi_2 = 0$, with parameters $\epsilon_i = K_{ij}\lambda_j$, $i, j = 1, 2$, ($\epsilon = K\lambda$), where K_{ij} are arbitrary real constant commuting numbers⁽⁷⁾. The resulting solution has the form given in (5) and (7) and is determined by

$$\chi^s = -(i\not{\partial}\phi_0 + V_0)K, \quad (9a)$$

$$\phi_2^s = (\det. K)V_0, \quad (9b)$$

where the superscript s is to remind us that this solution is obtained by supertranslating the bosonic sector. We remark is passing that for this case we have

$$\chi_{12}^s = K_{12}(\phi_{0,x} - V_0) - K_{22}\phi_{0,t}, \quad (10)$$

$$\chi_{21}^S = K_{11}\phi_{0_t} - K_{21}(\phi_{0_x} + V_0) \quad , \quad (10)$$

which does not satisfy, in general, the constraint $\chi_{12}^S = -\chi_{21}^S$ imposed by the parametrization (6). The subscripts x and t denote the derivatives of the function with respect to the corresponding variables.

The Lagrangian (1) does not depend explicitly on time so that the energy of the system is conserved. The expression for the energy associated with a general solution ϕ_0 , χ , ϕ_2 of the field equations is given by

$$E = \frac{1}{2} \int_{-\infty}^{+\infty} dx \left\{ \phi_{0_t}^2 + \phi_{0_x}^2 + V_0^2 + 2i\lambda_1\lambda_2[\phi_{0_x}\phi_{2_t} + \phi_{0_x}\phi_{2_x} + V_0V_0'\phi_2 + \right. \\ \left. + \frac{1}{2}(\chi_{11}\chi_{22_x} - \chi_{22}\chi_{11_x} + \chi_{21}\chi_{12_x} - \chi_{12}\chi_{21_x}) - (\det. \chi) V_0' \right\} \quad , \quad (11)$$

which in general is an even number of the Grassmann algebra.

3. STATIC SOLUTIONS

In this section we consider the case where all fields are time independent. We show that under this circumstance it is possible to fully integrate Eqs. (8b) and (8c) in terms of a solution $\phi_0(X)$ of Eq. (8a). In fact, this last equation can be integrated once to give the relation

$$V_0 = \pm \sqrt{\phi_{0_x}^2 - \alpha} \quad , \quad (12)$$

where α is an integration constant. From here we can write

$$V_0' = \pm \frac{\phi_{0_{xx}}}{\sqrt{\phi_{0_x}^2 - \alpha}} = \pm \frac{d}{dx} \ln \left(\phi_{0_x} + \sqrt{\phi_{0_x}^2 - \alpha} \right) \quad , \quad (13)$$

which allows us to directly integrate the fermionic equation,

$$i\gamma' \frac{dx}{dx} = V_0' \chi \quad , \quad (14)$$

for each component χ_{ij} separately because γ' is a diagonal matrix. The result can be cast in the form

$$= -(i' \phi_{0_x} + V_0)C \quad , \quad (15)$$

which is valid for both signs of Eq. (12), and where $C = [C_{ij}]$ is a real 2×2 matrix arising from the corresponding integration constants. That χ in (15) is a solution of Eq. (14) can be verified by direct substitution using the properties $(i\gamma')^2 = 1$, $V_0 V_0' = \phi_{0_{xxx}}$ (obtained from the first equality in Eq. (13)) and $V_{0_x} = V_0' \phi_{0_x}$. The fact that we have four real independent constants of integration, C_{ij} , assures us that χ is indeed the most general solution to Eq. (14).

Now we consider Eq. (8c) for the bosonic component ϕ_2 , which can be written as

$$\left\{ -\frac{d^2}{dx^2} + V_0'^2 + V_0 V_0'' \right\} \phi_2 = -\alpha(\det. C) V_0'' \quad , \quad (16)$$

where we have used Eq. (12) in the calculation of $\det. \chi$. The relation

$$V_0'^2 + V_0 V_0'' = \frac{d}{d\phi_0} (V_0 V_0') = \frac{1}{\phi_{0_x}} \phi_{0_{xxx}} \quad (17)$$

allows us to rewrite Eq. (16) in the form

$$\frac{d}{dx} \left[\phi_{0_x}^2 \frac{d}{dx} \left(\frac{\phi_2}{\phi_{0_x}} \right) \right] = \alpha(\det. C) V_0'' \phi_{0_x} \quad , \quad (18)$$

which can be completely integrated to produce

$$\phi_2 = (\det. C) V_0 + \phi_{0_x} \left(\beta \int \frac{dx}{\phi_{0_x}^2} + \gamma \right) \quad , \quad (19)$$

which is the general solution of Eq. (16). In fact, the first term on the right hand side of Eq. (19) corresponds to a particular solution of Eq. (16) while the other two terms are the general solution (two integration constants) of the homogeneous equation associated with it.

The expression (15) together with (19) constitute the general static solution for the system (8), written in terms of a given solution ϕ_0 of the non-linear equation (8a). Now we proceed to compare this

solution with the one obtained by supertranslating the purely bosonic sector ϕ_0 , $\psi = \phi_2 = 0$. Taking the matrix K in Eq. (9a) equal to C in Eq. (15) we see that it is possible to obtain χ and $\phi_2 - \phi_0 \left[\gamma + \beta \int \frac{dx}{\phi_0^2} \right]$ from the purely bosonic sector via a supertranslation. In other words, the only piece of the general solution in the static case which cannot be generated by a supersymmetry transformation from the purely bosonic sector is the contribution to ϕ_2 arising from the homogeneous equation related to Eq. (16). This way of generating solutions to the system (8), which are not supertranslations of the bosonic sector, was already suggested in Ref. 1, and here we have shown that it is the only possibility for the static situation.

The energy corresponding to the solution given by Eqs. (12), (15) and (19) is

$$E = \int_{-\infty}^{+\infty} dx \left(V_0^2 + \frac{\alpha}{2} \right) + i\lambda_1 \lambda_2 \alpha \beta \int_{-\infty}^{+\infty} \frac{dx}{V_0^2 + \alpha} \quad , \quad (20)$$

provided that $V_0 \rightarrow 0$ as $x \rightarrow \pm \infty$. Under this circumstance the contribution to E coming from the fermionic part χ together with the one coming from the non-homogeneous part of ϕ_2 vanish identically. The same happens with the contribution from the γ -dependent part of ϕ_2 . We then see from Eq. (20) that in order to have finite energy it is enough to require that $\alpha = 0$. Then E in (20) has only a non-Grassmann component even though the associated solution is not a rotation from the purely bosonic sector ($\beta \neq 0$, $\gamma \neq 0$). This result is consistent with the corresponding statement made in Ref. 1 for the case $\chi = 0 = \phi_2$ because the energy is invariant under supersymmetry transformations.

Now we discuss the asymptotic behavior of the solutions χ and ϕ_2 for the case $\alpha = 0$. When this happens we have that $\phi_0 \sim V_0$ in such a way that $\chi \rightarrow 0$ as $x \rightarrow \pm \infty$ according to Eq. (15). The Eq. (16) for ϕ_2 reduces to the corresponding homogeneous and this shows up in the explicit solution (19) in the fact that now the term proportional to $\det C$ is of the same type as the one proportional to γ . In fact we have

$$\phi_2 = V_0 \left[\gamma' + \beta' \int \frac{dx}{V_0^2} \right] \quad . \quad (21)$$

The asymptotic behavior of ϕ_2 is related to the asymptotic equation

$$\left(-\frac{d^2}{dx^2} + \mu^2\right)\phi_2 = 0 \quad , \quad (22)$$

where we have considered potentials such that $V'_0 = \mu$ ($\mu = \text{constant}$) when $x \rightarrow \pm\infty$ which is the case of the sine-Gordon and ϕ^4 theories. From Eq. (22) we have two possibilities for the asymptotic behavior of ϕ_2 : (i) either $\phi_2 \rightarrow e^{\pm\mu x}$ which blows up at $\pm\infty$ (this term must arise from the β' part in Eq. (21)); (ii) or $\phi_2 \rightarrow 0$ and $\phi_{2,xx} \rightarrow 0$ separately when $x \rightarrow \pm\infty$ which is the case of the γ' term in Eq. (21). Thus, contrary to what is stated in Ref. 1, not all solutions of the homogeneous equation related to (16) are non-localized (scattering states) as shown by setting $\beta' = 0$, $\gamma' \neq 0$ in Eq. (21).

4. STATIONARY SOLUTIONS

In this section we still require the bosonic field ϕ_0 to be time independent but now we look for fermionic solutions of the type

$$\chi(x,t) = e^{i\omega t} \Delta(x) \quad , \quad (23)$$

with real frequency $\omega \neq 0$. Because Eq. (8b) is linear and we restrict ourselves to real spinors it will be necessary to consider only the real part of χ at the end of the calculation.

The fermionic Eq. (8b) reduces to

$$\begin{aligned} \left(\frac{d}{dx} + V'_0\right)\Delta_{1k} &= i\omega \Delta_{2k} \quad , \\ \left(\frac{d}{dx} - V'_0\right)\Delta_{2k} &= i\omega \Delta_{1k} \quad , \end{aligned} \quad (k = 1,2) \quad (24)$$

after the substitution (23) is made. This is a simple system of coupled linear equations which can be disentangled in the usual way yielding two Schrödinger equations

$$\left(\frac{d^2}{dx^2} + \omega^2 + V_a\right)\Delta_{ak} = 0 \quad , \quad (25)$$

where $a = 1, 2$ is not summed. The corresponding potentials are

$$V_1 = \frac{dV_0'}{dx} - V_0'^2, \quad (26a)$$

$$V_2 = \frac{dV_0'}{dx} - V_0'^2. \quad (26a)$$

We will be interested in solutions to Eqs. (25) which give finite contributions to the classical energy (11). This can be guaranteed by requiring the solutions to go to zero fast enough when $x \rightarrow \pm\infty$ and we will take this behavior as our boundary conditions for Eqs. (25).

Contrary to the situation in the static case, we can no longer study the problem in general terms and we are forced to specialize to some specific potentials in order to find explicit solutions. We are going to consider the following two examples whose bosonic sector is well understood

$$V = 2 \sin \phi/2, \quad \phi_0 = 4 \operatorname{tg}^{-1}(e^x), \quad (27a)$$

$$V = \left(\frac{\lambda}{2}\right)^{\frac{1}{2}} \left(\phi^2 - \frac{1}{\lambda}\right), \quad \phi_0 = \frac{1}{\sqrt{\lambda}} \operatorname{tgh} \frac{x}{\sqrt{2}}, \quad (27b)$$

which are the one soliton solution to the sine-Gordon potential and the kink solution to the ϕ^4 theory respectively. The functions ϕ_0 chosen in (27) obviously satisfy Eq. (8a) and we want to look for the localized supersymmetric extension of such bosonic solutions. Besides, both functions correspond to the choice $\alpha = 0$ in Eq. (12).

In both cases (27a) and (27b) the potentials associated with the fermionic Schrödinger equations (25) can be written in the form

$$V_a = \rho_a + \mu_a \operatorname{sech}^2(vx), \quad (28)$$

where

$$\begin{aligned} \rho_1 &= \rho_2 = -1, \\ \mu_1 &= 0, \quad \mu_2 = 2, \end{aligned} \quad (29)$$

$$v = 1,$$

in the sine-Gordon theory, while

$$\begin{aligned} \rho_1 = \rho_2 &= -2 & , \\ \mu_1 = 3 & , \quad \mu_2 = 1 & , \\ v &= \frac{1}{\sqrt{2}} \end{aligned} \tag{30}$$

in the ϕ^4 case. That is to say, we have to look for solutions, with appropriate boundary conditions, of the generic equation

$$\left(\frac{d^2}{dx^2} + \omega^2 + \rho + \mu \operatorname{sech}^2 vx \right) \Delta = 0 \quad , \tag{31}$$

paying attention to the fact that we must find eigenfunctions Δ with the same frequency ω for the different values of μ in each case. Eq. (31) is solved in Ref. 10 and we adapt those results to our particular situation.

Let us begin with the sine-Gordon case. From the fact that $V_1 = -1$ ($\mu_1 = 0$) we see that Eq. (31) reduces to the Schrödinger equation for a free particle. Thus we conclude in this case that there are no solutions of the type (23) ($\omega \neq 0$), which satisfy the chosen boundary conditions, for the particular election of the function ϕ_0 made in Eq. (27a). Of course we can always go back to the static case $\omega = 0$, where equation (31) does not appear and where we can certainly find adequate solutions.

Now we turn to the ϕ^4 theory with ϕ_0 given by Eq. (27b). The spectra for solutions of Eq. (31) which go to zero as $x \rightarrow \pm \infty$ is

$$\begin{aligned} (\omega^2 - 2)_1 &= -2, \quad -\frac{1}{2} & , \\ (\omega^2 - 2)_2 &= -\frac{1}{2} & , \end{aligned} \tag{32}$$

where the subscripts 1 and 2 refer to the potentials V_1 and V_2 , respectively. We have then found a common real frequency $\omega^2 = 3/2$ whose corresponding eigenfunctions are given by⁽¹⁰⁾

$$\Delta_{1k} = A_k \frac{\sinh x/\sqrt{2}}{\cosh^2 x/\sqrt{2}} \quad , \tag{33a}$$

$$\Delta_{2k} = B_k \frac{1}{\cosh x/\sqrt{2}} \quad , \quad (33b)$$

which behave like $e^{\pm x/\sqrt{2}}$ as $x \rightarrow \pm \infty$, thus describing a highly localized fermionic configuration. The complex integration constants A_k and B_k are related through $A_k = i\sqrt{3} B_k$, which is a consequence of the first order equations (23). The final answer for the components χ_{ik} is

$$\chi_{1k} = -\sqrt{3} b_k \sin(\omega t + \varphi_k) \frac{\sinh x/\sqrt{2}}{\cosh^2 x/\sqrt{2}} \quad , \quad (34a)$$

$$\chi_{2k} = b_k \cos(\omega t + \varphi_k) \frac{1}{\cosh x/\sqrt{2}} \quad , \quad (34b)$$

which is obtained by taking the real part of Eq. (23) and where $B_k = b_k e^{i\varphi_k}$, $\omega^2 = 3/2$.

In order to complete the solution of system (8) for this particular case of the ϕ^4 theory we need to find ϕ_2 from Eq. (8c), which can be written as

$$\left(\frac{d^2}{dx^2} - 2 + 3 \operatorname{sech}^2 \frac{x}{\sqrt{2}} \right) \phi_2 = \sqrt{6\lambda} b_1 b_2 \sin(\varphi_1 - \varphi_2) \frac{\sinh x/\sqrt{2}}{\cosh^3 x/\sqrt{2}} \quad , \quad (35)$$

under the assumption that ϕ_2 is time independent. Eq. (35) can be fully integrated in terms of a particular solution plus the general solution of the corresponding homogeneous equation which was written in terms of ϕ_0 in Eq. (19) of Section 3. A particular solution is

$$\phi_2 = -\frac{\sqrt{3\lambda}}{2} b_1 b_2 \sin(\varphi_1 - \varphi_2) \frac{1}{\cosh^2 x/\sqrt{2}} \quad , \quad (36)$$

which also describes a localized excitation. In the following we will set the homogeneous term equal to zero because, in general, it leads to non-localized contributions to ϕ_2 as shown at the end of Section 3.

The main point to be noticed regarding the solution of system (8) given by the fields (27b), (34) and (36) is that, contrary to

the situation in the static case, the fermionic part of the solution together with the non-homogeneous part of ϕ_2 cannot be obtained from the purely bosonic sector via a supersymmetry transformation. This can be easily seen for the time dependent fermionic field (34) because a supersymmetry rotation with constant parameters from a time independent purely bosonic sector cannot produce a time dependent field. It is also a simple matter to see that ϕ_2 in Eq. (36) is not of the rotated form (9b). In this sense the above solution differs drastically from those already found in the literature for the system described by the Lagrangian (1). The fermionic part of this solution also fails to satisfy the requirement $\chi_{12} = -\chi_{21}$ imposed by the parametrization (6) used in Ref. 1.

Finally we have calculated the energy associated to the above solution according to the expression (11). The finite result

$$E = \frac{2\sqrt{2}}{3\lambda} + i\lambda_1\lambda_2 \frac{2}{\sqrt{3}} b_1 b_2 \sin(\psi_1 - \psi_2) \quad (37)$$

is an even number of the Grassmann algebra and the purely numerical part comes from the contribution of the kink ϕ_0 .

5. SUMMARY

In this work we study the classical finite-energy solutions of a two-dimensional "spin" $0-\frac{1}{2}$ globally supersymmetric system given by the Lagrangian (1) and characterized by the superpotential $V(\phi)$. Special emphasis has been made in the search of solutions for the bosonic and fermionic fields which cannot be generated from a purely bosonic field via a supersymmetry transformation.

We have taken the point of view that the fermionic field components must be considered as anticommuting numbers (elements of a Grassmann algebra) in order to realize the supersymmetry invariance with transformations (3) in the Lagrangian (1). Notice for example that $\bar{\psi}\psi$ would be identically zero if the components of ψ were commuting numbers. This forces us to extend the meaning of the bosonic and fermionic fields to even and odd elements of the algebra respectively. In order to implement this idea in a simple way we follow the suggestion of expanding such fields, according

to Eqs. (5) and (7), in terms of a basis which generates a real two-dimensional Grassmann algebra⁽¹⁾. This expansion provides us with the set of equations (8) for the non-Grassmann components of the fields, ϕ_0 , χ_{ij} , ϕ_2 which can be solved in a sequential manner.

The static case is considered in the first place. We show that the linear equations (8b) and (8c) can be fully integrated for an arbitrary superpotential in terms of any solution ϕ_0 of the non-linear equation (8a). Having found the general solutions for χ_{ij} and ϕ_2 we then prove that the fermionic components χ_{ij} can always be rotated to zero by a global supersymmetry transformation. However such transformation does not reduce to zero the general expression of the corresponding component ϕ_2 of the bosonic field. The remaining part comes from the solution of the homogeneous equation related to Eq. (8c), as suggested in Ref. 1, and may be a localized function according to the choice of parameters. The energy associated to such static solutions can always be made finite by the choice $\alpha = 0$ in (12). Besides, it is a pure non-Grassmann number even though the ϕ_2 component of the bosonic field cannot be completely rotated to zero.

The next thing we do is to look for solutions of the system (8) which are static in the bosonic field but stationary in the fermionic field. Here we cannot proceed in full generality and we discuss the separate cases where the non-linear equation (8a) corresponds to a sine-Gordon theory or a ϕ^4 theory. We need to specify further the solution ϕ_0 of Eq. (8a) which is chosen to be the one-soliton and the kink respectively. In the former case we find no localized solution for the fermionic equation (8b). In the latter situation a single frequency (up to \pm signs) is found such that a localized fermionic solution exists. This solution cannot be rotated to zero by a supersymmetry transformation. The ϕ_2 component of the bosonic field can also be determined exactly and for the sake of simplicity we consider only a particular solution of Eq. (8c). This piece of the bosonic field is localized also and cannot be transformed to zero either. The energy of such complete solution is finite but this time it is an even number of the Grassmann algebra.

As we mention in the Introduction we expect that the physical meaning of such solutions will be revealed only after a semiclassical approximation to the full quantum theory is made. Such approximation

would use these solutions as a starting point and would take proper care of the anticommuting parameters already introduced.

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