AN INTEGRAL REPRESENTATION OF THE NAVIER-STOKES EQUATION-I

Ramon Peralta-Fabi[*]

Departamento de Física
Universidad Autónoma Metropolitana-Iztapalapa
Apdo Postal 55-534, México 09340, D.F.

ABSTRACT

An integral representation of the Navier-Stokes equations for an incompressible viscous fluid is given. Making use of standard integral transform methods and considering the longitudinal components of the velocity field, thereby eliminating the pressure field, the Navier-Stokes equations are cast in integral form. The structure of the resulting equation for the velocity field, in time and space variables, is then discussed. The stationary case is also considered.

RESUMEN

Se obtiene una representación integral de las ecuaciones de Navier-Stokes para un fluido viscoso incompresible. Usando métodos convencionales de transformadas integrales y estudiando la componente longitu dinal del campo de velocidad, eliminando así el campo de presión, se formu

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[*] Present Address: Departamento de Física, Facultad de Ciencias, UNAM
Apdo. Postal 70-542, México 04510, D.F.

lan las ecuaciones en forma integral. Se analiza la estructura de la ecua
ción resultante para el campo de velocidad, así como la dependencia espa-
cial y temporal explícita de los núcleos (kernels). Se considera también
el caso estacionario.

1. INTRODUCTION

The formulation of the equations describing the dynamics of a
fluid dates back to the first half of the past century. The first
formulation for a viscous incompressible fluid is due to C.L.M.H. Navier
(1822)[1]. A few years later, the same equations were derived, without
the special assumptions regarding molecules and their interactions as
introduced by Navier, by Saint-Venant (1843)[2] and by Stokes (1845)[3].
The foundations of the resulting equations are now well established and
stem from the conservation principles of mass, momentum and energy. When
these are supplemented by constitutive equations the result is a system
of partial differential equations characteristic of the classical theory
of fields for a system with an infinite number of degrees of fredom[4].

The intrinsically non linear character of the equations has
proved to be an unsurmountable difficulty that has severely restricted
their practical use. The limited understanding of the turbulent motion of
fluids and the lack of a comprehensive theory of turbulence is a consequence
of this mathematical complication[5].

Here, an equivalent formulation of the equations for a viscous
and incompressible fluid is presented. The motivation is that an
alternative point of view might help to gain new insights on this formida-
ble problem. The final result is a non linear integral equation for the
velocity field alone, involving a single convolution over the space and
time variables.

2. INTEGRAL REPRESENTATION OF THE NAVIER-STOKES EQUATIONS

As it is well known[4], the equations describing the space-time
behavior of a viscous and incompressible fluid are the so called Navier-
Stokes equations:

$$\nabla \cdot \vec{U} = 0 \quad , \tag{1}$$

$$\frac{\partial \vec{U}}{\partial t} + (\vec{U} \cdot \nabla)\vec{U} = -\nabla(p/\rho) + \nu\nabla^2\vec{U} + \vec{F} \quad , \tag{2}$$

where $\vec{U}(\vec{r},t)$ is the velocity field at time t and at the point $\vec{r}$, p/ρ is the pressure field divided by the (constant) density, ν is the kinematic viscosity coefficient and $\vec{F}$ a given external force density. This set of four, coupled, nonlinear parabolic partial differential equations must be solved on a certain domain Ω with given initial and boundary conditions. Let $\vec{U}(\vec{r},0)$ be the initial velocity field and assume that $\vec{U}(\vec{r},t)$ vanishes on all solid boundaries; the no-slip boundary condition holds on the surface $\delta\Omega$, *i.e.*,

$$\vec{U}(\vec{r},t) = 0 \quad , \qquad \vec{r} \; \varepsilon \; \delta\Omega \quad .$$

In order to transform the above posed problem into an integral equivalent problem one can proceed as follows: The non-linear term is expressed in terms of a suitably defined second rank tensor and the resulting equations are then Laplace-Fourier transformed. Considering only the longitudinal components, thereby eliminating the pressure field, the equations are then transformed back by using the convolution theorem. The final step is to restore the bilinear term and to integrate by parts. The final result is a vector integral equation for the velocity field in which the initial condition is incorporated in a natural way. All the remaining quantities are either given or fully known.

The above sketched procedure is carried out explicitly in what follows.

Define the symmetric second rank tensor $\underline{V}(\vec{r},t)$ by

$$\underline{V} = \vec{U}\vec{U} \quad , \qquad \textit{i.e.} \; , \qquad V_{ij} = U_i U_j \quad . \tag{3}$$

Equation (2) can then be rewritten, using the solenoidal character of the velocity field, as

$$\frac{\partial \vec{U}}{\partial t} + \nabla \cdot \underline{V} - \nu\nabla^2\vec{U} = \vec{F} - \nabla(p/\rho) \quad . \tag{4}$$

Let f_k be the Fourier transform of any of the components of the fields:

$$f_k = \int_{-\infty}^{\infty} e^{i\vec{k}\cdot\vec{r}} f(\vec{r},t) d\vec{r} \; .$$

The fields are assumed to vanish at infinity if the fluid is unbounded. For a finite domain, which is usally the case, the assumption is that the fields vanish on the boundaries and remain null on to infinity. In the case of an unbounded turbulent flow the required conditions can be relaxed by using instead generalized harmonic analysis[5,6]. Taking the space Fourier transform of Eqs. (1) and (4) leads to

$$\vec{k}\cdot\vec{U}_k = 0 \quad , \tag{5}$$

$$\frac{\partial}{\partial t}\vec{U}_k - i\vec{k}\cdot\vec{V}_k + \nu k^2\vec{U}_k = \vec{F}_k + i\vec{k}\,p_k/\rho \; . \tag{6}$$

Solving for p_k/ρ by eliminating the transverse components of $\vec{U}_k$ one gets

$$p_k/\rho = \frac{i\vec{k}}{k^2}\cdot\vec{F}_k - \frac{\vec{k}}{k}\cdot\vec{V}_k\cdot\frac{\vec{k}}{k} \quad .$$

Upon substitution of this result into Eq. (6) gives

$$\frac{\partial}{\partial t}\vec{U}_k + \nu k^2\vec{U}_k = (i\vec{k}\cdot\vec{V}_k + \vec{F}_k)\cdot\nu k^2 T_k \; , \tag{7}$$

where T_k is the Fourier transform of the Oseen tensor:

$$(T_k) = \frac{1}{\nu k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \quad . \tag{8}$$

Equation (7) is a well known integro-differential equation for $\vec{U}_k$,[7,8,9] since V_k is explicitly given by

$$V_k = \int_{-\infty}^{\infty} \vec{U}_{k-k'}\cdot\vec{U}_{k'}\cdot d\vec{k}' \quad .$$

By taking the one sided Laplace transform of Eq. (7) it follows that

$$s\vec{U}_{ks} - \vec{U}_{k0} + \nu k^2\vec{U}_{ks} = (i\vec{k}\cdot\vec{V}_{ks} + \vec{F}_{ks})\cdot\nu k^2 T_k \; ,$$

here

$$f_{ks} = \int_0^\infty e^{-st} f_k(t)\, dt \quad ,$$

and $\vec{U}_{k0}$ is the Fourier transform of the velocity field at time $t = 0$. Next, one solves for $\vec{U}_{ks}$:

$$\vec{U}_{ks} = \frac{1}{\nu k^2 + S}\, \vec{U}_{k0} + \frac{\nu k^2}{\nu k^2 + S}\, \underset{\sim}{T}_k \cdot (\vec{F}_{ks} + i\vec{k} \cdot \underset{\sim}{V}_{ks}) \quad ,$$

and takes the inverse Fourier transform:

$$\vec{U}_s = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{-i\vec{k}\cdot\vec{r}}\, \vec{U}_{ks}\, d\vec{k} \quad .$$

The result, in three dimensions, is

$$\vec{U}_s = \int_{-\infty}^{\infty} dr'\, \overline{I}_0(\vec{r} - \vec{r}', S)\vec{U}(\vec{r}', 0) + \int_{-\infty}^{\infty} d\vec{r}'\, \overline{\underset{\sim}{G}}(\vec{r} - \vec{r}', S)\, (\vec{F}_s(r') - \nabla'\cdot\underset{\sim}{V}_s(\vec{r}')), \tag{9}$$

where

$$\overline{I}_0(\vec{r}, s) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{-i\vec{k}\cdot\vec{r}}\, \frac{d\vec{k}}{\nu k^2 + S} = \frac{e^{-r\sqrt{s/\nu}}}{4\pi\nu r} \quad , \tag{10}$$

and

$$\overline{\underset{\sim}{G}}(\vec{r}, s) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\nu k^2 e^{-i\vec{k}\cdot\vec{r}}}{\nu k^2 + S}\, \underset{\sim}{T}_k\, d\vec{k}$$

$$= \underset{\sim}{I}\, \overline{I}_0(\vec{r}, s) + \frac{\nu}{S}\, \nabla\nabla \left[\overline{I}_0(\vec{r}, s) - \overline{I}_0(\vec{r}, 0) \right] \quad , \tag{11}$$

$\underset{\sim}{I}$ is the unit second rank tensor and $\nabla\nabla$ is the dyadic representation of the tensor that results from applying twice the gradient operator. Note that

$$\left[\nabla\nabla\, f(r) \right]_{ij} = \delta_{ij}\, \frac{1}{r}\, \frac{d}{dr}\, f + \frac{x_i x_j}{r^2} \left(\frac{d^2}{dr^2}\, f - \frac{1}{r}\, \frac{d}{dr}\, f \right) \quad . \tag{12}$$

Then, by taking the inverse Laplace transform of Eq. (9) and again making use of the convolution theorem and of Eq. (12), one obtains

$$\vec{U}(\vec{r},t) = \int_{-\infty}^{\infty} d\vec{r}' \ I_0(\vec{r}-\vec{r}',t)\vec{U}(\vec{r}',0) \ +$$

$$\int_0^t dt' \int_{-\infty}^{\infty} d\vec{r}' \ \underset{\sim}{G}(\vec{r}-\vec{r}', \ t-t') \cdot \left[\vec{F}(\vec{r}',t') - \nabla' \cdot \underset{\sim}{V}(\vec{r}',t')\right],$$

where

$$I_0(\vec{r},t) = \frac{\phi(3/2, \ 3/2; \ -r^2/4\nu t)}{(4\pi\nu t)^{3/2}} = \frac{e^{-r^2/4\nu t}}{(4\pi\nu t)^{3/2}} \quad , \tag{13}$$

$$\underset{\sim}{G}(\vec{r},t) = \underset{\sim}{I}\left[\left(1+\frac{2\nu t}{r^2}\right) I_0(\vec{r},t) - 2I_1(\vec{r},t)\right] +$$

$$\frac{\vec{r}\vec{r}}{r^2}\left[6I_1(\vec{r},t) - \left(1 + \frac{6\nu t}{r^2}\right)I_0(\vec{r},t)\right] \tag{14}$$

and

$$I_1(\vec{r},t) = \left(\frac{\nu t}{r^2}\right)\frac{\phi(1/2,3/2; \ -r^2/4\nu t)}{(4\pi\nu t)^{3/2}} = \frac{1}{8\pi r^3} \ erf\left(\frac{r}{2\sqrt{\nu t}}\right) \quad , \tag{15}$$

$\phi(a,b; \ x)$ is the confluent hypergeometric function[10]. The final step is to integrate by parts the last term in the second integral in order to eliminate the differential operator. The boundary conditions are then again implicitly incorporated into the equations. The expressions given by Eqs. (13)-(15) are valid for an unboundend fluid. Should this not be the case, the kernels would show an explicit boundary dependence (their geometry).

The resulting integral equation for the velocity field is

$$\vec{U}(\vec{r},t) = \int_{-\infty}^{\infty} d\vec{r}' \ I_0(\vec{r}-\vec{r}',t)\vec{U}(\vec{r}',0)$$

$$+ \int_0^t dt' \int_{-\infty}^{\infty} d\vec{r}' \ \underset{\sim}{G}(\vec{r}-\vec{r}', \ t-t') \cdot \vec{F}(\vec{r}',t')$$

$$- \int_{0,t}^t dt' \int_{-\infty}^{\infty} dr' \ \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} \vec{U}(\vec{r}',t') \cdot \underset{\sim}{P}(\vec{r}-\vec{r}', \ t-t') \cdot \vec{U}(\vec{r}',t') \tag{16}$$

$$- \int_0^t dt' \int_{-\infty}^{\infty} d\vec{r}' \ \vec{U}(\vec{r}',t') \ \underset{\sim}{Q}(\vec{r}-\vec{r}', \ t-t') \cdot \vec{U}(\vec{r}',t') \ ,$$

where

$$P_{ij}(\vec{r},t) = \delta_{ij}\left[\frac{6}{r} I_1(r,t) - \frac{1}{r}\left(1 + \frac{6\nu t}{r^2}\right) I_0(r,t)\right]$$
$$+ \frac{x_i x_j}{r^2}\left[\frac{5}{r}\left(\frac{r^2}{10\nu t} + 1 + \frac{6\nu t}{r^2}\right) I_0(r,t) - \frac{30}{r} I_1(r,t)\right],$$
(17)

and

$$Q_i = \frac{x_i}{r}\left[\frac{12}{r} I_1(r,t) - \frac{2}{r}\left(\frac{r^2}{4\nu t} + 1 + \frac{6\nu t}{r^2}\right) I_0(r,t)\right],$$
(18)

for $i,j = 1,2,3$; the remaining terms have been previously defined. Again, these last two expressions hold for an unbounded fluid.

The physical basis for Eq. (16) is clearly the same as for the Navier-Stokes equations with the prescribed boundary and initial conditions. However, from the mathematical point of view the equivalence is not altogether simple. There is a single integral over each independent variable left to be done. This means that the class of fields $\vec{U}(\vec{r},t)$ that can be solutions of Eq. (16) is larger than those satisfying Eqs. (1) and (2). The latter being of second order in the spatial derivatives require additional regularity conditions. The former is therefore what is called a weak formulation of the problem.

3. DISCUSSION

The above formulation is general and should be adequate to describe a stationary flow. The corresponding integral equation in this case is obtained as follows.

The first term in the right hand side of Eq. (16) is the one that involves the initial condition. The kernel I_0, given by Eq. (13), exhibits the well known $t^{-3/2}$ ($t^{-d/2}$, in d dimensions) persistence of the initial condition for a diffusion process. The structure of the kernel follows directly from the difussion operator $(\partial_t - \nu\nabla^2)$. In the opposite limit of very short times the kernel is exactly a delta function, as it should. The remaining three integrals vanish identically and the flow

is that given at time $t = 0$. In order to get rid of the initial state consider the limit of very long times. Assume that the velocity field and the external force density do not depend on time. Then, from Eq. (16),

$$\vec{U}(\vec{r}) = \int\limits_{-\infty}^{\infty} d\vec{r}' \left(\int\limits_0^{\infty} dt \ \underset{\sim}{G}(\vec{r} - \vec{r}',t) \right) \cdot \vec{F}(\vec{r}')$$

$$- \int\limits_{-\infty}^{\infty} d\vec{r}' \ \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|} \ \vec{U}(\vec{r}') \cdot \left(\int\limits_0^{\infty} dt \ \underset{\sim}{P}(\vec{r} - \vec{r}',t) \right) \cdot \vec{U}(\vec{r}')$$

$$- \int\limits_{-\infty}^{\infty} d\vec{r}' \ \vec{U}(\vec{r}') \left(\int\limits_0^{\infty} \vec{Q}(\vec{r} - \vec{r}',t)dt \right) \cdot \vec{U}(\vec{r}') \quad .$$

Hence

$$\vec{U}(\vec{r}) = \int\limits_{-\infty}^{\infty} d\vec{r}' \ \underset{\sim}{T}(\vec{r} - \vec{r}') \cdot \vec{F}(\vec{r}')$$

$$+ \int\limits_{-\infty}^{\infty} d\vec{r}' \ (\vec{r} - \vec{r}') \ \vec{U}(\vec{r}') \cdot \underset{\sim}{D}(\vec{r} - \vec{r}') \cdot \vec{U}(\vec{r}') \quad , \tag{19}$$

where

$$\underset{\sim}{T}(\vec{r}) = \frac{1}{8\pi\nu r} \left(\underset{\sim}{1} + \frac{\vec{r}\,\vec{r}}{r^2} \right) \quad ,$$

is the Oseen tensor and

$$8\pi\nu \ \underset{\sim}{D}(\vec{r}) = - \frac{1}{r^3} \left(\underset{\sim}{1} - \frac{3\vec{r}\,\vec{r}}{r^2} \right) \quad ,$$

is the dipole tensor. The proof that

$$\underset{\sim}{T}(\vec{r}) = \int\limits_0^{\infty} \underset{\sim}{G}(\vec{r},t) \ dt \quad , \tag{20}$$

$$\underset{\sim}{D}(\vec{r}) = - \int\limits_0^{\infty} \underset{\sim}{P}(\vec{r},t) \ dt \quad , \tag{21}$$

and

$$\int\limits_0^{\infty} \vec{Q}(\vec{r},t) \ dt = 0 \quad , \tag{22}$$

is sketched in the Appendix. Equation (19) is satisfied by any steady state flow with the prescribed non-slip boundary condition. It can also be derived by carrying out the same procedure followed in the previous section by dropping out the time derivative at the outset.

It is clear that the Oseen tensor appearing in Eq. (19) is the Green's function for the linearized steady-state Navier-Stokes equations[11,12]. The tensor $\mathcal{G}$ plays the same role for the time dependent case. The long range effect of an external force can clearly be seen from the spatial structure of the corresponding kernels.

The last two terms in Eq. (16) are the contribution of the convective term, hence their bilinear form. The vector structure is different on each term as expected on general grounds. The original nonlinear term, being quadratic in $\vec{U}$, must lead to terms, in view of the transformations that were used, of bilinear character. Integrands of the form $U_i U_j S_{ijk}$, with the summation convention, are to be expected. Symmetry arguments lead to a spherically symmetric third rank tensor:

$$S_{ijk} = x_i x_j x_k f(r,t) + (x_i \delta_{jk} + x_j \delta_{ik} + x_k \delta_{ij}) g(r,t) ,$$

which, upon contraction with a bilinear form, gives rise to terms whose tensor nature is precisely that found in Eq. (16).

The time and spatial structure of the kernels, defined by Eqs. (17) and (18), stems essentially from the overall structure of the differential equation; the linear part defines the basic features of the kernels.

In summary, the integral equation for the velocity field, Eq. (16), is an alternative formulation of the Navier-Stokes equations. As such, it has a richer structure in view of the explicit details amenable to analysis; it might be usefull to tackle certain formal problems connected with the global existence of solutions and it seems to be specially suited to obtain numerical solutions. The use of an integral equation, such as the one presented here, to study certain aspects of turbulent flow, could prove to be more than an academic exercise.

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APPENDIX

In order to prove relations (20)-(21) one can proceed as follows[14,13].

Let

$$K(r,t) = I_1(r,t) - \frac{vt}{r^2} I_0(r,t) \quad ,$$

with I_1 and I_0 given by Eqs. (13) and (15), then

$$
\begin{aligned}
\int_0^\infty K(r,t) \, dt &= \int_0^\infty dt \left[\frac{1}{8\pi \, r^3} \, \text{erf} \left(\frac{r}{2\sqrt{vt}} \right) \right. \\
&\quad \left. - \left(\frac{vt}{r^2} \right) \frac{e^{-r^2/4vt}}{(4\pi vt)^{3/2}} \right] \\
&= \frac{1}{12\pi\sqrt{\pi}vr} \int_0^\infty \phi(3/2,5/2; -s^2) \, ds = (16\pi vr)^{-1} \quad ,
\end{aligned}
\tag{A1}
$$

also

$$\int_0^\infty I_0(r,t) \, dt = (4\pi vr)^{-1} \quad . \tag{A2}$$

From Eqs. (14), (17) and (18) one can see that

$$\underset{\sim}{G}(\vec{r},t) = \underset{\sim}{1} \left[I_0(r,t) - 2K(r,t) \right] + \frac{\vec{r}\vec{r}}{r^2} \left[6K(r,t) - I_0(r,t) \right] \quad ,$$

$$
\begin{aligned}
\underset{\sim}{P}(r,t) = \underset{\sim}{1} & \left[\frac{6}{r} K(r,t) - \frac{1}{r} I_0(r,t) \right] \\
& + \frac{\vec{r}\vec{r}}{r^2} \left[\frac{1}{r} \left(5 + \frac{r^2}{2vt} \right) I_0(r,t) - \frac{30}{r} K(r,t) \right] \quad ,
\end{aligned}
$$

$$\vec{Q}(r,t) = \frac{\vec{r}}{r} \left[\frac{12}{r} K(r,t) - \frac{2}{r} \left(1 + \frac{r^2}{4vt} \right) I_0(r,t) \right] \quad .$$

Using relations (A1) and (A2) to evaluate the integrals in $\underset{\sim}{G}$, $\underset{\sim}{P}$ and $\vec{Q}$ one gets Eqs. (20), (21) and (22).

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